4. THEORY

4.1. THREE BODY FORMALISM FOR ALPHA-DEUTERON INTERACTION

As stated earlier, the six nucleon system - bound states (resonances) of $^6$Li ($^6$He), d-α scattering, d-α break up - can be considered a three body system, if alpha is regarded as a spinless structureless boson which is assumed to be correct below alpha break up threshold. The T-matrix for the α-d break up reaction is expressed schematically as

$$T = T_{\text{2 body}} + T_{\text{3 body}} + \cdots$$

where $\times$ represents a two body interaction.

In impulse approximation and R-matrix theory, one of the nucleons in the deuteron is assumed to be a spectator and the interaction between the remaining nucleon and the alpha
particle is taken into account. In that case the first term only of (1) is considered. The break up reaction is then assumed to proceed via only one pair interaction (n-α or p-α). This assumption is thought to be valid for QFS at high incident energies but at low incident energies where all relative energies between two particles become small compared to their interaction energies, the other pair interactions can play a significant role and multiple processes are expected to become appreciable\(^6\). An improved impulse approximation, called new impulse approximation (NIA), was then introduced by Nakamura\(^6\) where S- and P-state amplitudes were subtracted completely in both exit channels (\(^5\)He-p and \(^5\)Li-n systems) from the matrix elements of off-energy shell impulse approximation (OIA). This subtraction was previously\(^6\) considered to be due to the strong interaction between spectator nucleon and α-particle. Later it was found\(^6\) that the OIA matrix elements for low angular momenta of the incident or entrance channel (α-\(^2\)H system) tend to exceed the unitarity limit and the subtraction was therefore attributed to the strong suppression of specific partial amplitudes in the entrance channel. This results in the suppression in S- and P-states of the exit channels since both \(^5\)He and \(^5\)Li systems are in S- and P-states in the energy region under consideration. This was the reason for the success of NIA. But still it was an approximate theory which considered only the effects of the final state interaction between the spectator nucleon and the alpha particle whereas the initial state interaction between the deuteron and the incoming α-particle was not treated properly.
It was Nakamura again\textsuperscript{21} who took into account this $\alpha$–D initial state interaction, which results in the reduction of partial amplitudes in the entrance channel. The idea was to eliminate completely the $S$, $P$, and $D$-state interactions in the entrance channel ($\alpha-^2\text{H}$ system) from OIA. The formalism of this modified impulse approximation (MIA) proceeds as follows.

The matrix element $M$ for the $^2\text{H}(\alpha, \alpha)p n$ reaction is given by\textsuperscript{63,65},

$$M = M^{(n)} + M^{(p)}$$

where the first (second) term includes all the processes in which neutron (proton) interacts first with the $\alpha$-particle. The initial-state interaction was not considered earlier\textsuperscript{63,65}. The exit channel is treated as $^5\text{He}-p(^{5}\text{Li}-n)$ in the first (second) term.

The matrix element $M^{(n)}$ can be written as\textsuperscript{21},

$$M^{(n)} = M^{(I)} + \sum_Y C(Y) \left[ r(Y) - 1 \right] P(Y)$$

where,

$$r(Y) = \frac{c(Y)}{C(Y)}$$

$M^{(I)}$ matrix element of OIA

$Y$ = Possible angular momentum state.

$C(Y)$ = Partial amplitude of $M^{(I)}$ in state $Y$.

$P(Y)$ = Projection operator to state $Y$.

$c(Y)$ = Partial amplitude which is corrected by the initial state interaction between the deuteron and the incident $\alpha$ particle as well as by the final state interaction between spectator proton and alpha.
The Feynman diagrams\textsuperscript{21} of the first and second terms of Eq. (3) are illustrated in Fig. 4.1. According to the ordinary off-shell impulse approximation (OIA)\textsuperscript{22},

\[ M^{(I)} = \sum_{Y} C(Y) P(Y) \]  

(5)

The second term of (3) is obtained by multiplying each term of (5) with the correction factor \( r(Y) - 1 \). The initial state \( \alpha-d \) interaction and the exit channel interaction between the alpha particle and the spectator proton (neutron) are introduced through the reducing factor, \( r(Y) \). \( M^{(p)} \) has a similar expression. This method (MIA) gives much better results\textsuperscript{21, 66} compared to NIA (Fig. 4.2).

The modified impulse approximation (described above) explains the data quite well at higher energies\textsuperscript{21} (Fig. 4.2) but for \( E_{c.m.} \lesssim 6 \) Mev, this theory fails. Some reaction mechanisms such as triton transfer reaction and the \( n-p \) FSI were introduced to reproduce the data, which, according to Koike\textsuperscript{25}, are not valid at low energies. The reason for failure of MIA analysis at low energies was explained by Koike in the following manner.

