2. FORMULATION

Choose the cartesian frame of reference such that x-axis is the axis of the rectangular pipe, (y-z) plane its crossection. The equations governing the unsteady flow of a second order Riv lin-Erickson fluid through the rectangular pipe under prescribed flux are

$$\frac{\partial U}{\partial T} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \gamma \left(\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2}\right) + \beta^* \frac{\partial}{\partial T}\left(\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2}\right) \quad (1)$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + (2\beta^* + \gamma)\left(\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial Y}\right)^2 + \frac{\partial}{\partial Z}\left(\frac{\partial U}{\partial Z}\right)^2\right) \quad (2)$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial Z} + (2\beta^* + \gamma)\left(\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial Y}\right)^2 + \frac{\partial}{\partial Z}\left(\frac{\partial U}{\partial Z}\right)^2\right) \quad (3)$$

where u is the axial velocity, P is the pressure, T is the time ρ is the density of the fluid, ν is the coeff of Kinematic viscosity, β^* is the viscoelasticity, γ is the kinematic cross viscosity.

The governing initial and boundary conditions are

$$U(Y,Z,0) = U_0$$

$$U(Y,Z,T) = 0 \quad \text{on} \quad Y = \pm a \quad \text{and} \quad Z = \pm b \quad (4)$$

The prescribed discharge condition is

$$\int_0^t \int_{-a}^{a} \int_{-b}^{b} U(Y,Z,T)\, H(T)\, dydzdt = Q(T) \quad (5)$$

Where H(T) is the Heaviside function

$$H(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases}$$

We introduce the following non-dimensional variables

$$x = \frac{X}{a}, \quad y = \frac{Y}{a}, \quad z = \frac{Z}{a}, \quad t = \frac{T\nu}{a^2}, \quad u = \frac{Ua}{\nu}$$

$$P = \frac{pa^2}{\rho\nu^2}, \quad u_0 = \frac{U_0 a}{\nu}, \quad c = \frac{b}{a} \qquad (6)$$

Substituting (6) in (1) the governing equations in the non-dimensional form reduce to

$$\frac{\partial u}{\partial t} = f + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (7)$$

where $\beta = \left(\frac{\beta^*}{a^2}\right)$ is the viscoelastic parameter

$\frac{-\partial p}{\partial x} = f$ to be determined

The initial, boundary conditions are

$$u(\pm 1, z, t) = u(y, \pm c, t) = 0, \quad t>0$$

$$\text{and } u_o = u(y, z, o) \tag{8}$$

The prescirbed discharge in the non-dimensional form is

$$\int_o^t \int_{-1}^{1} \int_{-c}^{c} u(y, z, t)\ H(t)\ dydzdt = q(t) \tag{9}$$

Taking Laplace Transforms in equation (7), the equation for the transformed variable $\bar{u}$ is

$$(s\beta+1)\nabla^2\bar{u}-s\bar{u} = -\bar{f}-u_o+\beta(\nabla^2 u)_{t=o} \tag{10}$$

The boundary conditions (8) reduces to

$$\bar{u}(\pm 1, z) = \bar{u}(y, \pm c) = 0 \tag{11}$$

The flux condition (9) under the transformation is

$$\int_{-1}^{1} \int_{-c}^{c} \bar{u}(y, z, s)\ dydz = \bar{q}(s) \tag{12}$$

3. SOLUTION BY GALERKIN's METHOD

Under Galerkin's method, we take the N^{th} order solution in the form $\bar{u} = \phi_o + \sum_{i=1}^{N} C_i \phi_i$. If E is the residual (error) then $E = L\bar{u}+F$ where L is the differential operator $[(s\beta+1)\nabla^2-s)]$ and $F=\bar{f}+u_o-\beta(\nabla^2 u_o)$, we now determine c_i's using the fact that the weighted residual integrated over the domain is zero. i.e. $\int_\Omega \psi_i E \; d\Omega = 0$ where ψ_is are the Weight function. In Galerkin we choose ψ_i's as the approximation functions ϕ_i's itself.

