Chapter - 1

INTRODUCTION
In this chapter we give certain basic definitions of some algebraic concepts which are indispensable for our work. A brief survey of the work done by Herstein, Bell, Johnsen, Outcalt, Yaqub, Quadri and Abu-khuzam on the commutativity of associative rings and non-associative rings is also given.

**Associative Ring** : An associative ring $R$, sometimes called a ring in short, is an algebraic system with two binary operations addition $+$ and multiplication $\cdot$ such that

(i) the elements of $R$ form an abelian group under addition and a semi group under $\cdot$ multiplication,

(ii) multiplication $\cdot$ is distributive on the right as well as on the left over addition i.e., $(x+y)z = xz+yz, z(x+y) = zx+zy$ for all $x, y, z$ in $R$.

**Non-associative Ring** : A non-associative ring $R$ is an additive abelian group in which multiplication is defined, which is distributive over addition on the left as well as on the right, i.e., $(x+y)z = xz+yz, z(x+y) = zx + zy$, for all $x, y, z$ in $R$.

A non-associative ring differs from an associative ring in that the full associative law of multiplication is no longer assumed to be associative, i.e., it is not necessarily associative. Strictly speaking the associative law of multiplication has not been done away with, it has merely weakened.

The well known examples of non-associative rings are alternative rings, Lie rings and Jordan rings. In 1930 Artin and Max Zorn defined alternative rings.
Alternative Ring: An alternative ring \( R \) is a ring in which \((xx)y = x(xy), y(xx) = (yx)x\) for all \( x, y \) in \( R \). These equations are known as the left and right alternative laws respectively.

Lie Ring: A Lie ring \( R \) is a ring in which the multiplication is anti-commutative, i.e., \( x^2 = 0 \) (implying \( xy = -yx \)) and the Jacobi identity \((xy)z + (yz)x + (zx)y = 0\) for all \( x, y, z \) in \( R \) is satisfied.

Jordan Ring: A Jordan Ring \( R \) is a ring in which products are commutative, i.e., \( xy = yx \) and satisfy the Jordan identity \((xy)x^2 = x(yx^2)\) for all \( x, y \) in \( R \).

Associator: The associator \((x, y, z)\) is defined by \((x, y, z) = (xy)z - x(yz)\) for all \( x, y, z \) in a ring.

This plays a key role in the study of non-associative rings. It can be viewed as a measure of the non-associativity of a ring. This definition is due to Maxzorn wherein he proved that a finite alternative division ring is associative.

In terms of associators, a ring \( R \) is called left alternative if \((x,x,y)=0\), right alternative if \((y, x, x)=0\) for all \( x, y \) in \( R \) and alternative if both the conditions hold.

Commutator: The commutator \([x, y]\) is defined by \([x, y] = xy - yx\) for all \( x, y \) in a ring.

Commutative Ring: If the multiplication in a ring \( R \) is such that \( xy = yx \) for all \( x, y \) in \( R \) then we call \( R \) a commutative ring.
A non-commutative ring differs from commutative ring in that the multiplication is not assumed to be commutative. i.e., we do not assume \( xy = yx \) for all \( x, y \) in \( R \) as an axiom. However, it does not mean that there always exist elements \( x, y \) in \( R \) such that \( xy \neq yx \).

The ring of 2x2 matrices over rationals and the ring of real quaternions due to Hamilton are the examples of non-commutative rings.

**Periodic element**: An element \( x \in R \) is called a periodic element if there exist distinct \( m, n \in \mathbb{Z}^+ \) such that \( x^n = x^m \).

**Potent element**: An element \( x \) of \( R \) is called potent if \( x^n = x \) for some positive integer \( n = n(x) > 1 \).

**Assosymmetric Ring**: An Assosymmetric ring \( R \) is one in which

\[
(x, y, z) = (P(x), P(y) P(z)), \text{ where } P \text{ is any permutation of } x, y, z \text{ in } R.
\]

**Standard ring**: A ring \( R \) is defined to be standard if it satisfies the following two identities:

i) \( (wx, y, z) + (xz, y, w) + (wz, y, x) = 0 \)

ii) \( (x, y, z) + (z, x, y) - (x, z, y) = 0 \), for all \( w, x, y \) and \( z \) in \( R \).

**Accessible ring**: A ring \( R \) is called accessible in case it satisfies the identities:

i) \( (x, y, z) + (z, x, y) - (x, z, y) = 0 \)

ii) \( (w, x), y, z) = 0 \), for all \( w, x, y \) and \( z \) in \( R \).
**Periodic ring**: A ring $R$ is called a periodic ring if for every $x$ in $R$, there exists distinct positive integers $m=m(x)$, $n=n(x)$ such that $x^m=x^n$. Due to Chacron $R$ is periodic if and only if for each $x \in R$, there exists a positive integers $k=k(x)$ and a polynomial $f(\lambda)=f_x(\lambda)$ with integer co-efficients such that $x^k=x^{k+1}f(x)$.

**s-Unital Ring**: A ring $R$ is called a left (respectively right) s-unital ring if $x \in Rx$ (respectively $x \in xR$) for each $x \in R$. Further $R$ is called s-unital if it is both left as well as right s-unital, i.e., if $x \in xR \cap Rx$, for each $x \in R$.

