CHAPTER I
FLOWS THROUGH POROUS MEDIA - GENERAL DESCRIPTION
AND
GOVERNING EQUATIONS
1. GENERAL DESCRIPTION OF POROUS MEDIUM

Flow through a porous medium is a topic encountered in many branches of Engineering and Science, e.g. ground water hydrology, reservoir engineering, soil science, soil mechanics and chemical engineering. Flows through porous media are important in petroleum engineering to study the movement of natural gas, oil and water through the oil reservoirs and in chemical engineering for filtration and water purification processes.

Examples of porous materials are numerous. Soil, porous or fissured rocks, ceramics, fibrous aggregates, filter paper, sand filters are a few.
A porous medium may be considered as a solid with holes in it. The pores are interconnected with several continuous paths within. The details of the shapes, sizes and interconnections of these holes are seldom known, but the medium can be characterized by its average properties, such as its average resistance to flow of fluids. The solid phase of the matter is called the solid matrix. The space within the porous medium domain, that is not part of the solid matrix, is referred to as void space or porous space. The solid phase should be distributed throughout the porous medium within each domain occupied by the porous medium; solid must be present inside the representative elementary volume. An essential characteristic of a porous medium is that the specific surface (defined later) of the solid matrix is relatively high. In many respects, this characteristic dictates the behaviour of fluids in porous media. Another basic feature of the porous medium is that the various openings comprising the void space are relatively narrow. To represent the flow in a porous medium, usually the continuum approach is adopted. Accordingly, the porous medium is replaced by a fictitious continuum; a structureless substance, to any point of which we can assign kinematic and dynamic variables and parameters that are continuous functions of the spatial coordinates of the
points and of time. Sometimes, the porous medium may be thought of as consisting of overlapping continua and interactions may take place between these continua. The variables and parameters of the fictitious continuum, averaged over a representative elementary volume enable us to describe flow and other phenomena within a porous medium domain, by means of partial differential equations. Such equations describe what happens at every physical point in space and at every physical instant of time. Even if we describe and solve problems at the microscopic level (i.e., obtain as a solution a description of what happens at points within a single pore), such solutions would be of no practical value. In fact, there would be no easy way even to verify these solutions as no instruments are available for measuring values of dependent variables at the microscopic level. Hence the importance of the continuum hypothesis. In applying the continuum approach to the dynamics of fluids in porous media we shall feel the need to introduce macroscopic medium parameters or coefficients to accommodate the observed phenomena and to enable us to make the passage from the microscopic to the macroscopic continuum level. Such parameters are porosity, permeability, etc. In principle, it is possible to calculate these coefficients from information supplied from the lower levels of treatment, like
microscopic level. In practice, however, they are deduced from actual experiments in which various phenomena related to these parameters are observed.

2. PARAMETERS

(i) Permeability:

The permeability of a porous medium is its most useful fluid flow property. The permeability is a measure of the ease with which a fluid flows through the medium. If the permeability is higher, then the flow rate will be higher for a given pressure gradient. The permeability is a statistical average of the fluid conductivities of all the flow channels in the medium. The average conductivity takes into account the variations in size, shape, direction and interconnections of all the flow channels in the medium. While obviously a number of pores or flow channels must be considered in obtaining a statistically average permeability, it is often convenient for mathematical purposes to consider the permeability as the property of a point in the medium. In a homogeneous medium the permeability at each point coincides with the average
permeability. In a heterogeneous medium the permeability varies from point to point.

The most commonly used unit of permeability is the Darcy, defined as follows:

'A porous medium has a permeability of one darcy when a single phase fluid of one centipose viscosity that completely fills the voids of the medium will flow through it under conditions of viscous flow at the rate of 1 cubic centimeter per second per square centimeter of cross-sectional area under a pressure equivalent to hydraulic gradient of one atmosphere per centimeter.

When a large porous medium is homogeneous, the average permeability coincides with the permeabilities of each of its small pieces. A clear and concise discussion of methods of averaging permeabilities is given by Cardwell and Parsons (1945).