For the present case find the first and third order Galerkin solution. In view of the homogenous essential boundary conditions ϕ_0 may choosen to be zero.

Hence $\bar{u} = \lambda_1\phi_1+\lambda_2\phi_2+\lambda_3\phi_3$

where $\lambda_1,\lambda_2,\lambda_3$ are to be determined. The approximation function relevant to the rectangular configuration are

$\phi_1= (1-y^2)(c^2-z^2)$; $\phi_2= y^2(1-y^2)(c^2-z^2)$

$\phi_3= z^2(1-y^2)(c^2-z^2)$

Substituting for $\bar{u}$ in the residual E and performing the integration we obtain

$$\int_{-1}^{1}\int_{-c}^{c} \psi_i E \, dydz = 0 \qquad i = 1,2,3 \qquad (13)$$

Solving the equations (13) for $\lambda_1,\lambda_2,\lambda_3$ and using the discharge condition (12) the transform of the pressure gradient $\bar{f}$ is obtained in terms of the discharge $\bar{q}$.

Taking $\bar{u}(y,z,s) = \lambda\ (1-y^2)(c^2-z^2)$ (14)

as the first order solution and substituting in (10) the error function E is

$$E = (s\beta+1)\lambda\ [-2-2c^2+2y^2+2z^2] - sD\lambda\ (1-y^2)(c^2-z^2)]$$

$$+\bar{f} + u_o - \beta\nabla^2 u_o \qquad (15)$$

From (13) λ is choosen such that

$$\int_{-1}^{1}\int_{-c}^{c} E\ (1-y^2)(c^2-z^2)dydz = 0 \qquad (16)$$

$$\Rightarrow \int_{-1}^{1}\int_{-c}^{c} \{(s\beta+1)\lambda(-2-2c^2+2y^2+2z^2) - sD\lambda\ (1-y^2)(c^2-z^2)]+\bar{f}+$$

$$+u_o-\beta\nabla^2 u_{t=o}\}(1-y^2)(c^2-z^2)dydz = 0$$

which gives

$$F(1,c) = 5\lambda \ (s\beta+1)(1+c^2)+2sc\lambda - \frac{25}{8}\bar{f}$$

$$= \lambda\{5(s\beta+1)(1+c^2)+2sc^2\} - \frac{25}{8}\bar{f} \qquad (17)$$

where

$$F(1,c) = \frac{15^2}{128\, c^3} \int_{-1}^{1} \int_{-c}^{c} (u_o - \beta\nabla^2 u_{t=o}) \times$$

$$\times \ (1-y^2)(c^2-z^2)\,dydz \qquad (18)$$

From the discharge condition (12)

$$\bar{q}(s) = \frac{16c^3\lambda}{9} \qquad (19)$$

Eliminating λ from (17) and (19) we get

$$\bar{f} = \frac{9}{50}\,\frac{\bar{q}(s)}{c^3}\{2sc^2+5(s\beta+1)(1+c^2)\} - \frac{8}{25}F(1,c) \qquad (20)$$

Once $\bar{q}(s)$ is prescribed, taking the inverse transform of $\bar{f}$, we obtain the pressure gradient f.

PARTICULAR CASES

A1) **CASE OF CONSTANT DISCHARGE**

Supposing the fluid moves with an uniform velocity u_o so that the flux along the channel is constant and is equal to q_o. Then $\bar{q}(s) = \frac{q_o}{s}$

The initial and boundary conditions are

$$u(o,y,z) = u_o(y,z) = \frac{q_o}{4c} \qquad (21)$$

$$u(\pm 1,z,t) = u(y,\pm c,t) = 0 \quad t > 0$$

The discharge condition is

$$q(t) = \int_{-1}^{1} \int_{-c}^{c} \bar{u}(y,z,s)\, dydz = q_o \qquad (22)$$

Hence from (18) we get

$$F(1,c) = \frac{25q_o}{32c}$$

and from (20)

$$\bar{f} = \frac{9q_o}{10sc^3}(s\beta+1)(1+c^2) + \frac{11}{100}\frac{q_o}{c} \qquad (23)$$

Taking the inverse Laplace transform we get, the pressure gradient

$$f(t) = \frac{9}{10}\frac{q_o\beta}{c^3}\delta(t)(1+c^2) + \frac{9}{10}\frac{q_o}{c^3}(1+c^2) + \frac{11}{100}\frac{q_o}{c}\delta(t)$$

where $\delta(t)$ is the Dirac delta function .