The break up amplitudes for the process \( d+\alpha \rightarrow \alpha+p+n \) are given by\textsuperscript{25},

\[ X_{2c,1c_0}^{(j\pi)}(q,q_o;E) = a_{2c,1c_0}^{(j\pi)}(q,q_o;E) Z_{2c,1c_0}^{(j\pi)}(q,q_o;E) \]

where,

\[ a_{2c,1c_0}^{(j\pi)} = \text{break up amplitude, } X \]

\[ Z_{2c,1c_0}^{(j\pi)} = \text{Impulse amplitude, } Z \]

\[ = \text{reduction factor in the MIA theory.} \]
Figure 4.1:

Feynman diagrams corresponding to
a) first term of equation (3)
b) second term of equation (3).
FIG. 4.1
Figure 4.2:

Absolute differential cross sections for $^2\text{H}(\alpha, \alpha\text{p})\text{n}$ reaction

a) experimental points are from ref. 50 together with theoretical calculations

b) experimental points are from ref. 29 together with theoretical calculations

--- fit based on MIA

--- fit based on NIA (New impulse approximation)
\( E_{\text{inc}} = 29.2 \text{ MeV} \)
\( \theta_\alpha = 21^\circ \)
\( \theta_\beta = 45^\circ \)

- MIA CALCULATIONS
- NIA CALCULATIONS

FIG. 4.2
This factor is, in general, a complex function of the three body momentum \( q \), and the two- and three-body angular momenta \((l, j; S, L)\) in both entrance and exit channels. According to MIA with some inherent simplified assumptions this factor does not depend on \( q \), which is seen to be approximately valid when multiple scattering terms (MST) are not much effective, i.e. at high energies. But in the lower energy region, relative energies of every two body subsystem are small and hence the effects of different MST become appreciable causing a large sensitivity of cross sections to the values of the reduction factor. That is why MIA fails to reproduce the data at lower energies.

Koike\(^{25,46}\), Charnomordic\(^{11}\) and others\(^{14,67}\) have, on the other hand, studied the \( \alpha \)-d system in the light of Faddeev equations. Calculations based on this theory are time consuming and difficult because of complicated co-ordinates and singularities in the scattering amplitude. Suitable methods for finding a solution of Faddeev equations are:

1) to solve the Faddeev equation with the theory of symmetric kernel\(^{68,69}\), and

2) to solve the Faddeev equation with a separable potential\(^{70,71}\).

Realistic potentials are used in method (1). By using method (2) one can get an exact solution of Faddeev equation by reducing it to one dimensional equations. This theory with two body separable potentials was applied to the analysis of \(^2\text{H}(\alpha, \alpha)p\) reaction and explained the data fairly well at 15-42 Mev energy region\(^{25}\) and
also at much lower energies\(^{55}\)). Several calculations of \(^6\)Li\(^{(6\)He\)} levels\(^{19,72,73}\) and \(d-\alpha\) elastic scattering\(^{11,14,46}\) have been done with fairly good results. Thus Faddeev formalism for the investigation of three body problem seems to be more complete compared to the other lines of theoretical approach discussed above.

Section 4.1.1 is concerned with Faddeev equations. These equations for the three particle system, though complete and solvable in principle, are too complicated for practical purposes and therefore need simplification to some manageable form. Transition operators of the two body subsystems occurring in Faddeev equations are split into separable (with respect to subsystem transition operators only) and non separable parts. Derivation of an exact one dimensional integral equations for the three particle transition amplitudes is done by Schmidt method\(^{74-76}\). These one dimensional equations due to E.O.Alt et al\(^{49}\) are described in section 4.1.2.

Faddeev treatment gets simplified when applied to the potential theory of final state interaction. In case of low relative energy for the interacting pair in the final state, FSI is expected to be predominant resulting typical formulas for the matrix element due to Watson\(^{77}\). This is shown in section 4.1.3. Section 4.2 deals with the theory of elastic scattering due to Charnomordic et al\(^{11}\).
4.1.1. Faddeev Equations

Interactions between three particles can be represented by three two body potentials (Fig. 4.3) and suspected three body potential \( V_{123} \) not obtainable from two body interaction. Thus the total potential is given by,

\[
V(1,2,3) = V_{23} + V_{31} + V_{12} + V_{123} \quad \text{(if present)}
\]
\[
= V_1 + V_2 + V_3 + V_{123}
\]

Effects related to the two body interaction in three body systems are very hard to disentangle. But to get the possible evidence of three body forces \( V_{123} \), if any, one should precisely determine the two body potentials. The formalism of three particles interacting mutually in the final state becomes simplified under the assumption that three body forces are negligible.