(ii) Porosity:

The porosity of a porous medium is defined as the void volume, or volume of pore space divided by the total volume
of the medium. Void, or pore volumes are usually determined by measuring either gravimetrically or volumetrically the amount of liquid needed to saturate the dry medium. Pore volumes are also determined by gas-expansion methods. Bulk volumes are determined from measurements of the external dimensions of the medium or from the volume of liquid displaced by immersion of the saturated medium. Porosities are expressed as either fractions or percent.

The average porosity of a very large porous medium such as an oil-bearing sand may be determined from the porosity of a number of small core samples of the reservoir rock. A simple arithmetic average will suffice when sufficient samples are available to get a statistical distribution of porosities in the core samples.

Many porous media are made of a mixture of discrete large and small grains or particles that are either loose (as dune sand) or held together by compression and cementing material (e.g., sandstone). The quantity of finer particles has a marked effect on the porosity. The porosity of consolidated materials depends mainly on the degree of cementation. The porosity of unconsolidated materials depends on the packing of the grains, their shape,
arrangement and size distribution. The particle size distribution may appreciably affect the resulting porosity, as small particles may occupy pores formed between the large particles, thus reducing porosity.

Many methods have been developed for determining porosity, like mercury injection method, compression chamber method, gas expansion method etc.

(iii) Specific Surface:

Many porous materials contain enormous surface areas per unit volume. For packs of perfectly uniform spheres, the surface area per unit bulk volume is inversely proportional to the radius of the spheres and is readily calculated. However, in most porous materials the surface area per unit volume must be determined experimentally.

The specific surface of a porous material is defined as the total interstitial surface area of the pores per unit bulk volume of the porous medium.

For example, the specific surface of a porous material made of identical spheres of radius R in a cubical packing
is \( M = \frac{4\pi R^2}{(2R)^3} = \frac{\pi}{2R} \). It thus becomes obvious that fine materials will exhibit a much greater specific surface than coarse materials. Some fine porous materials contain an enormous surface area per unit volume. For example the specific area of sandstone may be of the order of 1500 cm\(^2\)/cm\(^3\). Carman (1938) gives the range of 1.5 \( \times \) 10\(^2\) - 2.2 \( \times \) 10\(^2\) cm\(^1\) for the specific surface of sand.

The usual method of determining specific surfaces is from nitrogen absorption experiments at constant temperature given by Brannauer, Emmett and Teller (1938). A statistical method developed by Chalky (1949) is also used to determine specific surface of a consolidated porous material.

3. DARCY'S LAW:

The foundation on which laminar flow of fluids through porous media rests is Darcy's law. Darcy (1856), while experimenting with flow of water through sand filters, noted that the flow rate of water was proportional to the difference in head of water across the filter. His basic equation was \( q = -k(h_2 - h_1)/L \), where \( q \) is the flow rate of water per unit cross-section area, \( K \) is a constant for the system, \( h_2 - h_1 \) is the difference in fluid head across
length L. Since the work of Darcy, a number of experiments have studied the flow of various fluids through many types of porous solids. The basic relation of Darcy has been extended to cover flow of any fluid, and as with any law, its limitations and range of applicability have been defined. The use of Darcy's law is restricted to cases in which the flow is laminar or streamlined. The Darcy's law expresses that the seepage velocity is proportional to the pressure gradient and it does not have a convective acceleration of the fluid. This law is therefore considered to be valid for low speed flows.

To study the flows through porous media we must introduce a simplified porous medium model that will be amenable to a mathematical treatment and that will incorporate the main features of a porous medium. One of the essential features of a porous medium, in connection with the flow of a fluid through it, is that it restricts the transport of the fluid to well defined channels. The porous medium is in fact a non-homogeneous medium. For the sake of mathematical analysis it is replaced by a homogeneous fluid which has dynamical properties equal to those of non-homogeneous continuum. Thus one can study the flow of a hypothetical homogeneous fluid under the action of properly
averaged external forces (additional resistance due to the medium). The complicated problem of the flow through a porous medium thus reduces to the flow problem of a homogeneous fluid with resistance of the medium taken into account. When the porosity of the medium is very close to unity, the fluid occupies almost all parts of the porous medium.

