CHAPTER III

3. AN E.O.Q MODEL FOR DETERIORATING ITEMS WITH INVENTORY RETURNS AND SPECIALS SALES

3.1: INTRODUCTION

In this chapter we reconsider the economic order quantity model for inventory returns and special sales (Naddor [28]). Here the items are subjected to linear deterioration, which is of interest in the recent past. The focus of the discussion is to consider a situation where the optimal
stock level of inventory system is less than the amount on hand. Naddor[28] has consider this problem in case of E.O.Q inventory systems. Dave [10] has considered this problem in the context of order level inventory model. This model is presented in a novel manner by considering shortages with prescribed scheduling period for deterministic demand.

The problem of returning excessive inventory to a vendor or of special sales usually arises when the optimal lot size of an inventory system is less than the amount on hand. Obviously, if no loss is incurred when returning or selling this excess stock, there is no problem. However, if there is such a loss, the optimal amount to return (or) sell must be determined by minimizing the losses due to various costs involved in the inventory system.

An insight to this problem can be gained by studying a deterministic inventory system for deteriorating items. The effect of deterioration cannot be ignored in many inventory systems. An economic order quantity model for deteriorating is developed for a system with inventory
returns. The present work includes two inventory models viz., an infinite planning horizon model and finite horizon model.

3.2 ASSUMPTIONS AND NOTATIONS:

The following assumptions are made in order to develop the E.O.Q models for infinite and finite planning periods.

1. Demand is constant and uniform at a rate of ‘R’ units per unit time.

2. The carrying cost is $C_1$ per unit per unit time; shortage cost is $C_2$ per unit per unit time; the replenishment cost is $C_3$ per order and the cost of returning (or) selling is $C_4$ per unit.

3. A constant fraction ‘$\theta$’ of the on hand inventory deteriorates per unit time.

4. Replenishment rate is infinite; replenishment cost is constant and ‘$q$’ be the optimal lot size.

5. The system starts with an amount of ‘$Q$’ units on hand of which only ‘$P$’ units are retained after returning (or) selling the rest. The problem is to determine optimal value of ‘$P$’.
3.3 AN INFINITE HORIZON MODEL:
Consider the period ‘T’ with the on hand inventory ‘Q’ units and final inventory is assumed to be zero since (Q-P) units are returned (or) sold.
The retained ‘P’ units are to be consumed during the time ‘t₁’, during the remaining period (T-T₁) the optimum E.O.Q model for deteriorating items (Ghare and Schrader[13])will operate.

Now Q₁(t) denotes the inventory position at time ‘t’ (0 ≤ t ≤ t₁), then the differential equation that describes the instantaneous state of Q(t) when the items are deteriorating at the rate of ‘θ’ (0<θ<1) is given by

\[
\frac{d}{dt}Q₁(t) + θQ₁(t) = -R \quad 0 ≤ t ≤ t₁
\]  

...(3.3.1)

Using the boundary conditions

Q₁(0) = P and Q₁(t₁) = 0, the solution of (3.3.1) is

\[
Q₁(t) = -\frac{R}{θ} + \left(q + \frac{R}{θ}\right)e^{-θt}
\]  

...(3.3.2)

with

\[
t₁ = \frac{1}{θ} \log\left(1 + \frac{Pθ}{R}\right)
\]

Using second order approximation of logarithm function, we get
From (3.3.2) and (3.3.3) the total inventory carried during the period $t_i$ is

$$\int_{0}^{t_i} Q_i(t) \, dt$$

$$= \frac{1}{\theta} \left[ \left( P + \frac{R}{\theta} \right) \left( 1 - \left( \frac{P}{R} + \frac{R}{\theta} \right)^{-1} \right) - \frac{R}{\theta} \log \left( 1 + \frac{P}{R} \right) \right]$$

$$l_i(P) = \frac{P^2}{R} - \frac{P^3 \theta}{2R^2}$$

$$\cdots (3.3.4)$$

The total cost of the system is given by

$$K(P) = (Q - P) C_4 + C_i I_i(p) + (T - t_i) C(T_0)$$

$$\cdots (3.3.5)$$

where

$$T = \log \left( 1 + \frac{Q \theta}{R} \right)$$

$$\cdots (3.3.6)$$

Further we note that

$$T = \frac{Q}{R} - \frac{Q^2 \theta}{2R^2}$$

$$\cdots (3.3.7)$$

Which can be obtained by expanding logarithmic function up to second order in equation (3.3.6) where as $C(T_0)$ is the average cost of economic order quantity model with constant rate of deterioration and is given by
\[ C(T_0) = \frac{CR}{\theta} \left[ e^{\frac{\theta T_0}{T_0}} - 1 \right] - CR + \frac{CR e^{\frac{\theta T_0}{T_0}} - C_R}{2\theta} + \frac{C_3}{T_0} \] 