The non-interacting Hamiltonian corresponding to \( V = 0 \) is given by,

\[
H_0 = \sum_{i=1}^{3} \frac{k_i^2}{2M_i}
\]  

(6)

Taking into account the interaction, \( V = \sum_{i=1}^{3} V_i \) the interacting Hamiltonian is,

\[
H = H_0 + V
\]

(7)

State vectors are \( |p_i, q_i\rangle \).

Now the three body Green function operator for non-interacting particles is,
Figure 4.3:

Schematic of the interaction of three bodies through two body potentials \((V_1, V_2, V_3)\).

\(M_1, M_2, M_3\) = masses of three particles.

\(k_1, k_2, k_3\) = momenta of particles.
\[ G_0(s) = (H_0 - s)^{-1} \]  
\[ (8) \]

where \( s \) is an energy variable.

For interacting particles, the Green function operator becomes,

\[ G(s) = (H_0 + V - s)^{-1} \]  
\[ (9) \]

The operator form of the three-body Green function is

\[ G(s) = G_0(s) - G_0(s)VG(s) \]  
\[ (10) \]

It operates on asymptotic three-body channel wave functions \( \phi_j \) where the subscript \( j = 0, \ldots, 3 \) describes different asymptotic situations.

Now channel wave function for three free bodies,

\[ \phi_0 = \exp(i \sum_{i=1}^{3} k_i \cdot r_i) \]  
\[ (11) \]

Channel wave functions with two particles in a bound state of order 'n' and a free third particle are given by,

\[ \phi_{kn} = \exp(i q_k \cdot \rho_k) \psi_{kn}(\rho_{ij}) \]  
\[ (12) \]

where

\[ \rho_{ij} = \text{relative co-ordinate of } ij \text{-pair} \]
\[ = r_i - r_j \]

and

\[ \rho_k = \text{co-ordinate of the third particle with respect to the pair's barycentre.} \]
\[ = r_k - \frac{M_i r_i + M_j r_j}{M_i + M_j} \]

Now the iterative form of the T-matrix is given by (13).
\[ T(s) = V - V G_0(s) T(S) \quad \text{(in terms of } G_0) \]

\[ = V_1 + V_2 + V_3 - (V_1 + V_2 + V_3) G_0(s) (V_1 + V_2 + V_3) \]

\[ + (V_1 + V_2 + V_3) G_0(s) (V_1 + V_2 + V_3) G_0(s) (V_1 + V_2 + V_3) + \ldots \]  
(13)

This can be rearranged:

\[ T(s) = \sum_{i=1}^{3} \left[ V_i - V_i G_0(s) V_i + V_i G_0(s) V_i G_0(s) V_i - \ldots \right] \]

\[ + \sum_{j \neq k} \left[ V_j - V_j G_0(s) V_j + V_j G_0(s) V_j G_0(s) V_j + \ldots \right] G_0(s) \]

\[ \times \left[ V_k - V_k G_0(s) V_k + V_k G_0(s) V_k G_0(s) V_k + \ldots \right] + \ldots \]  
(14)

The series (14) contains within the brackets potentials corresponding to interactions of two particles while the third is non-interacting. They can be denoted by,

\[ T^3_1(s) = V_i - V_i G_0(s) V_i + V_i G_0(s) V_i G_0(s) V_i - \ldots \]

\[ = V_i - V_i G_0(s) T^3_1(s) \]  
(15)

They are three body operators acting on a state vector \( \{ p_i, q_i \} \).

There are two difficulties associated with (13) or (14).

1) Bound states yield homogeneous solutions to the Lippmann-Schwinger integral equations, which results in a non-uniqueness of the three body equations.

2) The kernel \( K = G_0 V \) is not completely continuous and therefore the condition of the Schmidt norm,

\[ \left| K^2 \right| = \text{tr} \left[ K K^+ \right] \leq \infty \]  
(16)

is not satisfied. The singularities associated with the kernel \( V_i G(s) \) in Eq. (14) give rise to disconnected diagrams given by,
The sum of these diagrams is the matrix $T_i^3(s)$ corresponding to particle $i$ free. Here $V_i \neq 0$ but $V_j = V_k = 0$ (or neglected).

