CHAPTER-IV

ULTRAHARMONIC SYNCHRONIZATION IN A NOISY ENVIRONMENT

4.1. Introduction:

Radiowave signals in their journey from one station to another, although radiated in a pure and noise-free state, do not remain so during propagation. This is because the incoming signal is invariably contaminated with some extraneous disturbances like additive atmospheric noise, the circuit noise, the galactic noise, thermal noise, shot noise etc. These disturbances have different characters, but for many practical situations, noise, corrupting the signal, may be characterised as a stationary Gaussian random process of zero mean and wide spectral power density at the radio frequency (IV-1, IV-2, IV-3). Fading and multipath propagation also constitute other types of disturbances.

It is for this reason, that the study of the performance of an injection synchronized oscillator in the face of a noise corrupted signal assumes importance and the design of a system with improved performance is sought for.

Quite a sizable amount of work (IV-4, IV-5, IV-6, IV-7) has been done since the fifties of this century on the phenomenon of entrainment of a harmonic oscillator when.
synchronized by a signal embedded in additive random Gaussian noise. In a comparatively recent work [17-3] the matter has been dealt with in a more rigorous way wherein the author has investigated the synchronization phenomenon for an ISO in additive noise process having a flat spectral power density of reasonably wide bandwidth as well as for a band limited noise process. The author has suggested there how faithfully an ISO can be employed as a reference carrier generator or as a wideband amplifier in such a situation. Unlike the case of harmonic synchronization, ultraharmonic synchronization of a class-A oscillator in the face of a noisy signal has not been touched upon by researchers. But it is quite likely that in the applications of ultraharmonic synchronization of an ISO for the design of a frequency synthesizer or for the construction of a hearing-aid device for partially deafened persons, the characteristics of signals carrying information may be altered unpredictably by the extraneous noise arising from sources, such as the circuit, the intervening media etc. Thus one cannot ignore the importance of studying the situation when a synchronizing signal corrupted by an additive random Gaussian process is injected to an ISO operating in the ultraharmonic mode. Again since an ultraharmonically locked oscillator is functionally similar to a multiplier and enjoys advantages in terms of circuit complexity and cost over a multiplier, a need for studying the performance of an ultraharmonically locked oscillator in a noisy environment is felt.
These are done in this Chapter of the dissertation. In particular, the Chapter addresses the problem with a bid to examine the capture capability of the ISO's under this condition.

4.2. Synchronization with a CW Signal Accompanied by an Additive Gaussian Noise:

Let us now discuss analytically the behaviour of a class-A oscillator in the third order ultraharmonic synchronization mode in the presence of a CW signal accompanied by an additive white Gaussian random noise. The level of synchronization, as in previous chapters, is restricted to a low value.

It is not difficult to see that the noise accompanying the signal renders the locking picture of an oscillator somewhat unpredictable in the sense that any random process creates fluctuations in the instantaneous phase or frequency and the amplitude of the oscillator. The locking characteristics of the oscillator hence forth needs to be looked upon from the statistical point of view. The fluctuations remaining small, however, the problem can be tackled by linearization technique [IV-3].

As in the analysis of ultraharmonic synchronization with CW signal in the previous chapters, we start with the same oscillator loop equation (2.41) and the expressions in (2.40) and (2.44). The only difference lies in the forms of oscillator
input and output voltages. Let the input signal to the oscillator be written in the form,

\[ e_i(t) = E \sin \omega t + n_c(t) \cos \omega t - n_s(t) \sin \omega t \quad (4.1) \]

\[ = E \sin \omega t + n(t) \quad (4.2) \]

where,

\[ n(t) = n_c(t) \cos \omega t - n_s(t) \sin \omega t \quad (4.3) \]