... (3.3.8)

Where \( T_0 \) is a solution of the following equation

\[ CR \left[ Te^{\frac{\theta T}{T}} - \frac{1}{\theta} (e^{\frac{\theta T}{T}} - 1) \right] + \frac{1}{2} C_R T^2 e^{\frac{\theta T}{T}} - C_3 = 0 \] 

... (3.3.9)

and we note that

\[ q_0 = \frac{R}{\theta} (e^{\frac{\theta T_0}{T_0}} - 1) = RT_0 + \frac{R \theta T_0^2}{2} \] 

... (3.3.10)

For whose derivation see Ghare and Schrader [13]. Using (3.3.3), (3.3.4), (3.3.7) in (3.3.5) and taking logarithm terms up to second order we get

\[ K(P) = (Q - P)C_4 + C_4 \left[ \frac{P^2}{2R} - \frac{P^2 \theta}{R^2} \right] + \left[ T - \left( \frac{P}{R} \right) \left( \frac{P^2 \theta}{2R^2} \right) \right] C(T_0) \] 

... (3.3.11)

To get optimal value of \( P \), the above cost function will be differentiated with respect to ‘\( P \)’ and equating the resulting expression to zero, the optimal vale of ‘\( P \)’ is a solution of the following equation
\[ P^2W_0 + PW + W_1 = 0 \] ...(3.3.12)

where \( W_0 = -3C_1\theta \), \( W_1 = C_1R + \theta C(T_0) \),

\[ W = -R[Rc_4 + C(T_0)] \] ...(3.3.13)

Substituting \( \theta = 0 \) in the above equation we get

\[ P = \frac{RC_4}{C_1} + \sqrt{\frac{2rC_1}{C_1}} \]

which agrees with Naddor [28]

The solution of (3.3.12) is given by

\[ P = \frac{-W \pm \sqrt{W^2 - 4(W_0)(W_1)}}{2W_0} \] ...(3.3.14)

Where \( W, W_0, W_1 \) are as defined in (3.3.13)

**Theorem:**

The cost function \( K(P) \) in the equation (3.3.11) is convex in \( P \) if

\[ \theta \leq \frac{RC_1}{6C_1P - C(T_0)} \]

Proof: The second derivative of \( K(P) \) with respect to \( P \) is

\[-6C_1P\theta + C_1R + \theta C(T_0) \geq 0\]
The value of $C(T_0)$ can be obtained from (3.3.8) by substituting optimum value of $T_0$, from equation (3.3.9).

$$\Rightarrow \theta \leq \frac{RC_i}{6C_iP - C(T_0)} \quad ...(3.3.15)$$

this completes the proof.

The convexity of $K(P)$ ensures that the minimum cost yielded by $P^0$ is global minimum whenever equation (3.3.15). This means that there is upper bound on $\theta$ that ensures the convexity of $K(P)$ given in (3.3.5).

The proposed model will be feasible with the input parameters satisfy the required quantification. To do so, the following proposition has been developed.

**PROPOSITION:** The solution given by equation (3.3.14) will be real iff

$$\theta \leq \left[12\{RC_4 + C(T_0)\} - C(T_0)\right]^{-1} \quad ...(3.3.16)$$

**PROOF:** - For $P_0$ to be real, the discriminant of equation (3.3.14) should be nonnegative forcing $P_0$ to be nonnegative directly gives (3.3.16) and necessary condition is established. Now reconsider (3.3.14) we have
From equation (3.3.17) implies that \( P_0 > 0 \) and from equation (3.3.18) we get

\[ RC_4 + C(T_0) \geq 0 \]

which is always positive, since all terms in the L.H.S of the above equation are positive hence the condition is sufficient and the proof is completed.

This proposition proposed here will ensure that the optimum quantity \( P^o \) given in equation (3.3.14) will always yield positive roots only.