If any two particles are assumed to interact at a time neglecting the third body (i.e., pairwise interaction), $V_i, V_j, V_k$ are all different from zero and the Faddeev equations remove the two difficulties mentioned above. The homogeneous equations when two bodies are asymptotically bound are eliminated by having coupled integral equations. Interactions of two body pairs \(78\) give rise to

\[
\begin{pmatrix}
\frac{1}{i} + \frac{1}{j} \\
\frac{1}{k} + \frac{1}{l}
\end{pmatrix} + \begin{pmatrix}
\frac{1}{j} + \frac{1}{k} \\
\frac{1}{l} + \frac{1}{i}
\end{pmatrix}
\]

The Eq. (14) may be written as

\[
T(s) = \sum T_1 + \sum_{i \neq j} T_iG_j T_j + \sum_{i \neq j \neq k} T_iG_j T_j G_k + \ldots
\]

The Eq. (19) is the iterative expansion of the coupled set of equations due to Faddeev given by,

\[
T(s) = T^{(1)}(s) + T^{(2)}(s) + T^{(3)}(s)
\]

such that

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3
\end{pmatrix} =
\begin{pmatrix}
T_1 \\
T_2 \\
T_3
\end{pmatrix} - G
\begin{pmatrix}
0 & T_1 & T_1 \\
T_1 & 0 & T_2 \\
T_3 & T_3 & 0
\end{pmatrix}
\begin{pmatrix}
T_1 \\
T_2 \\
T_3
\end{pmatrix}
\]

\[
(21)
\]
In symbolic vector representation,

$$T = T - G K F T$$  \hspace{1cm} (22)$$

The kernel $K_F$ satisfies (16).

4.1.2. The Alt-Grassberger-Sandhas (AGS) Equations

For three distinguishable particles (like $a$, $n$, $p$) with two particle pairwise interactions ($V_a = V_{ij}$; $i, j = 1, 2, 3$ and $\alpha \neq i, j$ as stated earlier in section 4.1.1), the Schrödinger equations with eigenstates $\left| \phi_{a,m} \right>$ are given by,

$$H_\alpha \left| \phi_{a,m} \right> = E_{a,m} \left| \phi_{a,m} \right>$$  \hspace{1cm} (23)$$

where $H_\alpha = H_0 + V_a$ are called the channel Hamiltonians.

Introducing the resolvent or Green's function (as described in section 4.1.1)

$$G(s) = (H-s)^{-1}$$  \hspace{1cm} (24)$$

and using resolvent equations

$$G(s) = G_\alpha(s) - G_\alpha(s)\overline{V}_a G(s)$$  \hspace{1cm} (25)$$

$$G(s) = G_\alpha(s) - G(s)\overline{V}_a G_\alpha(s)$$  \hspace{1cm} (26)$$

with

$$G_\alpha(s) = (H_\alpha-s)^{-1}$$  \hspace{1cm} (27)$$

the $S$-matrix representing the transition from an initial configuration $\left| \phi_{a,m} \right>$ to a final configuration $\left| \phi_{\beta,n} \right>$ is given by \hspace{1cm} (28)

$$S_{\beta n, \alpha m} = \delta_{\beta \alpha} \delta_{nm} - 2\pi i \delta_{E_{\beta n} - E_{\alpha m}} \left< \phi_{\beta,n} \mid U_{\beta \alpha} (E_{\alpha m} + i0) \mid \phi_{a,m} \right>$$
where, $U_{\beta\alpha}$ represents the transition operators given by,

$$U_{\beta\alpha}(s) = -(1-\delta_{\beta\alpha})(H_o-s) + \bar{V}_\alpha \bar{V}_\beta \delta_{\beta\alpha} \bar{V}_\alpha \bar{G}(s) \bar{V}_\alpha$$  \hspace{1cm} (29)

Only the on-shell matrix elements of the transition operators, $U_{\beta\alpha}(s)$, are needed to determine the S-matrix. The Faddeev-like equations for the transition operators

$$U_{\beta\alpha} = -(1-\delta_{\beta\alpha})(H_o-s) - \sum_{\gamma \neq \beta} T_{\gamma} G_{\beta\gamma} U_{\gamma\alpha}$$  \hspace{1cm} (30)

or

$$U_{\beta\alpha} = -(1-\delta_{\beta\alpha})(H_o-s) - \sum_{\delta \neq \alpha} U_{\beta\delta} G_{\delta\alpha} T_{\delta}$$  \hspace{1cm} (31)

may be used to calculate, in principle, the desired S-matrix elements but with considerable difficulties. These equations [(30) and (31)] are, however, transformed to a structure of multichannel two particle Lippmann-Schwinger equations by E.O. Alt et al [49].