A2) DISCHARGE VARYING LINEARLY WITH TIME

Let the fluid starts from rest so that the discharge q(t) is linearly varying with time. The initial and boundary conditions in this case are

$$u_o = u(y,z,o) = 0$$

$$u(\pm 1,z,t) = u(y,\pm c,t) = 0, \quad t > 0 \tag{24}$$

and the discharge condition is

$$q(t) = q_o t \tag{25}$$

where q_o is a constant.

From (18) we get F(1,c) = 0 and

from (20), $\bar{f} = \frac{9q_o}{50c^3}\left[\frac{1}{s}(5\beta(1+c^2)+2c^2) + \frac{5(1+c^2)}{s^2}\right]$

The inverse transform of $\bar{f}$ yields the pressure gradient

$$f = \frac{9q_o}{50c^3}\left[5\beta(1+c^2)+2c^2)+5t(1+c^2)\right)$$

A3) DISCHARGE IS AN EXPONENTIAL TIME DEPENDENT FUNCTION

Supposing the fluid starts from rest and the discharge q(t) is proportional to $q_o(1-e^{-kt})$, then the initial and boundary conditions are

$$u_o = u(y,z,o) = 0$$

$$u(\pm 1,z,t) = u(y,\pm c,t) = 0, \quad t > 0 \qquad (26)$$

and the discharge conditions is

$$q(t) = q_o(1-e^{-kt}) \qquad (27)$$

where q_o and k are positive constants

$$\bar{q}(s) = \frac{q_o k}{s(s+k)}$$

From equations (18) and (20) we get

$F(1,c) = 0$ and

$$\bar{f} = \frac{9q_o k}{50c^3}\left[\frac{1}{(s+k)}\left\{5\beta(1+c^2)+2c^2-\frac{5(1+c^2)}{k}\right\} + \frac{5(1+c^2)}{ks}\right] \quad (28)$$

The inverse transform gives the pressure gradient

$$f = \frac{9}{50}\frac{q_o}{c^3}\left[e^{-kt}\{5\beta k(1+c^2)+2kc^2-5(1+c^2)\}+5(1+c^2)\right]$$

B. **THIRD ORDER SOLUTION**

Taking $u(y,z,s) = (\lambda_1+\lambda_2 y^2+\lambda_3 z^2)(1-y^2)(c^2-z^2)$ (29)

as Galerkin's third approximate solution and substituting in (10) the third order function E_2 is obtained as

$$E_2 = (1-y^2)(c^2-z^2)[(2\lambda_2+2\lambda_3)(s\beta+1)-s(\lambda_1+\lambda_2 y^2+\lambda_3 z^2)]$$
$$-(1-y^2)[(s\beta+1)(2\lambda_1+2\lambda_2 y^2+10\lambda_3 z^2)]$$
$$-(c^2-z^2)[(s\beta+1)[2\lambda_1+10\lambda_2 y^2+2\lambda_3 z^2]$$
$$+\bar{?} + u_o-\beta\nabla^2 u_{t=o} \quad (30)$$