We now illustrate sequential procedure to adopt this model in the following steps

**Step 1:**

For the given hypothetical parameters \( C_1, C_2, C_4, T, R \) and \( \theta \), compute \( P \) value from equation (3.3.14). If the root is unique say \( P^o \) off \( P \) then compute corresponding minimum cost using the equation (3.3.11) say \( C(C(P^o)) \). Else, go to next step.
Step 2:  

If the roots obtained in step (1) are real and distinct, say $P_0$ and $P_1$ consider the following cases

**Case (i).** If two roots say $P_0$ and $P_1$ are less than $Q$ then compute costs of $P_0$ and $P_1$ say $C(P_0)$ and $C(P_1)$. If $C(P_0) < C(P_1)$, $P_0$ will be the optimum quantity to be retained that is $P^0$ of $P$. Otherwise $P_1$ will be the optimum quantity.

**Case (ii).** If both roots are greater than $Q$, then optimum quantity would be taken as $Q$.

**Case (iii).** If one root of $P$ say $P_0$ greater than $Q$ then take $P_0 = Q$ if the other root is happened to be less than $Q$ say $P_1$, we compare the cost of $C(Q)$ and $C(P_1)$, using equation (3.3.5). If $C(Q) < C(P)$ then the optimum quantity to be retained $Q$ units. Otherwise $P_1$ will be the optimum quantity to be retained.
NUMERICAL ILLUSTRATION 1:

To illustrate the model numerically we consider hypothetical values for the parameters as given here under

\[ C_1 = 0.56, \, C_2 = 5.04, \, C_4 = 0.28, \, R = 2400 \text{ and } Q = 4800 \]

all the parameters are expressed in consistent units per month. For different values of \( \theta \), we have determined the optimal quantity to be retained and the associated costs are summarized in the following table.
Table 3.1: SENSITIVITY OF THE MODEL WITH RESPECT TO DETERIORATION RATE THAT IS $\theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$q^0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$K(P^0)$</th>
<th>$P^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>597.3413</td>
<td>1839.609</td>
<td>78361.03</td>
<td>1044.08745</td>
<td>1839.609</td>
</tr>
<tr>
<td>0.02</td>
<td>594.7225</td>
<td>1885.427</td>
<td>38316.6</td>
<td>1037.136484</td>
<td>1885.427</td>
</tr>
<tr>
<td>0.03</td>
<td>592.1415</td>
<td>1934.736</td>
<td>24934.96</td>
<td>1030.060306</td>
<td>1934.736</td>
</tr>
<tr>
<td>0.04</td>
<td>589.5989</td>
<td>1989.759</td>
<td>18214.27</td>
<td>1022.456935</td>
<td>1989.759</td>
</tr>
<tr>
<td>0.05</td>
<td>587.095</td>
<td>2051.954</td>
<td>14153.08</td>
<td>1014.225892</td>
<td>2051.954</td>
</tr>
<tr>
<td>0.06</td>
<td>584.6231</td>
<td>2123.429</td>
<td>11415.08</td>
<td>1005.230973</td>
<td>2123.429</td>
</tr>
<tr>
<td>0.07</td>
<td>582.1885</td>
<td>2207.404</td>
<td>11415.93</td>
<td>995.283708</td>
<td>2207.404</td>
</tr>
<tr>
<td>0.08</td>
<td>579.7866</td>
<td>2309.189</td>
<td>9248.178</td>
<td>984.1039016</td>
<td>2309.189</td>
</tr>
<tr>
<td>0.09</td>
<td>577.4179</td>
<td>2438.606</td>
<td>7898.806</td>
<td>971.2434443</td>
<td>2438.606</td>
</tr>
<tr>
<td>0.1</td>
<td>575.0827</td>
<td>2617.682</td>
<td>6659.257</td>
<td>955.8855102</td>
<td>2617.682</td>
</tr>
</tbody>
</table>

These costs are obtained after comparing $K$ ($Q$: 4800). In the above table $P_1$ and $P_2$ are the roots of the equation (3.3.12) if either of the roots are both greater than $Q$, $Q$ will considered as optimum order quantity to be retained.
The last column of the table are optimal quantities to be retained form equation (3.3.12) it is to be noted that $P^o$ of $P$ increases as the value of rate of deterioration increases. It is obvious when ever rate of deterioration increases the prudent stockiest will keep more large quantity in the stock. Further higher the returning or selling cost, the smaller should be quantity to be returned or disposed. Thus, optimal value of $P$ sensitive with respect to parameter $C_4$. This promited the researcher to probe further the effect of fluctuation in the value of $C_4$ over the optimum $K (P)$. Such changes may take place due to uncertainties in any business scenario (or) they might be influenced by the decision maker him self. To verify implications of these actual changes, the sensitivity analysis suggested below would of great help.

