CHAPTER - II

SIMULATION OF A ROTATING FLOW WITH MIXED LAYERS
AT A BOUNDARY
1. INTRODUCTION:

The problems concerning the manner in which a rigidly rotating fluid bounded by one or two disks or confined in a cylinder adjusts from one state of rigid rotation to another state are models of several complicated problems of practical interest in astrophysics and geophysics and hence provide valuable insight into the dynamics of homogeneous rotating fluids. The importance and applicability of the theory of rotating fluids is described in the works of Greenspan (1969), Howard et al (1967) and Chawla (1972).

The development of the theory of rotating fluids started with the discoveries of Ekman (1905) who studied problems involving wind stress on ocean surface. The linear theory of rotating fluids has been well established through the works of Greenspan and Howard (1963), Stewartson (1957) and Claire Jacobs (1971) by considering idealised situations comprising a viscous rotating fluid bounded by one or two disks or confined in a cylinder. Fluid motion in a rigidly
rotating system is often induced by moving sections of the bounding surface at slightly different angular velocities. Because of the basic rotation of the earth most of the large scale motions of the atmosphere and the sea are of this type. In these problems it has been noted that near a rigid boundary of a rotating system, the important forces are the Coriolis force and the viscous stress. It is well established that shear layers are located along the bounding surface and are the regions where the tangential velocity is adjusted to its proper boundary value by viscosity. The main aspects of investigation in the unsteady motion of rotating fluids concern with the evaluation of transient time and structure of the boundary layers on the rigid bounding surfaces. The transient time is the time required for the transient motion to decay. In the case when the motion is created due to a differential rotation, the transient time is known as spin-up time (or spin-down time in some cases).

Initially, an unsteady boundary layer starts growing on a rigid surface bounding the fluid due to the interactions of Coriolis force and viscous force which ultimately results in the formation of steady boundary layers on the rigid bounding surface, for large times. The depth of penetration
depends on the underlying force balance. The steady boundary layers of thickness order \( E^{1/2}, E^{1/3}, E^{1/4} \) etc., usually occur in rotating fluid problems (\( E \) - Ekman number). Classical boundary layer theory using the method of matched asymptotic expansions is surely an analytical means of connecting various non-linear flow regimes. The Ekman number \( E \) and the Rossby number \( \varepsilon \) are two fundamental parameters in rotating flows. The Ekman number is a gross measure of how the typical viscous force compares to the Coriolis force and it is, in essence, the inverse Reynolds number for the flow. The Rossby number is the ratio of the convective acceleration to the Coriolis force and provides an overall estimate of the relative importance of non-linear terms in comparison with the linear terms. The Ekman number is very small in several cases of interest and linear theories presume a small value for the Rossby number \( \varepsilon \) also. In the case of rotating fluids which are temperature stratified, other parameters like Froude number and Prandtl number also arise.

The linear theory of rotating stratified fluid motion has been developed by Barcilov and Pedlosky (1967 a, b) and Pedlosky (1969). The nonlinear theory of incompressible viscous rotating fluids is developed by Chawla (1973, 1976).
both for hydrodynamic and hydromagnetic cases. Theory of incompressible rotating fluids has also developed through the investigation of unsteady motion over a state of rigid rotation, created due to torsional or nontorsional oscillations of the boundary. The works of Jones (1969) and Claire Jacobs (1971) deal with motion created by torsional oscillations. The important findings in these works are that, when the frequency of oscillation is much greater than the angular velocity of rigid rotation ($\omega >> \Omega$), the viscous force would balance the acceleration and the depth of penetration is a Stokes layer of thickness order $(\nu/\Omega)^{1/2}$. If $\omega << \Omega$, when the viscous force would balance the Coriolis force, the depth of penetration is an Ekman layer of thickness order $(\nu/\Omega)^{1/2}$. The frequencies $\omega = \pm 2 \Omega$ are also found to be resonant frequencies. The problem of unsteady motion in a rotating fluid due to non-torsionally oscillating disks is studied by Debnath (1974). The electromagnetic effects on rotating fluids are analysed by Benton and Loper (1969), Chawla (1976) and Venkatasiva Murthy (1979). The compressibility effects on rotating fluids have been investigated by Matsuda et al. (1975), Iwao Harada (1979) and Venkatasiva Murthy (1984).
In this chapter we consider a rotational flow of a viscous incompressible fluid past a porous layer of finite thickness. The motivation for considering such a problem is described now. In the case of differential rotation of an infinite disk over a state of rigid rotation of a slightly viscous fluid of uniform density, the Ekman layer formed on the disk transmits boundary information into the fluid by vortex tube stretching and can strongly control the inviscid interior. The effect of the spin-up of the disk is to produce additional azimuthal and radial velocities near the disk. To satisfy the continuity requirements, a small axial velocity comes into existence and persists in the far field.