The Eqs. (30) can be written in the form,

$$F = I - ITF$$  \hspace{1cm} (32)

where $F$, $I$ and $T$ are matrices, the elements of which represent operators in the Hilbert space

$$F_{\beta\alpha} = G_o U_{\beta\alpha} G_o$$  \hspace{1cm} (33)

$$I_{\beta\alpha} = -(1-\delta_{\beta\alpha})G_o$$  \hspace{1cm} (34)

$$T_{\beta\alpha} = -\delta_{\beta\alpha} T_{\alpha}$$  \hspace{1cm} (35)

Transition operators of the two particle subsystems occurring in Faddeev equations are now split into separable and non-separable parts,

$$T_{\alpha} = T^{s}_{\alpha} + T^{'}_{\alpha}$$  \hspace{1cm} (36)
where, \( T_{\alpha}^S \) is a sum of separable terms of the form,

\[
T_{\alpha}^S = -\sum_m \left| \alpha_m,s \right\rangle t_{\alpha,m}(s) \left\langle \alpha_m,s \right|
\]  \hspace{1cm} (37)

Due to this splitting the Eq. (32) takes the form,

\[
F = I - IT^S F - IT^S F
\]  \hspace{1cm} (38)

where \( T^S \) and \( T^T \) are matrices given by (35) with \( T_{\alpha}^S \) and \( T_{\alpha}^T \) instead of \( T_{\alpha} \) respectively.

Defining a similar equation,

\[
F' = I - IT^T F'
\]  \hspace{1cm} (39)

as (32) and multiplying it with \((1-T^S F)\) from the right, one obtains,

\[
\tilde{F} = I - IT^S F - IT^T \tilde{F}
\]  \hspace{1cm} (40)

where

\[
\tilde{F} = F'(1-T^S F)
\]  \hspace{1cm} (41)

Comparing (40) with (38) one gets,

\[
\tilde{F} = F
\]  \hspace{1cm} (42)

or

\[
F = F' - F'T^S F
\]  \hspace{1cm} (43)

Inserting (33) - (35) in (43), the following integral equation for \( U_{\beta \alpha} \) is obtained:

\[
G_0 U_{\beta \alpha} G_0 = G_0 U_{\beta \alpha} G_0 - \sum_{\gamma, \tau} G_0 U_{\beta \gamma} G_0 \left\langle \gamma, \tau \right| t_{\gamma, \tau} \left\langle \tau, \tau \right| G_0 U_{\gamma \alpha} G_0
\]  \hspace{1cm} (44)

where

\[
U'_{\beta \alpha} = -(1-\delta_{\beta \alpha})(H_0 - s) - \sum_{\gamma \neq \beta} T_{\gamma} G_0 U'_{\gamma \alpha}
\]  \hspace{1cm} (45)

(according to (39))
Considering the case of elastic or rearrangement collisions and defining the transition amplitudes as,

\[ X_{\beta n, \alpha m}(s) = \langle \beta, n; s | G_0(s)U_{\beta\alpha}(s)G_0(s) | \alpha, m; s \rangle \]  

(46)

One can have from (44),

\[ X_{\beta n, \alpha m} = Z_{\beta n, \alpha m} - \sum_{\gamma, r} Z_{\beta n, \gamma r} t_{\gamma r} X_{\gamma r, \alpha m} \]  

(47)

where the generalized potentials \( Z_{\beta n, \alpha m} \) are given by,

\[ Z_{\beta n, \alpha m}(s) = \langle \beta, n; s | G_0(s)U'_{\beta\alpha}(s)G_0(s) | \alpha, m; s \rangle \]  

(48)

Eq. (47) has a structure of a multichannel two particle Lippmann-Schwinger equation and \( Z_{\beta n, \alpha m} \) and \( t_{\gamma r} \) are respectively potentials and free propagators. After angular momentum decomposition, Eq. (47) are reduced to a system of one dimensional integral equations (called AGS equations) due to the fact that the splitting of the transition operator into separable and non-separable parts in three particle space is only performed in the subsystem operators unlike in the two particle case where the square of the Faddeev kernel is approximated by a series of terms separable with respect to all of its momentum variables. These AGS equations are more practical than those of Faddeev and have become the starting point for the theoretical investigation of the three body problem.

4.1.3. Faddeev Treatment Applied to Final State Interactions

Faddeev treatment gets simplified when applied to a potential theory of final state interactions. With the production poten-
tial $T_0$ and final state pair amplitude $T_{jk} = T_i$, the series (19) takes the form,

$$T = T_0 + \sum_i T_i G_0 T_0 + \sum_{i \neq j} T_{i} G_0 T_{j} G_0 T_0 + \ldots$$

(49)

This is shown in Eq. (50) with the additional series (51) corresponding to the three body interaction.