**SENSITIVITY ANALYSIS WITH RESPECT TO SPECIAL SALES:**

At first find the optimal values $P$ and $K (P)$ for various values of $C_4$ and keeping all other parameters fixed. These optimal values are denoted by
$P_0^i$ and $K (P_0^i)$ respectively. Then the following sensitivity measures are calculated with respect to changes in the value of $C_4$

$S_{QR} = \text{Sensitivity of the optimal quantity to be retained}$

$$S_{QR} = \left( \frac{P_0^i}{P_0} - 1 \right) \times 100$$

$STC = \text{Sensitivity of the minimum total cost}$

$$STC = \left( \frac{K(P_0^i)}{K(P_0)} - 1 \right) \times 100$$

where $P_0$ and $K (P_0)$ are optimal values of $P$ and $K(P)$ respectively. The pertinent results are exhibited in the following table.

**Table 3.2: SENSITIVITY ANALYSIS WITH RESPECT TO CHANGES IN THE SELLING COST THAT IS SPECIAL SALES COST ($C_4$)**

<table>
<thead>
<tr>
<th>$C_4$</th>
<th>$SQR$</th>
<th>$STC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.22</td>
<td>-14.5625</td>
<td>-17.7829</td>
</tr>
<tr>
<td>0.24</td>
<td>-9.7196</td>
<td>-11.68</td>
</tr>
<tr>
<td>0.26</td>
<td>-4.8654</td>
<td>-5.7565</td>
</tr>
<tr>
<td>0.30</td>
<td>5.118</td>
<td>5.4162</td>
</tr>
<tr>
<td>0.32</td>
<td>10.1716</td>
<td>10.6799</td>
</tr>
<tr>
<td>0.34</td>
<td>15.2428</td>
<td>15.7597</td>
</tr>
<tr>
<td>0.36</td>
<td>20.3256</td>
<td>20.6550</td>
</tr>
<tr>
<td>0.38</td>
<td>25.42013</td>
<td>25.3656</td>
</tr>
<tr>
<td>0.40</td>
<td>30.5263</td>
<td>29.8909</td>
</tr>
<tr>
<td>0.42</td>
<td>35.6445</td>
<td>34.2309</td>
</tr>
</tbody>
</table>
From above calculations we conclude that the optimum values of $P$ and $K (P)$ significantly sensitive with respect to the changes in the parameter $C_4$.

3.4 FINITE HORIZON MODEL:

The length of replenishment period is $t_p$ and is given by

$$t_p = \frac{(H-t_1)}{m} \quad \ldots (3.4.1)$$

where, $t_1$ is defined as in infinite horizon model and 'm' is the number of replenishments

$$t_1 = \frac{1}{\theta} \log(1 + \frac{P\theta}{R}) \quad \ldots (3.4.2)$$

Here 'H' is the planning horizon and fixed quantity and 'm' is the number of replenishments to be made during the period $(H - t_1)$.

Substituting $t_1$ value in (3.4.1) we get

$$t_p = \frac{1}{m} \left[ H - \frac{1}{\theta} \log(1 + \frac{P\theta}{R}) \right] \quad \ldots (3.4.3)$$
and 
\[ q_p = R \ t_p \]
\[ = \frac{R}{m} \left[ H - \frac{1}{\theta} \log(1 + \frac{P\theta}{R}) \right] \]
\[ = \frac{R}{m} \left[ H - \frac{P}{R} \left(1 - \frac{P\theta}{2R}\right) \right] \] ... (3.4.4)

During this period the initial on hand inventory is \( q \). Then, as given by Ghare and Schrader, it can be easily shown that the total number of units carried in inventory are

\[ l_1(q) = C_3 + C_1 \int_0^T Q(t) \ dt \]
\[ = C_3 + C_1 \left[ \left( q_p + \frac{R}{\theta} \right) \frac{1}{\theta} \left( 1 - e^{-\theta t} \right) - \frac{RT}{\theta} \right] \] ... (3.4.5)