On evaluating the integrals (16) the following three equations are obtained

$$F_1+ \frac{175}{8}\bar{?} = 7\lambda_1 p+\lambda_2(p+2c^2 d)+\lambda_3 c^2(p+2d) \quad (31)$$

$$F_2 + \frac{735}{8}\,\bar{\tau} = 21\ \lambda_1(p+2c^2d)+7\lambda_2(p+28c^2d)$$
$$+3c^2\lambda_3(p+2d+2c^2d) \qquad (32)$$

$$F_3 + \frac{735}{8}\,\bar{\tau} = 21\ \lambda_1(p+2d)+3\lambda_2(p+2d+2c^2d)$$
$$+7c^2\lambda_3(p+28d) \qquad (33)$$

where

$$p = 2sc^2+5(s\beta+1)(1+c^2), \quad d = (s\beta+1) \qquad (34)$$

$$F_1 = \frac{7\times 15^2}{128c^3}\int_{-1}^{1}\int_{-c}^{c} u_0(y,z)(1-y^2)(c^2-z^2)\,dydz \qquad (35)$$

$$F_2 = \frac{3\times 7^2\times 15^2}{128c^3}\int_{-1}^{1}\int_{-c}^{c} u_0(y,z)y^2(1-y^2)(c^2-z^2)\,dydz \qquad (36)$$

$$F_3 = \frac{3\times 7^2\times 15^2}{128c^5}\int_{-1}^{1}\int_{-c}^{c} u_0(y,z)z^2(1-y^2)(c^2-z^2)\,dydz \qquad (37)$$

Solving the equations (31), (33) and (34) for λ_1, λ_2, λ_3 we get

$$\frac{\lambda_1}{D_1} = \frac{\lambda_2}{D_2} = \frac{c^2\lambda_3}{D_3} = \frac{1}{D} \tag{38}$$

where

$$D = \begin{vmatrix} 7p & p+2c^2d & p+2d \\ 21(p+2c^2d) & 7(p+28c^2d) & 3(p+2d+2c^2d) \\ 21(p+2d) & 3(p+2d+2c^2d) & 7(p+28d) \end{vmatrix} \tag{39}$$

and D_1, D_2 and D_3 are the determinants obtained by replacing first, second and third columns respectively by A_1, A_2, A_3 in D where

$$A_1 = F_1 + \frac{175}{8}\,\text{₹} \tag{40}$$

$$A_2 = F_2 + \frac{735}{8}\,\text{₹} \tag{41}$$

$$A_3 = F_3 + \frac{735}{8}\,\text{₹} \tag{42}$$

The values D_1, D_2, D_3 and D interms of p can be obtained as

$$D_1 = 4A_1\{10p^2+334pd(1+c^2)+9586d^2c^2-9d^2(1+c^4)\}$$

$$-4A_2\{p^2+46pd+2pdc^2+95d^2c^2-3d^2\}$$

$$-4A_3\{p^2+2dp+46pdc^2+95d^2c^2-3d^2c^4\}$$

$$D_2 = 21A_1(4p^2+184dp+8dpc^2+380d^2c^2-12d^2)$$

$$-28A_2(p^2+46pd-3d^2)-84A_3d^2c^2$$

$$D_3 = -21A_1(4p^2+8dp+184dpc^2+380d^2c^2-12d^2c^4)$$

$$+84A_2d^2c^2+28A_3(p^2+46pdc^2-3d^2c^4)$$

$$D = 112\{p^3+46dp^2(1+c^2)+2116pd^2c^2-3pd^2(1+c^4)$$

$$-138d^3c^2(1+c^2)\}$$

The Laplace transform of the discharge condition is

$$\bar{q}(s) = \int_{-1}^{1}\int_{-c}^{c} \bar{u}(y,z,s)\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-c}^{c} (\lambda_1 + \lambda_2 y^2 + \lambda_3 z^2)(1-y^2)(c^2-z^2)\,dy\,dz$$

$$= \frac{16c^3}{45}(5\lambda_1 + \lambda_2 + c^2\lambda_3) \qquad (43)$$

$$\text{i.e.,} \quad \frac{45\bar{q}(s)}{16c^3} = \frac{5D_1 + D_2 + D_3}{D} \qquad (44)$$