The term \( n(t) \) signifies the additive white Gaussian noise with \( n_c(t) \) and \( n_s(t) \) being independent random variables with zero mean and one-sided spectral power density (say) \( N_0 \). The output of the oscillator may be written, exactly as in earlier cases, to be,

\[ e_o(t) = A \sin (\omega t + \Theta_1) + B \sin (3\omega t + \Theta_2) + n'(t) \cos (\omega t + \Theta_1) \]

\[ - n'(t) \sin (\omega t + \Theta_1) \quad (4.4) \]

\[ = A \sin (\omega t + \Theta_1) + B \sin (3\omega t + \Theta_2) + n'(t) \quad (4.5) \]

where,

\[ n'(t) = n_c'(t) \cos (\omega t + \Theta_1) - n_s'(t) \sin (\omega t + \Theta_1) \quad (4.6) \]

The first and third terms in (4.5) denote the directly transmitted components of the signal and noise inputs and the other term, the oscillator frequency component.

Now proceeding as in earlier chapters, the governing amplitude and phase equations of the ISO come out as,
\[
\frac{d\theta_1}{dt} = \frac{w_0}{2q(A-n_s')} \left( \gamma - n_s \right) \sin\theta_1 + n_c \cos\theta_1 + (a_1 - 1) n_c' + \left( \frac{3}{4} n_c'^3 + \frac{3}{2} n_s'^2 - \frac{3}{4} n_c n_s'^2 + \frac{3}{4} A^2 n_c' + \frac{3}{2} B^2 n_c' - \frac{3}{2} \Delta n_c n_s' + \frac{3}{2} ABn_c' \cos\psi - \frac{3}{2} Bn_c' n_c' \cos\psi - \frac{3}{4} ABn_c \sin\psi + \frac{3}{4} A^2 B \sin\psi \right) \]
\[
\frac{w_0 A}{2(A-n_s')} - \frac{w^2}{w_0} \]
\[
\frac{dB}{dt} = \frac{w_0}{2q} \left( a_1 - 1 \right) B - a_3 \left\{ \frac{3}{4} B^3 - \frac{1}{4} A^3 \cos\psi + \frac{3}{2} A^2 B + \frac{3}{2} Bn_c'^2 - \frac{3}{2} ABn_c' + \frac{n_c'^3}{4} - \frac{3}{4} n_c'^2 n_s' + \frac{3}{4} A^2 n_c' - \frac{3}{4} A^2 n_c'^2 \cos\psi - \left( \frac{c}{4} - \frac{3}{4} n_c n_s'^2 - \frac{3}{4} A^2 n_c' + \frac{3}{2} \Delta n_c n_s' \right) \sin\psi \right\} \]
\[
\frac{d\phi}{dt} = \frac{w_0}{2} \left( 9w^2 - w_0^2 \right) - \frac{w_0 a_3}{3qB} A^3 \sin\phi + \frac{w_0 a_3}{3qB} N_2(t) + \frac{3d\theta_1}{dt} \]
\[
where \phi = 3\theta_1 - \theta_2, \text{ the input-output phase difference and}
\]
\[
N_2(t) = (n_c'^3 \cos\phi + n_s'^3 \sin\phi) - 3n_c' n_s' (n_c' \cos\phi + n_s' \sin\phi) - 3A^2 (n_c'^3 \cos\phi + n_s'^3 \sin\phi) + 3A (n_c'^2 - n_s'^2) \sin\phi + 6 \Delta n_c n_s' \cos\phi \]
\[
N_2(t) \text{ actually stands for the total output noise.}
\]
The phase equation (4.9) can be written in the following compact form,
\[
\frac{d\varphi}{dt} = \Omega - K \sin \varphi + \beta N_2(t) \tag{4.11}
\]

wherein, we have dropped the \( \frac{d\Theta_1}{dt} \) term, \( \Theta_1 \) being nearly equal to \(-\frac{\pi}{2}\). And \( \Omega(= \frac{w_0}{2} \frac{3w^2-w_0^2}{5w_0}) \) is the instantaneous frequency detuning between the oscillator and the synchronizing signal frequency.

We have used,
\[
\beta = \frac{w_0^2}{8Q^2} \tag{4.12}
\]

and \( K(=6A^3) \) is, as in earlier calculations, the locking range of the ISO in the presence of \( \text{CW} \) signal only.