Natural flows where the turbulent processes transmit the direct effect of the boundary deeper into the fluid than the usual Ekman layer can be modeled by a geometry bounded at the bottom by a permeable medium. Such a model is proposed by Bretherton and Spiegel (1968). When the solid boundary at the bottom is replaced by a thin permeable layer, the vertical flux from the interior increases and the permeable medium imparts the new rotation rate to the incoming fluid every effectively. The entire process speeds up. Kroll and Veronis (1970) extended the ideas of Bretherton and Spiegel into a detailed theory valid for all values of permeability.
However, it should be noted that these authors used the Darcy's law to represent the motion in the porous medium and hence the angular velocity in the porous medium is uniform in space. Since the porous solid cannot deform as a fluid would, in response to changing angular velocity, the entire porous medium has to be regarded as a convection zone and the centrifugal pumping occurs throughout the entire convection zone. Therefore, a thin layer above the porous medium (in the free fluid) becomes a transition zone which is the locus of the centrifugal pumping and functions as an Ekman layer.

The Darcy's law expresses that the velocity is proportional to the pressure gradient. It does not have convective acceleration of the fluid and is valid for low speed flows. The speed in the rotating porous medium need not necessarily be small and the convective acceleration is important. Further, if the medium is of high permeability, which is more appropriate in several situations (Kroll & Veronis 1970), the fluid occupies all parts of the medium. Consequently, the viscous stress in the porous medium has to be accounted for (Yamamoto and Iwamura 1976).
In several astrophysical situations it is possible that the conditions exterior to a surface may influence the flow of a fluid in a layer adjacent to the surface which requires altogether different treatment than that of free motion of a viscous incompressible fluid. (For example, removal of angular momentum from the surface layers of sun by the solar wind). Kroll & Veronis (1970) suggested that several geophysical and astrophysical situations in which the permeable medium serves as an analogue to the convectively unstable surface layers, it is appropriate to simulate the flow near the boundary with a porous medium of high permeability. It is therefore necessary to model the flow near the boundary using Brinkman's law rather than Darcy's law. This allows the interstitial fluid in the porous medium to deform, in response to changing angular velocity, to the extent permitted by the porosity of the medium. This also allows the turbulent mixing to decrease towards the interface and to allow the shear to develop in the porous medium. Therefore a region on either of the interface will provide transition and shows the characteristics of an Ekman layer, in contrast to the earlier works.

Motivated by these ideas, a semi infinite expanse of viscous incompressible fluid bounded below by a permeable
A disk of finite thickness is considered. The entire system is in a state of rigid rotation initially. The unsteady flow over a state of rigid rotation is created by differentially rotating the porous disk, following an impulsive change. The Brinkman's law is used to represent the motion in the interstitial fluid in the porous medium and the Navier–Stokes equation represents the motion in the free fluid. We consider the steady problem corresponding to this flow.

The solution obtained in the two regions subject to boundary conditions and far field conditions are matched at the interface using the principle of conservation of momentum across the interface (Yamamoto and Iwamura 1976). In the case of classical problems of hydrodynamic spin-up, any theory that neglects non-linear terms is inadequate to give a solution because the non-linear terms eventually bound the far field growth of the solution (Jones 1969). The method of matched asymptotic expansions alone gives a uniformly valid solution in all regions of flow. However, when the lower rigid boundary is replaced by a porous disk of finite thickness, the solutions obtained in the porous medium and the free fluid medium are matched at the fluid porous medium interface to yield a solution uniformly valid in all regions satisfying the boundary condition. The solution obtained in this manner is found to be more
generally valid for rotational flows with mixed layers near the rigid surface in view of the fact that we have used Brinkman's law and not Darcy's law in the porous medium which allows deformation of the fluid and shear to develop.

2. FORMULATION OF THE PROBLEM:

We consider a steady flow of a viscous incompressible fluid of infinite extend bounded by a porous disk of thickness 'h' situated between \( z = 0 \) and \( z = h \) in a cylindrical polar coordinate system \((r,\theta,z)\). The entire system consisting of the porous layer with the interstitial fluid in \( 0 < z < h \) and the free fluid medium in \( z > h \) is in a state of rigid rotation (angular velocity \( \Omega \)) with the \( z \)-axis coinciding with the common axis of rigid rotation. The flow relative to the state of rigid rotation is created due to the differential rotation of the porous disk of finite thickness with an angular speed \( \Omega(1+\varepsilon) \) where \( \varepsilon \) is a constant (Rossby number).