From which we get

$$\frac{45D\bar{q}(s)}{256c^3} = 2s^2c^4(4A_1+A_2+A_3) + (s^2\beta c^2 + sc^2)(371A_1+56A_2+5A_3)$$
$$+ (s^2\beta c^4 + sc^4)(371A_1 + 5A_2 + 56A_3)$$
$$+ (s^2\beta^2 + 1 + 2s\beta)(882A_1 + 126A_2)$$
$$+ (s^2\beta^2c^2 + c^2 + 2s\beta c^2)\,14(910A_1 + A_2 + A_3)$$
$$+ (s^2\beta^2 + 1 + 2s\beta)c^4(882A_1 + 126A_3) \qquad (45)$$

Substituting the values of A_1, A_2, and A_3 in the above equation and expanding D we get

$$\frac{7\times 45\bar{q}(s)}{4c^3}\,[2s^3c^6 + 61\,s^2(s\beta+1)c^4(1+c^2) + s(s\beta+1)^2$$

$$\times\,\{266c^2(1+c^4) + 1593c^4 + 3563\,\beta\,c^2(1+c^2)$$

$$+ 315\,\beta(1+c^6)\} + (s\beta+1)^2\{3563\,c^2 + 315(1+c^6)\}$$

$$= 2s^2c^4(4F_1+F_2+F_3) + (s^2\beta+s)\{c^2(371F_1+56F_2+5F_3)$$

$$+ c^4(371F_1+5F_2+56F_3)\} + (s\,\beta+1)^2\{882\,F_1+126F_2)$$

$$+ c^4(882F_1+126F_3) + 14c^2(910F_1+F_2+F_3)\}$$

$$+ 35\bar{f}(s)[31s^2c^4 + (s^2\beta+s)\{784\,c^2(1+c^2)\}$$

$$+ 28(s\,\beta+1)^2\{63(1+c^4)+574\,c^2\}] \qquad (46)$$

Solving for $\bar{f}$ we obtain

$$\bar{f} = \frac{9\bar{q}(s)}{(s-\alpha)(s-\gamma)} [s^3c^3 + \frac{61}{2} s^2(s\ \beta+1)(c+c^3)$$

$$+ s(s\ \beta+1)^2\{\frac{133}{2}\ (\frac{1}{c} + \frac{c^3}{1}) + \frac{1593}{2} c(\beta+1)$$

$$+ \frac{1593}{2} \beta\ (1+\frac{1}{c}) + \frac{315}{2} \beta\ \ (c^3 + \frac{1}{c^3})\}$$

$$+ (s\ \beta+1)^2\{\frac{1593}{2}\ (c + \frac{1}{c}) + \frac{315}{2}\ (c^3 + \frac{1}{c^3})\}$$

$$- \frac{4}{70(s-\alpha)(s-\gamma)} [2s^2c^4(4F_1+F_2+F_3)$$

$$+(s^2\beta+s)\{c^2(371\ F_1+56F_2+5F_3)+c^4(371\ F_1+5F_2+56F_3)\}$$

$$+(s\ \beta+1)^2(882F_1+126F_3)+c^4(882\ F_1+126F_3)$$

$$+14c^2(910F_1+F_2+F_3)]$$

where

$$(s-\alpha)(s-\gamma) = 31s^2c^4 + 784c^2(s^2\beta + s) + 784c^4(s^2\beta + s)$$

$$+ 1764(s^2\beta^2 + 1 + 2s\beta) + 1764c^4(s^2\beta^2 + 1 + 2s\beta)$$

$$+ 16072c^2(s^2\beta^2 + 1 + 2s\beta) \qquad (47)$$

$$= s^2\{31c^4 + 784\beta(c^2 + c^4) + 1764\beta^2(1 + c^4) + 16072\, c^2\beta^2\}$$

$$+ s\{784(c^2 + c^4) + 3528\beta(1 + c^4) + 32144\, c^2\beta\}$$

$$+ \{1764 + (1 + c^4) + 16072\, c^2\}$$

Let the prescribed discharge be constant i.e., $q(t)=q_o$, so that the initial velocity