A process yielding three bodies in the final state is shown in Fig. 4.4. Amplitudes corresponding to the production potential ($V$) and the interaction potential ($U$) are called production amplitude $T_0$ and the single pair amplitude $T_{12}$. Hence the Eq. (49) would be limited to

$$T = T_0 + T_{12} G_0 T_0$$

(52)
Figure 4.4:

Schematic of a nuclear reaction producing three particles in the final state.

\( V \) = production potential.

\( U \) = potential of interaction of two particles in the final state.
corresponding to first two terms of Eq. (50), where the sum would consist of a single term with \( i = 1, j = 2, k = 3 \). The matrix element is given by,

\[
T_{1f} = \langle \Psi_f | (1 + T_{12} G_0) T_0 | \Psi_1 \rangle = \langle \Psi_f | T_0 | \Psi_1 \rangle + \langle \Psi_f | T_{12} G_0 T_0 | \Psi_1 \rangle
\]

(53)

The first term corresponds to the production amplitude \( T_0 \) (associated with \( V \)) and the second term corresponds to the two body interaction amplitude \( T_{12} \) (associated with \( U \)).

The second term is calculated using (49) with the approximations of Delbourgo. The amplitudes take the form:

\[
T_{1f} = T_{1f}^0 \left( 1 + \sum_{j \neq k} C_{jk}(q_{jk}) \frac{\exp(i\delta_{jk}) \sin\delta_{jk}}{q_{jk}} \right)
\]

(54)

where \( C_{jk} \) measures the strength of the \( jk \)-scattering relative to the unity of the production term. In the absence of FSI, \( C_{jk} = 0 \) and only production amplitude \( T_{1f}^0 \) (unenhanced) remains. If \( C_{jk} \gg 1 \), the final state formulas of Migdal and Watson are obtained, which are given by,

\[
|T|^2 = \frac{\sin^2 \delta_\ell}{k^{2\ell}}
\]

(55)

where

\[
k^{2\ell + 1} \cot \delta_\ell = -\frac{1}{a} + \frac{1}{2} r k^2
\]

(56)

\( a, r \) are effective range parameters called scattering length and effective range respectively.

Final state calculations using this effective range formula were performed by Dasgupta et al. for only \( \alpha p \) and \( \alpha n \) interactions.
and provide fair fits to the data. A similar calculation\(^{80}\) for the \(\alpha\)-d break up reaction at 15 Mev lab. energy shows that (\(\alpha p+\alpha n\)) FSI-values are at least as good as the rigorous Faddeev calculations up to 11.3 Mev. The discrepancy between the data and such final state interaction may even be explained by a contribution of possible three body forces. At 15 Mev, however, \(\alpha p\) FSI alone gives better fit to the data (Fig. 4.5). Of course, Coulomb interaction for \(\alpha\) FSI should be included for a much better fit.

4.2. ELASTIC SCATTERING

Three particles (i, j, k) are assumed to interact pairwise with masses \(m_i\), \(m_j\) and \(m_k\). The AGS equations (described in section 4.1.2) for the three body process \(j+(ki)\rightarrow i+(jk)\) are given by\(^{11}\),

\[
U_{ij}(s) = (1-\delta_{ij})(s-H_0) + \sum_{k=1}^{3} (1-\delta_{ik})T_k(s)G_0(s)U_{kj}(s)
\]

where, \(s\) refers to total energy,

\(U_{ij}\) = transition operators,

\(T_k\) = two body scattering amplitude for particles i and j,

\(G_0(s) = (s-H_0)^{-1}\) = resolvent of free Hamiltonian.

By choosing separable two body interactions, the corresponding scattering amplitudes are made separable.

\[
V_i = \sum_n |in\rangle \lambda_{in} \langle in| \tag{58}
\]
Figure 4.5:

Arc length projected coincidence spectrum.
Experimental data are from ref. 22.
Theoretical calculations (solid and dashed lines)
are based on effective range approximation.
\[ E_\alpha(\text{inc}) = 15 \text{ MeV} \]

\[ \theta_\alpha = 16.9^\circ, \ \phi_\alpha = 0.0^\circ \]

\[ \theta_p = 30.2^\circ, \ \phi_p = 180.0^\circ \]

\[ \frac{d^2\sigma}{d\Omega dE_\alpha} \text{ in mb. sr}^{-2} \text{ MeV}^{-1} \]

Fig. 4.5
\[ T_i(s) = \sum_n \left| \text{in} \right> R_{\text{in}}(s) \left< \text{in} \right| \]  

(59)

where, \( \lambda_{\text{in}} \) is called strength and \( \left| \text{in} \right> \) is the form factor in the two body channel denoted by \( n \).