The two simple concepts - Darcy's law and the law of conservation of mass - are used to describe the laminar flow of a fluid through porous medium. In some cases the equations for steady state flow can be simplified into Laplace equation. In these cases, many of the theorems of classical hydrodynamics for flow of an ideal fluid can be applied. The fundamental aspects and representative equations for flows through porous media are extensively described by Muskat (1937), Scheidegger (1957), Bear (1972) and Streeter (1961). Considerable literature is available for solutions of flow in one dimension. An important class of one-dimensional steady flows past porous beds are studied by Beavers and Joseph (1967), Saffman (1971), Taylor (1971) and Richardson (1971). Some of these authors derived boundary conditions at the fluid porous medium interface useful for solving problems of flows past porous media.
4. BEAVERS AND JOSEPH CONDITION:

Beavers and Joseph (1967) considered the rectilinear flow of a viscous fluid through a two-dimensional parallel channel formed by an impermeable upper wall \((y=h)\) and a permeable lower wall \((y=0)\). The plane \(y = 0\) defines a nominal surface for the permeable material. A uniform pressure gradient is maintained in the longitudinal direction in both the channel and the permeable material. The flow through the body of the permeable material, which is homogeneous and isotropic, is assumed to be governed by Darcy's law. As described already, this law is of the nature of a statistical result giving the empirical equivalent of the Navier-Stokes equation as averaged over a very large number of individual pores. It has been applied and tested on a very broad class of fluid flows, and is valid for common viscous liquids at low Reynolds number. In the absence of the body forces, Darcy's law may be written as

\[
Q = -k \frac{dp}{dx}\]

where \(k\) is the permeability of the material and \(Q\) is the volume flow rate per unit cross-sectional area. As such, \(Q\) represents the filter velocity rather than true velocity of the fluid in the pores. The permeability parameter \(k\) is a property of the medium alone. All the irregularities in size and shape of the pores and the
complexities of the interconnections of pores are represented in the $k$ of Darcy's law. Thus $k$ represents a statistical average of the fluid flow conductivity through the porous medium, at a given point. Conceivably, the permeability of the medium could vary from point to point along the flow path. When a Newtonian fluid flows over a porous surface, it is necessary, if the governing differential system is not to be under determinate, to specify some condition on the tangential component of the velocity of the free fluid at the porous interface. There exists an extensive analytical literature [see Joseph and Tao (1966)] which describes coupled fluid motions satisfying the Navier-Stokes equations in the free fluid, some empirical or semi-empirical set of equations (typically Darcy's law) in the permeable material, and matching conditions at the common boundaries. It is usual in these analyses to approximate the fluid motion near the true boundary by an adherence condition for the tangential component of velocity of the free fluid at some boundary. Of course, a certain ambiguity is implied by the notion of a 'true' boundary for a permeable material and for this reason, it is usual to define a nominal boundary. Usually, a nominal boundary is fixed by defining a smooth geometric surface and then assuming that the outermost perimeters of
all the surface pores of the permeable material are in this surface. Thus if the surface pores were filled with solid material to the level of their respective perimeters a smooth impermeable boundary of the assumed shape would result. This definition is precise when the geometry is simple (planes, spheres, cylinders etc.) but may not be fully adequate in more complicated situations.

Though the adherence condition is valid at an impermeable surface it is not clear that it is valid at the nominal surface of a permeable material. In the latter case, there is a migration of fluid tangent to the boundary within the porous matrix and the requirement that there should be no migration of fluid immediately outside the boundary is approximate at best. It can in fact be argued that there is some net tangential drag due to the transfer of forward momentum across the permeable interface. Indeed, if we were dealing with the true velocity in the porous material, there could be no slip between the free fluid and the fluid immediately within the porous boundary. In this case, a discontinuity of the tangential velocity component could not be allowed.
Beavers and Joseph postulated that the slip velocity at the permeable interface differs from the mean filter velocity within the permeable material and that shear effects are transmitted into the body of the material through a boundary layer region. Across this boundary layer the velocity changes rapidly from its value $u_B$ at the interface to the Darcy value given by equation $Q = -\frac{k}{\mu}\frac{dp}{dx}$. Beavers and Joseph assumed that the slip velocity for the free fluid is proportional to the shear rate at the permeable boundary, and related the slip velocity to the exterior flow by the adhoc boundary condition 