Hence from (3.4.4) and (3.4.5), the total cost of system during the planning horizon 'H' is given by

\[ K(P,m) = (Q - P)C_4 + C_1 \left[ \frac{P^2}{2R} - \frac{P^2\theta}{2R^2} \right] \]
\[ + m \left[ C_3 + C_1 \left\{ \left( q_p + \frac{R}{\theta} \right) \frac{1}{\theta} \left( 1 - e^{-\theta t} \right) - \frac{RT}{\theta} \right\} \right] \] ... (3.4.6)
The above equation is a function of two variables namely ‘m’ and ‘P’. Fixing ‘m’ to m*, the corresponding value of K (m*) of P is the solution of

\[ \frac{\partial K(P, m^*)}{\partial P} = 0 \]  

... (3.4.7)

The above equation yields to

\[ P = \frac{R^2 C_4 + R^2 \frac{C_1}{\theta} (1-e^{-\theta T})}{RC_1 (2 - e^{-\theta T}) - C_1 \theta} \]  

...(3.4.8)

which is independent of m*.

Now the corresponding minimum cost K (m*) is the value of K (P(m*), m*) in equation (3.4.6), which in view of (3.4.8),(3.4.5) and (3.4.4) is given by

\[ K(m^*) = A_1 C_4 + A_2 C_1 + mA_3 \]  

...(3.4.9)

where \( A_1 = \frac{R^2 C_4 + C_1 R^2 [H - \frac{P}{R} (1 - \frac{P \theta}{2R})]}{Q - \frac{m}{RC_1 (1 + \theta T) - C_1 \theta}} \).
The optimal number of replenishment \( m_0 \) is then the value of \( m^* \) that minimizes \( K(m^*) \). Since \( m^* \) be non negative integer the necessary condition for \( K(m^*) \) to be minimum at \( m^* = m_0^* \) is

\[
\Delta K(m_0^* - 1) \leq 0 \leq \Delta K(m_0^*)
\]

where
\[
\Delta K(m_0^*) = K(m_0^* + 1) - K(m_0^*)
\]

Using Taylor series expansion form of logarithmic terms of second and higher order powers of \( \theta \) under the assumption \( 0 < T \) and \( \theta q < 1 \) in equation (3.4.9) and substituting this equation in (3.4.11) along with equation (3.4.10), the condition for optimality \( m^* = m_0^* \) becomes

\[
m(m-1)(K_4 - C_3) - mK_5 - K_3 \frac{2m-1}{m(m-1)} < K_2 - K_1 < m(m+1)(K_4 - C_3) - mK_5 - K_1 \frac{2m+1}{m(m+1)}
\]

\[
\ldots (3.4.12)
\]
where \( K_1 = \frac{C_2 C_3 R^2 W_1}{W_2} \)

\[
K_2 = \frac{W_0 2^R C_4 C_4 W_1}{W_2^2}
\]

\[
K_3 = W_0 \left( \frac{C_4 R^2 W_1}{W_2^2} \right)
\]

\[
K_4 = \frac{C_4 R}{\theta} \left( 1 - e^{-\theta} \right)
\]

\[
K_5 = \frac{C_4 R}{\theta^2} W_1
\]

and \( W_6 = \frac{C_1 (1 - \frac{\theta}{R})}{2R} \)

\[
W_1 = \left[ H - \frac{P}{R} \left( 1 - \frac{P \theta}{2R} \right) \right]
\]

\[
W_2 = RC_1 (1 + \theta T) - C_1 \theta
\]

**NUMERICAL ILLUSTRATION 2:**

Reconsider the hypothetical values of parameters of numerical illustration 1 with \( H = 3 \) months as the length of planning horizon. Then using equation (3.4.12) the optimal number of replenishments is \( m_0 = 2 \).

From equation (3.4.8) the optimum quantity to be retained is \( P^0 = 2281.706 \) per month for \( \theta = 0.02 \). The lot size is 1416 units and the minimum total cost \( K (m_0) = Rs 2390.33 \) during \( H = 3 \) months.
3.5 Conclusions and scope for extension:

In this chapter we have considered E.O.Q model for inventory returns and special sales in the context of deteriorating items. Both, infinite horizon and finite horizon models are developed the novelty of the model compared to earlier work lies in the determination of optimum quantity to be retained. In general the problem of returning or selling arises whenever the on hand inventory is more than optimal order quantity. It would be interesting to deal this situation in the context of order level inventory model that too with probabilistic demand would be a challenging one.