Following Yamamoto and Iwamura (1976) we take the following generalised Darcy's law to represent the motion through the homogeneous, isotropic porous medium \((0 < z < h)\) which is also rotating with an angular velocity \( \Omega \).
\[ \frac{\rho \, dq}{\eta \, dt} = -\nabla \cdot P + \mu \nabla^2 q - \frac{\mu}{k} (q - q_p) \]

where \( q = (U,V,W) \) is the velocity in the porous medium, \( q_p = r \, \Omega \) is the velocity of the medium itself, \( P \) is the pressure, \( \rho \) is the density, \( \mu \) is the dynamic viscosity coefficient, \( k \) is the permeability of the medium and \( \eta \) is the porosity of the medium. We assume that the medium is highly porous (like a fibrous material or a tangle of infinitely thin wires) so that the fluid occupies almost all parts of it and \( \eta \sim 1 \). Consequently, it is assumed that the viscous stress in the porous medium is expressed in the same form as in the free fluid. The steady equations of motion for axisymmetric flow in the porous medium in cylindrical polar coordinates are

\[
\rho \left[ U_U + U_W \frac{V^2}{r} \right] = -P_r + \mu \nabla^2 U - \frac{\mu}{k} U
\]

\[
\rho \left[ U_U + W_W \frac{V^2}{r} \right] = \mu \nabla^2 V - \frac{\mu}{k} (V - r \, \Omega)
\]

\[
\rho \left[ W_U + W_W \right] = -P_z + \mu \nabla^2 W - \frac{\mu}{k} W
\]

\[ U_r + \frac{U}{r} + W_z = 0 \]
where $\nabla^2 = \nabla^2 - \frac{1}{r^2}$, $\nabla^2(\cdot) = (\cdot)_{rr} + \frac{1}{r} (\cdot)_r + (\cdot)_{zz}$

and the subscripts $r$ and $z$ stand for differentiation with respect to the corresponding variable. If $\vec{V} = (u, v, w)$ is the velocity and $p$ is the pressure in the free fluid region $(z > h)$ the equations of motion are obtained from the above equations by replacing $U, V, W, P$ by $u, v, w, p$ in the absence of the Darcy body force term.

It may be observed that the governing equations in the porous medium are satisfied by

$$U = 0, \ V = r \Omega, \ V = 0$$

$$p = p_e = p_o + \frac{1}{2} \rho r^2 u^2$$

where $p_o$ and $\rho$ are constants. This shows that a rigid body rotation of the entire system together with the porous medium and the interstitial fluid is possible initially.

The boundary conditions are
\[ u = 0, \quad v = r \Omega (1 + c), \quad w = 0 \text{ at } z = 0 \]

\[ u \to 0, \quad v \to r \Omega \quad \text{as } z \to \infty. \]

The conservation of momentum across the interface \( z = h \) of the free fluid and the porous medium gives

\[ s_{rz} - S_{rz} = \rho w (u - U) \quad (1) \]

\[ s_{z\theta} - S_{z\theta} = \rho w (v - V) \quad (2) \]

\[ s_{zz} - S_{zz} = p - P + \rho (w^2 - W^2) \quad (3) \]

where \( s_{rz}, s_{z\theta}, s_{zz} \) are the viscous stress components on the interface in the free fluid region and \( S_{rz}, S_{z\theta}, S_{zz} \) are the viscous stress components on the interface in the porous medium. Since the mass flow across the interface must be continuous we have

\[ w = W \text{ at } z = h \quad (4) \]

Also because of viscosity, the tangential component of the velocity at the surface in pores should be continuous so that

\[ u = U \text{ and } v = V \text{ at } z = h \quad (5) \]
In view of the differential rotation it is natural to describe the motion relative to an appropriate rotating coordinate system, so that we can develop physical concepts from the point of view of observers fixed in the coordinate system. Accordingly, we introduce the following non-dimensional variables.

\[
\begin{align*}
\bar{q} &= q^* \Omega L \\
P &= P_0 + \frac{1}{2} \rho r^2 \Omega^2 + \rho \Omega^2 L^2 P^*(r,z) \\
r &= L r^*, \quad z = Lz^*, \quad h = Lh^* \\
q^* &= (U^*, V^*, W^*) \\
U^* &= r^* \frac{\partial F^*}{\partial z^*} \\
V^* &= r^* (1 + G(z^*)) \\
W^* &= -2F(z^*)
\end{align*}
\]

The non-dimensional variables are defined similarly to represent the flow in the free fluid region (using lower case letters). Using these non-dimensional variables, the equations of motion, on omitting *, are