Using (59) in (57), some calculations\(^4^9\) lead to the equation for three body scattering amplitudes, \( X_{\text{in}, jm} \):

\[ X_{\text{in}, jm}(s) = Z_{\text{in}, jm}(s) + \sum_{k, r} Z_{\text{in}, kr}(s) R_{kr}(s) X_{kr, jm}(s) \]  

(60)

where, \( R_{kr} \) = propagator for the (ij)-pair in the three-body Hilbert space and the Born term is given by

\[ Z_{\text{in}, jm}(s) = (1 - \delta_{ij}) \left< \text{in} \right| G_0(s) \left| jm \right> \]  

(61)

According to the formalism of Afnan and Thomas\(^8^1\), in case of \( \alpha-d \) system, particles 1 and 2 always refer to nucleons while \( \alpha \) is particle 3. Following three processes occur.

\[
3+(12) \rightarrow 3+(12) \\
\rightarrow 1+(23) \\
\rightarrow 2+(13)
\]  

(62)

The first term is the \( d-\alpha \) scattering while two others are settings for \( d+\alpha \rightarrow N+(N\alpha) \). Considering Eq. (61) one can write the corresponding three body equations:

\[
X_{3n, 3m} = \sum_{\alpha} Z_{3n, 1} \alpha R_{1} \alpha X_{1\alpha, 3m} + \sum_{\alpha} Z_{3n, 2} \alpha R_{2} \alpha X_{2\alpha, 3m}, \\
X_{1\alpha, 3m} = Z_{1\alpha, 3m} + \sum_{\beta} Z_{1\alpha, 2} \beta R_{2} \beta X_{2\beta, 3m} + \sum_{\beta} Z_{1\alpha, 3n} R_{3n} X_{3n, 3m}, \\
X_{2\alpha, 3m} = Z_{2\alpha, 3m} + \sum_{\beta} Z_{2\alpha, 1} \beta R_{1} \beta X_{1\beta, 3m} + \sum_{n} Z_{2\alpha, 3n} R_{3n} X_{3n, 3m}
\]  

(63)
where \( n, m \) refer to \((N-N)\) pairs and \( \alpha, \beta \) refer to \((N-\alpha)\) pairs.

Considering identity of nucleons 1 and 2 and introducing the anti-symmetric amplitudes, the three coupled equations (63) can be combined as follows:

\[
X_{nm} = \sum_{\alpha} Z_{n\alpha} R_{\alpha} X_{\alpha m} \\
X_{\alpha m} = Z_{\alpha m} + \sum_{\beta} Z_{\alpha \beta} R_{\beta} X_{\beta m} + \sum_{n} Z_{\alpha n} R_{n} X_{nm}
\]

\[
(64)
\]

where,

\[
Z_{\alpha \beta} = -Z_{1\alpha, 2\beta}
\]

\[
Z_{n\alpha} = Z_{3n, 1\alpha}
\]

\[
Z_{\alpha m} = Z_{1\alpha, 3m}
\]

\[
R_{n} = 2R_{3n}
\]

Eq. (64) can be written in the form,

\[
X_{ij} = Z_{ij} + \sum_{k} Z_{ik} R_{k} X_{kj}
\]

\[
(66)
\]

4.2.1. Two Body Potential

Most of the theoretical investigations\(^{25,82-86}\) of three body problem use two body potentials which are of separable form mainly because of simplifications introduced in the three body equations. Realistic two nucleon potentials are of a local type\(^{87-89}\) and not expected to be of separable form. When non-local separable potentials are used in the Lippmann-Schwinger
integral equation for the off-energy shell two nucleon partial wave scattering amplitude, the kernel of the equation becomes separable and hence the equation can be solved algebraically. The choice of the appropriate form factor is done in such a way that the resulting off-shell amplitude is made to satisfy the following requirements\(^{90,91}\):

1) reduction to the correct on-shell amplitudes,
2) off-shell two particle elastic unitarity,
3) reasonable analyticity properties,
4) time reversal invariance,
5) proper threshold behaviour in both energy and momentum variables.

Reduction to the correct on-shell amplitude is met by choosing suitable potential parameters and the unitarity requirement for off-shell scattering amplitude is automatically met if the amplitude is generated from a potential by the Lippmann-Schwinger equation. The other three requirements are satisfied by a proper choice of the functional form of the separable potential which generates the scattering amplitude.