$$u_o = \frac{q_o}{4c} \qquad (48)$$

The initial and boundary condition are

$$u(o,y,z) = u_o \quad \text{a constant} \qquad (49)$$

$$u(\pm 1,z,t) = 0 = u(y,\pm c,t), \quad t > 0 \qquad (50)$$

and the discharge condition is

$$q(t) = q_o \qquad (51)$$

From the equations (B.7),(B.8) and (B.9) we get

$$F_1 = \frac{5^2 \times 7}{32} \frac{q_o}{c} \quad ; \quad F_2 = \frac{7^2 \times 15}{32} \frac{q_o}{c}$$

and

$$F_3 = \frac{7^2 \times 15}{32} \frac{q_o}{c}$$

From equation (B.19)

$$\bar{P} = \frac{9q_0}{(s-\alpha)(s-\gamma)} [s^2 c^3 + (s^2\beta + s)\frac{61}{2}(c+c^3)$$

$$+ 133(\frac{1}{c} + c^3)(s^2\beta^2 + 1 + 2s\beta) + \frac{1593}{2} c (s^2\beta^2$$

$$+ 1 + 2s\beta) + \frac{7\times509}{2}\beta(s^2\beta^2+1+2s\beta)(c+\frac{1}{c})$$

$$+ \frac{7\times509}{2}(s^2\beta^2\frac{1}{s}+2\beta)(c+\frac{1}{c}) + \frac{315}{2}(c^3+\frac{1}{c^3})(s^2\beta^2$$

$$+ 1+2s\beta) + \frac{315}{2}(c^3+\frac{1}{c^3})(s\beta^2+\frac{1}{s}+2\beta) - \frac{q_0}{(s-\alpha)(s-\gamma)}[\frac{31}{4}s^2c^3$$

$$+ 196c(s^2\beta+s)+196c^3(s^2\beta+s)+\frac{441}{c}(s^2\beta^2+1+2s\beta)$$

$$+ \frac{441}{c}\times c^4(s^2\beta^2+1+2s\beta)+\frac{2009}{2}c(s^2\beta^2+1+2s\beta)] \quad (52)$$

Taking the inverse Laplace transform of (52) we obtain the pressure gradient

$$f = \frac{q_o}{2(\alpha-\gamma)} [\delta(t)(\alpha-\gamma)+\alpha^2 e^{\alpha t} - \gamma^2 e^{\gamma t})(\frac{5c^3}{2}$$

$$+ 157\beta(c+c^3)+1512\beta^2(c^3+\frac{1}{c}) + 12328\ c\beta^2$$

$$+ 32067\ \beta^3(\frac{1}{c} + c)+315\ \beta^3(\frac{1}{c^3} + c^3))$$

$$+ (\alpha e^{\alpha t} - \gamma e^{\gamma t})(157(c^3+c)+3024(\frac{1}{c} + c^3)$$

$$+ 24656\ \beta c+96201\ \beta^2(\frac{1}{c} + c)+945\ \beta^2(\frac{1}{c} + c^3))$$

$$+ (e^{\alpha t} - e^{\gamma t})(1512(\frac{1}{c} + c^3)+12328\ c+96201\ \beta(\frac{1}{c} + c)$$

$$+ 945\ \beta(c^3 + \frac{1}{c^3}))+(\frac{\alpha-\gamma}{\alpha\gamma} + \frac{e^{\alpha t}}{\alpha} - \frac{e^{\gamma t}}{\gamma})$$

$$\times (32067\ (c + \frac{1}{c})+315\ (c^3+\frac{1}{c^3}))]$$

The variation of pressure gradient w.r.t. the governing parameters are given in tables 1-2.

Let the fluid starts from rest i.e., $u_b(y,z) = 0$ and the discharge is equal to $q_b t$. Where q_b is constant.