(i) In the porous medium (0<z<h)

\[
EF_{zzz} - F_z^2 + 2FF_{zz} + G^2 + 2G - HF_z = \frac{1}{r} P_r 
\]
\[ E_{\text{zz}} - 2F_z (1 + G) + 2FG_z - H(G-c) = 0 \]  \hspace{1cm} (7)

\[ 2EF_{zz} + 4FF_z - 2HF = -P_z \]  \hspace{1cm} (8)

(ii) In the free fluid medium \((z > h)\)

\[ E_{fzz} - f_z^2 + 2ff_{zz} + g^2 + 2g = \frac{1}{r} \rho \]  \hspace{1cm} (9)

\[ E_{qzz} - 2f_z (1 + g) + 2fg_z = 0 \]  \hspace{1cm} (10)

\[ 2Ef_{zz} + 4ff_z = -p_z \]  \hspace{1cm} (11)

where \( H = \frac{\mu}{k\rho} \) is the permeability parameter and \( E = \frac{\mu}{\rho G L^2} \) is the Ekman number. From equations \((6)\) and \((8)\) it follows that \( P = \frac{1}{2} A r^2 + p_z(z) \) where \( A \) is a constant in space. The R.H.S. in equation \((6)\) can be replaced by the constant \( A \). The equations of motion in the free fluid medium are obtained from equations \((6)\) to \((8)\) by putting \( H = 0 \). Using the continuity of the velocity components (eqs. 4 and 5) and the eqs. \((1)\) to \((3)\) for the conservation of momentum across the interface, it can be found that the viscous stress components and pressure are continuous across the interface. Consequently \( A \) is the same throughout the flow region \( 0 \leq z \leq 1 \). The boundary conditions in the non-dimensional form are
The far field condition implies that \( A = 0 \). The matching conditions at the interface \( z = h \) in the non-dimensional form are obtained from the equations (1) to (5) as follows:

\[
\begin{align*}
&f = F, f_z = F_z, f_{zz} = F_{zz}, g = G, g_z = G_z \quad \text{at} \quad z = h \\
&f \rightarrow 0, g \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty
\end{align*}
\]  

The eqs.(8) and (11) determine pressure after finding the velocity. The linearised equations of motion are

(i) In the porous medium \( 0 < z < h \),

\[
\begin{align*}
&EF_{zzz} + 2G - HF_z = 0 \\
&Eg_{zz} - 2f_z - H(G - \varepsilon) = 0
\end{align*}
\]

(ii) In the free fluid medium \( z > h \),

\[
\begin{align*}
&F_{zzz} + 2g = 0 \\
&G_{zz} - 2f_z = 0
\end{align*}
\]

For the sake of convenience in obtaining the solution of the above equations, we introduce a complex function

\[ Q = F_z + iG \quad \text{in} \quad \text{the porous medium} \quad 0 < z < h \]

\[ q = f_z + ig \quad \text{in} \]
the free fluid medium $z > h$. In terms of $Q(z)$ and $q(z)$ the linearised equations of motion are

(i) In the porous medium ($0 < z < h$)

$$E_{Q_{zz}} - (2i + H)Q + iHe = 0 \tag{19}$$

(ii) In the free fluid medium ($z > h$)

$$E_{q_{zz}} - 2iq = 0 \tag{20}$$

The boundary conditions reduce to

$$Q = i\epsilon \quad \text{on} \quad z = 0$$

$$q \to 0 \quad \text{as} \quad z \to \infty.$$  

The matching conditions are

$$Q = q \quad \text{and} \quad Q_z = q_z \quad \text{on} \quad z = h.$$  

The solution of eqs. (19) and (20) satisfying the above boundary conditions and matching conditions is
The axial velocity can be obtained by integrating the above equation and taking the real part. The arbitrary constants have to be evaluated by using the appropriate boundary condition on $F$ and far field condition on $f$ and matching conditions at the interface $z=h$.

The solution is

$$F = \text{Re} \left\{ \frac{2\alpha \sinh \alpha(h-z) + 2\beta \cosh \alpha(h-z) - i\epsilon \beta H \cosh \alpha z}{\alpha(H+2i)(\alpha \cosh \alpha h + \beta \sinh \alpha h)} \right\}$$

$$+ \frac{2\epsilon H}{H^2+4} \text{Re} \left\{ \frac{2\alpha \sinh \alpha h + 2\beta \cosh \alpha h - i\epsilon \beta H}{\alpha(H+2i)(\alpha \cosh \alpha h + \beta \sinh \alpha h)} \right\}$$

(23)
DISCUSSION OF THE RESULTS:

The appearance of the term \( \cosh \alpha z \) in the solution suggests that there is a boundary layer of thickness of \( \sqrt{\frac{\sqrt{E}}{H + \sqrt{H^2 + 4}}} \) on the surface \( z = 0 \). Similarly, the appearance of the terms \( \cosh \alpha(z-h) \), \( \sinh \alpha(z-h) \) suggests that there is a boundary layer of depth \( \sqrt{\frac{\sqrt{E}}{H + \sqrt{H^2 + 4}}} \) near the interface \( z = h \). Thus the theory
suggests the formation of boundary layers in the flow not only at the rigid surface at \( z=0 \) but also at the interface \( z=h \). This is because we used Brinkman's law to represent the fluid motion in the porous layer which accounts for viscous stress in the fluid and the fluid deformation is permitted. As a result, shear layers develop at either face of the porous medium in contrast to the theories which used the Darcy's law for fluid motion in the porous medium, in which case the angular velocity is uniform in space within the porous medium with no possibility for development of shear layers in the porous medium. In the present problem, the transition to interior starts to ensue in the porous medium itself around the interface. The solution is numerically evaluated and the velocity components are shown graphically. Since rotating flows have physical relevance only for small Ekman numbers, we have chosen \( E = 0.03 \) in the numerical work. The permeability parameter \( H \) is of order one.

We define functions \( \overline{f} \) and \( \overline{g} \) as follows.

\[
\overline{f} = F \quad \text{if } 0 < z < h \quad \text{(porous medium)} \\
\quad f \quad \text{if } z > h \quad \text{(free fluid medium)}
\]

\[
\overline{g} = B \quad \text{if } 0 < z < h \quad \text{(porous medium)} \\
\quad g \quad \text{if } z > h \quad \text{(free fluid medium)}
\]
Then the radial, azimuthal and axial velocity components in the entire flow are represented in terms of the function $f_z$, $g$ and $f$.

Figure 1 shows the radial velocity $f_z$ plotted with $z$ for various values of the permeability parameter $H$. It is found that as $H$ increases, the velocity decreases at distances near the rigid surface $z=0$ in the porous medium. The boundary layer behaviour is suppressed at $z=0$ whereas the boundary layer behaviour shows up at the interface. In the central regions of the porous layer, the velocity remains nearly constant as $H$ increases. The manner in which $f_z$ approaches zero as $z$ increases is shown in figure 1. In the case of rotational velocity component, it is observed from figure 2 that as the permeability parameter $H$ increases, the rotational velocity increases in the porous layer and remains nearly constant in the porous medium if $H$ is sufficiently large. This is because the porous medium is itself rotating and a rotating porous medium imparts more rotational velocity to the fluid as $H$ increases. The axial velocity represented in figure 3 is small in magnitude compared to radial and azimuthal velocity components. The axial velocity decreases as $H$ increases. Figures 4 and 5
show the azimuthal and radial velocity components plotted with distance $z$ for various values of $h$, the thickness of porous disk. If the disk is sufficiently thick, it is found that the velocity remains nearly constant in the central regions of the porous layer (away from its faces). Figure 2 and 4 show that the angular velocity decreases slowly with distance $z$ in the porous layer except near the interface. This zone of nearly constant angular velocity is essentially a convection zone, which corresponds to the ideal uniform angular velocity zone of Kroll and Veronis (1970). In comparison with earlier works wherein the angular velocity is uniform in space in a neighbourhood of the boundary, the present model is more appropriate to simulate a flow with mixed layers near the surface.
Figure 1: Radial velocity when $E = 0.03$, $h = 0.4$, $\varepsilon = 0.05$

(a) $H = 1$  (b) $H = 3$  (c) $H = 5$  (d) $H = 7$
Figure 2: Azimuthal velocity when $E = 0.03$, $h = 0.4$, $e = 0.05$

(a) $H = 1$  (b) $H = 3$  (c) $H = 5$  (d) $H = 7$. 
Figure 3: Axial velocity when \( \varpi = 0.03, h = 0.4, \varepsilon = 0.05 \)

(a) \( H = 1 \)  (b) \( H = 3 \)  (c) \( H = 5 \)  (d) \( H = 7 \).
Figure 4: Azimuthal velocity when $E = 0.03$, $H = 5$, $\varepsilon = 0.05$

(■) $h = 0.4$ (■) $h = 0.6$ (■) $h = 0.8$ (■) $h = 1.0$
Figure 5: Radial velocity when $E = 0.03$, $H = 5$, $t = 0.05$

(a) $h = 0.4$ (b) $h = 0.6$ (c) $h = 0.8$.
REFERENCES