$\left(\frac{du}{dy}\right)_{y=0^+} = \beta (u_B - Q) \quad \text{where} \quad 0^+ \text{ is a boundary limit point from the exterior fluid.}$

At the same time they retained the parallel flow assumption which leads, through the continuity equation, to the independence of $u$ upon $x$. It follows through the above boundary condition that $\beta$ also does not depend on $x$, and so depends only on the properties of the fluid and the permeable material. Using physical aspects and dimensional analysis, they concluded that $\beta$ is independent of the viscosity of the fluid and 

$\beta = \frac{a}{\sqrt{k}} \quad \text{where} \quad a \text{ is a dimensionless quantity depending on the material parameters which characterizes the structure of the permeable material within the boundary region.}$

This result is also strongly implied by their experimental observations.
Thus, though it is customary to use the no slip boundary condition at the porous surface where the effect of porosity is taken care of by the continuity of the normal component of velocity, Beavers and Joseph (1967) have investigated and experimentally established for the first time, this class of flows past a naturally permeable belt with slip at the nominal surface. Subsequently Beavers et al (1970), Taylor (1971), Richardson (1971) and Rajasekhara (1974) have confirmed experimentally, the Beavers and Joseph boundary condition. A rigorous theoretical justification for Beavers and Joseph boundary condition has been given by Saffman (1971). In the above investigations the thickness of the permeable bed does not enter the analysis.

The boundary condition at the fluid-porous medium interface derived by Beavers and Joseph has been extensively used in literature to solve problems of one dimensional flows through porous media.

5. Extended Brinkman’s Model:

We have seen that the Darcy’s law expresses that the velocity in a porous medium is proportional to the pressure gradient and it does not have convective acceleration of the fluid. This law is therefore considered to be valid for low
speed flows, whereas the speed in the porous medium is not always small and the convective force may be important. To analyse this kind of flow one should employ a generalised equation of Darcy's law, in which convective term is taken into account. Several studies (Yih (1965) and Yamamoto and Yosida (1974)) have been made on the basis of a generalised Darcy's law accompanied by convective term. It is shown that vorticity emerges at the discontinuity surface of permeability when the fluid flows across the surface and that vorticity, if any, decays steadily in the flow when the permeability is constant. This interesting phenomena cannot be explained by the usual Darcy's law.

Yamamoto and Iwamura (1976) considered a steady flow through an air filter, when the velocity is not always small. Most filters are made of fibrous materials and their porosity is very close to unity. They regarded the porous medium as an assemblage of small spherical particles fixed in space and this concept is widely used (Scheidegger, 1957). The Reynolds number is not so small that the convective term is taken into consideration in the macroscopic equation of motion. The viscous term due to the distortion of the viscosity is also taken into account for a general flow. Considering that the porosity of the medium is
unity, i.e., the fluid occupies almost all parts of the porous medium, the viscous stress $\tau_{ij}$ is expressed in the same form as in a pure fluid. With the neglect of compressibility of the fluid, the fundamental equations in the porous medium are given by

$$ \text{div} \, \bar{v} = 0 $$

$$ \rho (\bar{v} \cdot \text{grad}) \bar{v} = -\text{grad} \, p - \frac{\mu}{k} \bar{v} + \mu \nabla \, \nabla \, \bar{v} $$

where $p$ is the pressure, $\rho$ is the density, $k$ is the permeability and $\nabla \, \nabla$ is the Laplacian. The non-linear convective terms neglected by Brinkman are taken into effect and so the equation of motion can be called the extended Brinkman model [see Yamamoto and Iwamura (1976)]. Taking $v_*, \rho v_*^2$ and $L$ as a reference speed, pressure and length respectively, the above equations are rewritten in non-dimensional form as

$$ \text{div} \, \bar{v} = 0 $$

$$ (\bar{v} \cdot \text{grad}) \bar{v} = -\text{grad} \, p - K^{-1} \bar{v} + R_L^{-1} \nabla \, \nabla \, \bar{v} $$

where $K = R_L k/L^2$ and $R_L = v_* L/\nu$ are the permeability parameter and the Reynolds number respectively. The equations of motion in the pure fluid medium are

$$ \text{div} \, \bar{u} = 0 $$

$$ (\bar{u} \cdot \text{grad}) \bar{u} = -\text{grad} \, p + R_L^{-1} \nabla \, \nabla \, \bar{u} $$