To obtain an expression for the locking zone of the ISO we take the time average of (4.11) and equate \( \langle \frac{d\varphi}{dt} \rangle \) to zero in the locked state of ISO. This gives,

\[
\Omega - \langle K \sin \varphi \rangle + \beta \langle N_2(t) \rangle = 0 \tag{4.13}
\]

It can be shown that, if the noise process \( n(t) \) be passed through a predetection filter, having a bandwidth large compared with that of the oscillator, \( N_2(t) \) assumes the character of a nearly white Gaussian noise with zero mean. This leads (4.13) to the form,

\[
\frac{\Omega}{\langle K \rangle} = \langle \sin \varphi \rangle \tag{4.14}
\]

The factor \( \langle K \rangle \) in this equation is related to the average values of the amplitudes \( \langle A \rangle \) and \( \langle B \rangle \) in the presence of the noise process. Their values can be determined by
taking time-average of equations (4.7) and (4.8) and making
some minor approximations. Thus,

$$\langle A \rangle = \frac{E Q}{w w_0} \frac{w w_0}{w^2 - w_0^2} \quad \ldots (4.15)$$

and

$$\langle B^2 \rangle = B_o^2 - 2 \langle A^2 \rangle \quad \ldots (4.16)$$

where, $B_o^2$ is the square of the oscillator free-running
amplitude. Constancy of $\langle A \rangle$ and $\langle B \rangle$ shown in (4.15) and
(4.16) makes the time-average of $K$ to be also constant in
nature. One can then easily write,

$$\frac{Q}{K} = \langle \sin \phi \rangle \quad \ldots (4.17)$$

Now to arrive at a value of $\langle \sin \phi \rangle$, one may reasonably
assume a solution of the phase equation (4.11) to be of the
form,

$$\phi = \phi_0 + \phi_n(t) \quad \ldots (4.18)$$

where $\phi_0$ is the dc component of the instantaneous phase error
caused by the initial frequency detuning and $\phi_n(t)$ represents
the random phase error due to the incoming noise and has a
zero mean. With this value of $\phi$, the average of $\sin \phi$ has the
form,

$$\langle \sin \phi \rangle = \langle \sin(\phi_0 + \phi_n) \rangle = \sin \phi_0 \langle \cos \phi_n \rangle + \cos \phi_0 \langle \sin \phi_n \rangle \quad \ldots (4.19)$$

$\phi_n(t)$ in (4.18) has a probability density function which is
almost Gaussian in nature. This occurs when the input signal to noise power ratio is not too low. In such a case, one may express \( \sin \varphi_n \) and \( \cos \varphi_n \) in the following simple forms, applying quasilinearisation technique (cf. Appendix - 4B),

\[
\sin \varphi_n = \varphi_n \exp(-\frac{\varphi^2}{2}) \quad \text{..(4.20)}
\]

and

\[
\cos \varphi_n = \exp(-\frac{\varphi^2}{2}) \quad \text{..(4.21)}
\]

where \( \varphi^2 \) is the variance of the noise process \( \varphi(t) \).

The average values of \( \sin \varphi_n \) and \( \cos \varphi_n \) then become,

\[
\langle \sin \varphi_n \rangle = 0 \quad \text{..(4.22)}
\]

and

\[
\langle \cos \varphi_n \rangle = \exp(-\frac{\varphi^2}{2}) \quad \text{..(4.23)}
\]

With the help of these relations (4.22) and (4.23), equation (4.19) gives the result,

\[
\langle \sin \varphi \rangle = \sin \varphi_0 \exp(-\frac{\varphi^2}{2}) \quad \text{..(4.24)}
\]

Obviously, the value of \( \varphi_0 \) on the verge of unlocking at the upper-side is given by,

\[
\varphi_0 = \frac{\pi}{2} - \varphi \quad \text{..(4.25)}
\]

and this results in,

\[
\frac{\varphi}{\bar{N}} = \cos \varphi \exp(-\frac{\varphi^2}{2}) \quad \text{..(4.25)}
\]