From (35), (36) and (37), we get

$$F_1 = 0, \quad F_2 = 0, \quad F_3 = 0$$

and from (47) the Laplace transform of the pressure gradient is obtined as

$$\bar{f} = \frac{9q_0}{s^2(s-\alpha)(s-\gamma)} [s^3 c^3 + s^2(s\beta+1) \frac{61}{2} (c+c^3)$$

$$+133\, s\, (s\beta+1)^2(\frac{1}{c} + c^3) + \frac{1593}{2} sc(s\beta+1)^2$$

$$+ \frac{7 \times 509}{2} \beta s(s\beta+1)^2(c+\frac{1}{c}) + \frac{7 \times 509}{2} (s\beta+1)^2(c+\frac{1}{c})$$

$$+ \frac{315}{2} \beta s(s\beta+1)^2(c^3+\frac{1}{c^3}) + \frac{315}{2} (s\beta+1)^2(c^3+\frac{1}{c^3})]$$

$$\bar{f} = \frac{9q_0}{s^2(s-\alpha)(s-\gamma)} [s^3(c^3 + \frac{61}{2}\beta (c+c^3) + 133\beta^2(c^3+\frac{1}{c})$$

$$+ \frac{1593}{2}\beta^2 c + \frac{7\times509}{2}\beta^3(c + \frac{1}{c}) \frac{315}{2} \times \beta^3(c^3+\frac{1}{c^3}))$$

$$+ s^2 (\frac{61}{2} (c^3+c)+2\times133\beta(c^3+\frac{1}{c}) + 1593 BC + 3\beta^2$$

$$+ \frac{7\times509}{2} (c + \frac{1}{c}) + \frac{3\times315}{2}\beta^2(c^3+\frac{1}{c^3})) + s(133 (\frac{1}{c} + c^3)$$

$$+ \frac{1593}{2} c + \frac{3\times7\times509}{2}\beta (\frac{1}{c} + c) + \frac{315\times3}{2}\beta (c^3+\frac{1}{c^3}))$$

$$+ (\frac{7\times509}{2} (\frac{1}{c} + \frac{c}{1}) + \frac{315}{2} (c^3+\frac{1}{c^3}))$$

Taking inverse Laplace transform of which the pressure gradient is obtained as

$$f = 9q_0 [(\frac{\alpha e^{\alpha t} - \gamma e^{\gamma t}}{\alpha-\gamma})(c^3 + \frac{61}{2}\beta (c+c^3) + 133\beta^2 (\frac{1}{c} + c^3)$$

$$+\frac{1593}{2}\,\alpha\beta^{2}+1781.5\,\beta^{3}\left(\frac{1}{c}+c\right)+157.5\,\beta^{3}\left(c^{3}+\frac{1}{c^{3}}\right))$$

$$+\left(\frac{e^{\alpha t}-e^{\gamma t}}{\alpha-\gamma}\right)(30.5\ (c+c^{3})+266\ \beta\left(\frac{1}{c}+c^{3}\right)+1593\ \beta c$$

$$+5344.5\ \beta^{2}\left(\frac{1}{c}+c\right)+630\ \beta^{2}\left(c^{3}+\frac{1}{c^{3}}\right)+472.5\ \beta^{2}\left(c^{3}+\frac{1}{c^{3}}\right))$$

$$+\left(\frac{1}{\alpha\gamma}+\frac{1}{\alpha-\gamma}\left(\frac{e^{\alpha t}}{\alpha}-\frac{e^{\gamma t}}{\gamma}\right)\right)133\ \left(\frac{1}{c}+c^{3}\right)$$

$$+796.5\ c+5344.5\ \beta\ \left(\frac{1}{c}+\frac{c}{1}\right)+472.5\ \beta\ \left(c^{3}+\frac{1}{c^{3}}\right)$$

$$+\left(\frac{t}{\alpha\gamma}+\frac{1}{\alpha-\gamma}\left(\frac{e^{\alpha t}}{\alpha^{2}}-\frac{e^{\gamma t}}{\gamma^{2}}\right)+\left(\frac{\alpha+\gamma}{\alpha^{2}\gamma^{2}}\right)(1781.5\ \left(c+\frac{1}{c}\right)\right.$$

$$+157.5\ \left(c^{3}+\frac{1}{c^{3}}\right))]$$

The variation of pressure gradient w.r.t. the governing parameters are in tables 3-4.