The above sets of equations are to be solved subject to
appropriate boundary conditions at the boundary surface of the porous bed. The conservation of momentum across the surface leads to

\[
R^T_{\text{t}}(\tau^+ - \tau^-) = \rho v_0 (u_t - v_t)
\]

\[
R^T_{\text{n}}(\tau^+ - \tau^-) = p - P + \rho (u_n^2 - v_n^2)
\]

where the subscripts \( t \) and \( n \) denote the tangential components of the quantities to the surface and the sign + and − mean the values of stress evaluated in the pure fluid region and in the porous medium respectively. Considering the fluid motion just outside and just inside the surface, the mass flow across the surface must be continuous. This gives \( v_n = u_n \) where \( v_n \) and \( u_n \) are the normal components of the velocity in the two media. The tangential velocity at the surface in pores should be continuous because of viscosity and this gives one more condition \( v_t = u_t \).

Yamamoto and Iwamura considered the governing equations and boundary conditions described above and studied the flow past a porous medium with special emphasis on large Reynolds number flows.

In the presence of a temperature distribution in the fluid, the velocity field and the temperature field mutually interact, which means that the temperature and the velocity depend on one another. In the special case when
buoyancy forces may be disregarded, and when the properties of the fluid may be assumed to be independent of temperature, mutual interaction ceases, and the velocity field no longer depends on the temperature field, although the converse dependence of the temperature field on the velocity still persists. This happens at large velocities (large Reynolds numbers) and small temperature differences, such flows being termed forced. The process of heat transfer in such flows is described as forced convection. Flows in which buoyancy forces are dominant are called free, the respective heat transfer being known as free convection. This case occurs at small velocities of motion in the presence of large temperature differences. The state of motion which accompanies free convection is evoked by buoyancy forces in the gravitational field of earth, the latter being due to density differences and gradients. For example, the field of motion which exists outside a vertical hot plate belongs to this class.

The equations governing fluid motion in the presence of a temperature distribution in the fluid are as follows. The general form of the Darcy law governing the motion of an incompressible fluid through a homogeneous isotropic porous medium, namely
\[ \frac{\rho}{\varepsilon} \frac{D \bar{v}}{D t} = -\nabla p - \frac{\mu}{k} \bar{v} + \mu \nabla^2 \bar{v} + \rho \bar{x} \]

where \( \rho, \varepsilon, \bar{v}, p, \mu, k \) and \( \bar{x} \) denote fluid density, porosity of the medium, filtration velocity, fluid pressure, dynamic viscosity coefficient, permeability and body force per unit mass respectively. For very fluffy metal materials or fibrous materials \( \varepsilon \) is very close to unity and in beds of packed spheres, \( \varepsilon \) is in the range of \( 0.25 - 0.50 \) [Joseph (1976)]. Although the viscous term \( \mu \nabla^2 \bar{v} \) is generally neglected in slow motion through a porous medium, it should be taken into account for the general flow particularly in the case \( \varepsilon \ll 1 \), when the fluid occupies all parts of the porous medium. The equation of continuity is \( \nabla \cdot \bar{v} = 0 \) while the energy equation is (Caltagirone 1976)

\[ (\rho c)_t^* \frac{\partial T}{\partial t} + (\rho c)_f \bar{v} \cdot \nabla T = \lambda^* \nabla^2 T \]

The porous medium formed by the porous matrix and the interstitial fluid (which is the fluid in the pores) is regarded as a fictitious isotropic fluid with heat capacity

\[ (\rho c)_t^* = \varepsilon (\rho c)_f + (1 - \varepsilon) (\rho c)_s, \]

where \( (\rho c)_f \) and \( (\rho c)_s \) denote the heat capacity of the fluid and the solid respectively. The physical properties of the medium with an effective thermal conductivity \( \lambda^* \) are
assumed constant in particular with respect to temperature dependence. A linear density-temperature relation is assumed as the equation of state. The variations in density with temperature may be neglected except with regard to their influence on the buoyancy force. This is the well known Boussinesq approximation whose validity has been described in detail by Chandrasekar (1961).