It is clear, that for evaluation of the locking ratio in this
situation one requires a knowledge of the variance $\sigma_\varphi^2$. Let us now proceed to obtain $\sigma_\varphi^2$ in terms of the signal to noise power ratio ($\rho$). For this, we insert the solution of $\varphi$ as in (4.18) to the phase equation (4.11). This gives the relation,

$$\frac{d\varphi_0}{dt} + \frac{d\varphi_n}{dt} = \Omega - K\sin(\varphi_0 + \varphi_n) + \beta N_2(t)$$

Replacing $\sin\varphi_n$ and $\cos\varphi_n$ by the approximate values of (4.20) and (4.21) and putting $\varphi_0 = \frac{\pi}{2} - \delta_\varphi$, for the upper-side lock point, one obtains by equating the noise parts from both sides of this equation, the relation,

$$\varphi_n(t) = \frac{\beta N_2(t)}{\rho - K\sin \delta_\varphi \exp(-\delta_\varphi^2/2)} \quad \text{(4.27)}$$

where $\rho$ is the Heaviside operator.

With $\varphi_n(t)$ expressed in this way, it is not difficult to write for its variance,

$$\delta_\varphi^2 = \frac{\beta^2}{2\pi} \int_{-\infty}^{\infty} \frac{N_{02} \frac{dp}{\rho - K_1}}{|\rho - K_1|^2} \quad \text{(4.28)}$$

where,

$$K_1 = K\sin \delta_\varphi \exp(-\delta_\varphi^2/2) \quad \text{(4.29)}$$

and $N_{02}$ represents the single sided spectral power density
of the noise process \( N_2(t) \). It is related with \( N_0 \), the corresponding spectral power density of the input noise \( n(t) \) shown in Appendix - 4A/ by the expression,

\[
N_{o2} = A_1 \int \frac{1}{B_1^2 + w^2} - \frac{1}{w^2 + \frac{w_0^2}{Q^2}} + A_2 \int \frac{w_0}{Q(w^2 + 4B_1^2)} - \\
+ \frac{2B_1}{w^2 + \frac{w_0^2}{Q^2}} + \frac{2(B_1 + \frac{w_0}{2Q})}{w^2 + (B_1 + \frac{w_0}{2Q})^2} + \frac{1}{A_3} \int \frac{w^2}{Q^2(w^2 + 9B_1^2)} - \\
- \frac{B_1^2}{w^2 + \frac{9w_0^2}{4Q^2}} - \frac{w_0}{2Q} \frac{(2B_1 + \frac{w_0}{2Q})}{w^2 + (2B_1 + \frac{w_0}{2Q})^2} + \frac{B_1(B_1 + \frac{w_0}{Q})}{w^2 + (B_1 + \frac{w_0}{Q})^2} \\
\ldots (4.30)
\]

where,

\[
A_1 = \left\{12R^2(0) + 9A^4 + 12A^2R(0) \exp(-2\sigma^2)\right\} \frac{N_0w_0^2B_1^2}{8q^2(-\frac{w_0}{4q^2} - B_1^2)} \ldots (4.31)
\]

\[
A_2 = \frac{9}{16} A^2 \frac{N_0^2w_0^2B_1^3}{Q^3(w_0^2 - B_1^2)^2} \ldots (4.32)
\]

\[
A_3 = \frac{9}{64} \frac{N_0^3w_0^3B_1^4}{Q^4(w_0^2 - B_1^2)^3} \ldots (4.33)
\]

\[
R(0) = \frac{N_0w_0B_1}{8Q(B_1 + \frac{w_0}{2Q})} \ldots (4.34)
\]
and $B_1$ represents the single-sided bandwidth of the input noise $n(t)$.