A separable potential includes an attractive and a repulsive term for the core and may contain spin orbit and tensor effects\(^{92}\). Coulomb effects\(^{82}\) may be incorporated in this potential treatment when charged particles are involved.
Rank one separable interaction can be written as \((67)\),

\[ V_{LL'}(p,p') = g_L(p)c_L \Lambda c_L'g_L'(p') \]

where, \(c_L, c_L'\) are strength parameters, \(\Lambda\) is +1(-1) when the corresponding partial wave is repulsive (attractive), and \(g_L(p)\) and \(g_L'(p')\) are the form factors.

A similar expression for the two body T-matrix according to \((67)\) is given by,

\[ T_{LL'}(p,p',s) = g_L(p)c_L R(s)c_L'g_L'(p') \]

where, \(R(s) = \text{two-body propagator} = \left[\frac{\Lambda}{1 - G(s)}\right]^{-1} \)

A. N-\(\alpha\) interaction

For practical purposes, only \(S_{1/2} , P_{1/2} \) and \(P_{3/2} \) partial waves for the N-\(\alpha\) system are considered. Rank - 1 one term separable interactions (denoted by CFL) for the above partial waves for the N-\(\alpha\) system are constructed by Charnomordic et al \((11)\). Form factors are,

\[ g_L(p) = p^L(1+\gamma_L p^2) \prod_{j=0}^{L+1} (1+\beta_j p^2) \]

\(\beta\) and \(\gamma\) parameters \((11)\) are determined from the least square fit of the experimental data.

B. N-N interaction

Only \(1S_0\) and coupled \(3S_1-3D_1\) partial waves for the N-N system are taken into account. Two term separable potential (denoted by CFL_{np}) for \(1S_0\) partial wave are constructed for N-N system.
Attractive (A) and repulsive (R) form factors are given by\(^{11}\),

\[
g^A(p) = \frac{1 + \gamma^A p^2}{(1 + \beta_1^A p^2)(1 + \beta_2^A p^2)}
\]

\[
g^R(p) = \frac{\gamma^R p^2}{(1 + \beta_1^R p^2)(1 + \beta_2^R p^2)}
\]

CFL\(_{np}\) parameters for the \(^1S_0\) partial wave are shown in Ref. \(^{11}\).

Several parametrizations were used by Charnomordic et al\(^{11}\) for the description of \(^3S_1 - ^3D_1\) state of the N-N system. They are due to Doleschall\(^{93}\) (rank-1, one term), Serduke\(^{94}\) (rank-2, one term) and Mongan\(^{90}\) (rank-2, two terms).

Form factors are of Yamaguchi type\(^{11}\):

\[
g_L(p) = \frac{p^L}{(p^2 + \beta^2)(L+m)/n}
\]

We are interested with potentials due to Serduke\(^{94}\), which are denoted by ACS\(_X\) where \(X = D\)-state percentage value or \(P_D\) value.

4.2.2. Calculation of Differential Cross Section

Calculations for the differential cross section are considered with CFL interactions (described above) for the N-\(\alpha\) system and ACS\(_4\) (4\% \(P_D\) value) for the \(^3S_1 - ^3D_1\) for N-N system. Due to conservation of isospin, \(^3S_1 - ^3D_1\) partial waves of the interaction contribute only since for d-\(\alpha\) scattering the total isospin of the system, \(I = 0\) and hence \(^1S_0\) does not come into play.
The Eq. (66) is solved to obtain the physical scattering amplitudes $X_{3n,3m} = 2X_{nm}$, the knowledge of which is necessary for the evaluation of d-α scattering observables. A system of coupled equations is solved by Padé approximant technique. From the physical amplitudes one can construct the scattering matrix, $M(\theta)$. The differential cross section is finally evaluated using following trace of matrix products,

$$\sigma(\theta) = \text{Tr} \left[ M(\theta)M^+(\theta) \right]$$

(73)

Fig. 4.6 shows a comparison of the theoretical fit with the experimental data at several energies. Calculated cross sections are seen to be in good agreement with experiments showing the validity of the three body model using separable N-N and N-α forces to predict scattering properties of the six nucleon system. Discrepancies at small angles at low energies are explained to be due to lack of Coulomb interactions in the calculations. Disagreements at energies above 20 Mev are explained to be due to the limitation of the three body model considering structureless alpha particles.
Figure 4.6:

Differential cross section in the centre of mass system for d-α elastic scattering at various deuteron laboratory energies. The experimental data are from ref. 11. Solid lines (---) represent Charnomordic's Faddeev type calculations with ACS4-CFL interactions as described in sections 4.2.1A and 4.2.1B.
Fig. 4.6