**Weakly Periodic Ring**: A ring $R$ is called a weakly periodic ring if every element of $R$ is expressible as a sum of a nilpotent element and a potent element of $R$. $R=N+P$, where $N$ is the set of nilpotent elements of $R$ and $P$ is the set of potent elements of $R$. It is well-known that if $R$ is periodic, then it is weakly periodic.

**Quasi-periodic ring**: A ring $R$ is called quasi-periodic if for each $x \in R$ there exist integers $n, m$ with $n \geq m \geq 1$ such that $x^n=lx^m$ for some integer $k$.

**Generalised quasi-periodic ring**: A ring $R$ is called generalised quasi-periodic if for each $x \in R$ there exist distinct positive integers $m, n$ and non-zero integers $r, s$ with $(r, s)=1$, for which $rx^m=rx^n$.

**Prime Ring**: A ring $R$ is called a prime ring if whenever $A$ and $B$ are ideals of $R$ such that $AB=0$, then either $A=0$ or $B=0$.

**Semi Prime Ring**: A ring $R$ is semi prime if for any ideal $A$ of $R$, $A^2=0$ implies $A=0$. These rings are also referred to as rings free from trivial ideals.
**Simple Ring**: A ring $R$ is said to be simple if whenever $A$ is an ideal of $R$, then either $A = R$ or $A = 0$.

**Semi-Simple Ring**: A ring is semi simple in case the radical (i.e., the maximal ideal consisting of all nilpotent elements) is the zero ideal.

Obviously a simple ring is prime, which in turn is free from trivial ideals.

**Primitive Ring**: A ring $R$ is defined as primitive in case it possesses a regular maximal right ideal, which contains no two-sided ideal of the ring other than the zero ideal.

**Division Ring**: A ring $R$ is said to be a division ring if its non-zero elements form a group with respect to multiplication.

**Flexible Ring**: If in a ring $R$, the identity $(x, y, x) = 0$ i.e., $(xy)x = x(yx)$ for all $x, y$ in $R$ holds then $R$ is called flexible.

Alternative, commutative, anti-commutative and thereby Jordan and Lie rings are flexible.

**Nilpotent Ring**: A ring is called nilpotent if there is a fixed positive integer $t$ such that every product involving $t$ elements is zero.

**Torsion-free Ring**: A ring $R$ is said to be $m$-torsion free if $mx = 0$ implies $x = 0$ for all $x$ in $R$. 

Reduced Ring: A ring $R$ is called reduced if $N=\{0\}$, where $N$ is the set of nilpotent elements of $R$.

Center: In a ring $R$, the center denoted by $Z(R)$ is the set of all elements $x \in R$ such that $xy = yx$ for all $y \in R$.

Derivation: A derivation of a ring $R$ is an additive group homomorphism $d:R \to R$ satisfying $d(r_1 r_2) = (d(r_1))r_2 + r_1 (d(r_2))$.

It is important to note that this definition does not depend on the associative of multiplication and in fact, we shall have occasion to deal with derivations of non-associative algebras.

In 1963 [39] Herstein proved several results on the commutativity of rings in his book, "Topics in ring theory". In 1976 [41] he proved that if $R$ is a ring having no non-zero nil ideals in which for every $x, y$ in $R$ there exist integers $m = m(x, y) \geq 1$, $n=n(x, y) \geq 1$ such that $[x^n, y^m] = 0$, then $R$ is commutative. Further Bell [8] proved results on some commutativity theorems of Herstein. In 1986 Quadri and others studied semiprime rings and proved that a semiprime ring in which $(xy)^2 - xy$ is central for every $x, y$ in $R$ is commutative. In all these results, the ring is associative.

In 1968 [45] Johnsen, Outcalt and Yaqub proved an elementary commutativity theorem for non-associative rings. They showed that a ring $R$ with unity satisfying $(xy)^2 = x^2y^2$ for all $x, y$ in $R$ is necessarily commutative. In this
direction, Giri, Modi, Rakhunde and others proved some commutativity theorems for certain non-associative rings.

In 1976 Bell [9] studied some commutativity results for periodic rings. He proved that if \( R \) is a periodic ring such that \( N \), the set of nilpotent elements of \( R \) is commutative, then the commutator ideal of \( R \) is nil and \( N \) forms an ideal of \( R \). He presented a variety of restrictions on nilpotent elements which imply some measure of commutativity in periodic rings. In [10] Bell proved the commutativity of periodic rings satisfying \( xy = yxs \), where \( s = s(x, y) \) is an element in the center of the subring generated by \( x \) and \( y \). In [12] Bell presented new proofs of two basic results-Chacron's result that co-algebraic rings are necessarily periodic, and Herstein's result that periodic rings with central nilpotent elements are commutative. Abu-Khuzam [2] proved some commutativity results for periodic rings with polynomial identities. Abu-Khuzam, Bell and Yaqub [6] proved commutativity theorems for s-unital rings satisfying polynomial identities.

In this direction many mathematicians studied the commutativity of certain associative and non-associative rings. In the next chapter we will study the commutativity of semiprime associative and non-associative rings satisfying some identities.