The evaluation of the phase variance now rests on performance of the definite integral in (4.28) using the relations (4.30) to (4.34). It is easy to show that the phase variance is expressed by the following relation,

$$\delta^2 \varphi = \frac{b_1}{\tau^3} + \frac{b_2}{\tau^2} + \frac{b_3}{\tau} \ldots (4.35)$$

where $\frac{1}{\tau}$ indicates the input noise to signal power ratio, i.e.,

$$\frac{1}{\tau} = \frac{B_1N_0}{E^2/2} \ldots (4.36)$$

and the parameters $b_1$, $b_2$ and $b_3$ are given by,

$$b_1 = \frac{64}{243} \frac{Q^2 B_1 w_0^4}{(4Q^2 - B_1^2)^2} \left\{ \frac{1}{w_0^2} \left( \frac{w_0}{B_1 + 2Q} \right)^2 \right\} \left\{ \frac{1}{K_1 B_1 (K_1 + B_1)} \right\}$$

$$- \frac{1}{w_0^2} \frac{K_1 (K_1 + 2Q)}{2Q} \right\} + \frac{1}{\left( \frac{w_0}{4Q} - B_1^2 \right)^2} \right\} \frac{w_0^2}{2K_1 B_1 Q^2 (K_1 + 3B_1)}$$

$$- \frac{4Q B_1^2}{K_1 w_0 (K_1 + 3Q)} \right\} - \frac{3w_0}{QK_1 (K_1 + 2B_1 + \frac{w_0}{2Q})} + \frac{6B_1}{K_1 (K_1 + B_1 + \frac{w_0}{Q})} \right\} \ldots (4.37)$$
\[ b_2 = \frac{16}{27} \frac{QB_1w_0^3K^2}{w_0^2 - B_1^2} \left\{ \frac{1}{K_1B_1(K_1+B_1)} - \frac{1}{w_0 \frac{K_1}{2Q}(K_1+\frac{w_0}{2Q})} \right\} \]

\[ \frac{\exp(-2\delta_\phi^2)}{w_o^2} + \frac{3}{w_o^2} \left\{ \frac{2QB_1}{w_o K_1(K_1+\frac{w_0}{2Q})} + \frac{2}{K_1(K_1+B_1+\frac{w_0}{2Q})} \right\} \]

\[ b_3 = \frac{w_o^2B_1}{w_o^2 - B_1^2} \left\{ \frac{1}{K_1B_1(K_1+B_1)} - \frac{1}{w_o \frac{K_1}{2Q}(K_1+\frac{w_0}{2Q})} \right\} \]

To find out the nature of variation of the phase variance \((\delta_\phi^2)\) with that of the noise-to-signal power ratio \((i.e., \frac{1}{Q})\) we keep the Q-value of the oscillator, the locking band of the oscillator \((i.e., K/y_0)\) and the input noise band \((i.e., B_1/K)\) fixed. Then for a certain value of the parameter \(\frac{1}{Q}\), the equation (4.33) is solved to furnish the value of phase variance. The curve showing this variation is illustrated in Fig.4.1.

Determination of the locking ratio \((Q/K)\) for a particular value of the noise-to-signal power ratio \((\frac{1}{Q})\) is now
Fig. 4.1. The plot of phase-variance with the noise-to-signal power ratio.
quite easy. Using the relationship in equation (4.25) and the known variation of \( \phi^2 \) vs. \( \frac{1}{f} \) we have obtained Fig. 4-2 depicting the variation of locking range with the noise-to-signal strength ratio.

4.3. Ultraharmonic Oscillator Compared to A Frequency-Multiplier:

To find out the variance of the output phase \( \Theta_3 \) of an ultraharmonic oscillator we start with the phase equation (4.11). In order to simplify the analysis we consider the in-tuned case (i.e., \( \Omega = 0 \)), assume \( \sin \varphi = \varphi \) and neglect the \( \frac{d\Theta}{dt} \) term. This leads to the following expression for the variance of the oscillator phase, viz.,

\[
\sigma^2_{\Theta_3} = \frac{\sigma^2}{4\pi f} \int_{-\infty}^{\infty} \frac{N_{02}}{|p+K|^2} dp \quad \ldots (4.40)
\]

where \( N_{02} \) stands for the single sided spectral power density of the noise process \( N_2(t) \) and as has been already shown, is expressed by equation (4.30).

On substituting the value of \( N_{02} \) in (4.40), we arrive at a relation between the variance \( \sigma^2_{\Theta_3} \) and the signal-to-noise power ratio \( \rho \), given by,

\[
\sigma^2_{\Theta_3} = c_1 \frac{1}{\rho^3} + c_2 \frac{1}{\rho^2} + c_3 \frac{1}{\rho} \quad \ldots (4.41)
\]
Fig. 4.2. The variation of locking-ratio with the noise-to-signal power ratio.

Noise-to-signal power ratio (l/p)

Locking-ratio (Q/K)
where,

\[
C_1 = \frac{64}{243} \frac{Q^2 B_1 w_0^4 K}{w_0^2 \left( \frac{C_0}{4Q^2} - B_1^2 \right)} - \frac{1}{w_0 \left( \frac{C_0}{2Q} + B_1 \right)^2} \left\{ \frac{1}{B_1 (K+B_1)} \right. \\
- \frac{4Q^2}{w_0 (2KQ+w_0)} \right\} + \frac{1}{w_0^2 \left( \frac{C_0}{4Q^2} - B_1^2 \right)^2} + \frac{2B_1 Q^2 (K+2B_1)}{2B_1 Q^2 (K+3B_1)} \\
- \frac{3w_0^2 B_1^2}{w_0 (2KQ+3w_0)} + \frac{6w_0}{(2KQ+4B_1 Q+w_0)} + \frac{6B_1 Q}{(KQ+B_1 Q+w_0)} \right\} \right)
\]

\[\ldots(4.42)\]

\[
C_2 = \frac{16}{27} \frac{QB_1 w_0^3 K}{w_0^2 \left( \frac{C_0}{4Q^2} - B_1^2 \right)} - \frac{1}{B_1 (K+B_1)} - \frac{4Q^2}{w_0 (2KQ+w_0)} \right\}
\]

\[
\exp(-2 \phi^2) \frac{w_0}{\left( \frac{C_0}{2Q} + B_1 \right)} \right\} + \frac{3}{w_0^2 \left( \frac{C_0}{4Q^2} - B_1^2 \right)} \left\{ \frac{w_0}{2QB_1 (K+2B_1)} + \frac{2B_1 Q^2 B_1}{w_0 (KQ+w_0)} + \frac{4Q}{(2KQ+2B_1 Q+w_0)} \right\} \right)
\]

\[\ldots(4.43)\]

and

\[
C_3 = \frac{w_0^2 B_1 K}{w_0^2 \left( \frac{C_0}{4Q^2} - B_1^2 \right)} \left\{ \frac{1}{B_1 (K+B_1)} - \frac{4Q^2}{w_0 (2KQ+w_0)} \right\}
\]

\[\ldots(4.44)\]
Now for a multiplier with the same noisy input as assumed for ultraharmonic locking, the phase-variance can be written as,

\[ \sigma_M^2 = n_1^2 \frac{N_0 B_1}{E^2} \]  

...(4.45)

where \( n_1 \) denotes the order of frequency multiplication. To compare with the case of ultraharmonic synchronization of order three, we are to put \( n_1 = 3 \) in (4.45). This gives,

\[ \sigma_M^2 = \frac{9N_0 B_1}{E^2} = 4.5 \]  

...(4.46)

With the help of equations (4.41) and (4.46) we can easily compare the variation of output phase variance with the noise-to-signal power ratio for an ultraharmonic oscillator and a frequency multiplier, both of order three. This has been shown in Fig.4.3.

4.4. Experiment:

The experimental set up for the determination of locking range of an ISO with varying strengths of noise compared to that of a CW signal, in the case of ultraharmonic synchronization, was not much different from the others used for pure CW signal or a CW signal associated with interference. The oscillator built up for this purpose was an a-f. LC oscillator shown in Fig.4.4. The transistor used was AF 116, as in
Fig. 4.3. Comparison of variation of phase-variance with the noise-to-signal power ratio for an ultraharmonic oscillator and a frequency multiplier.
Fig. 4.4. The arrangement of the experimental set-up.
previous cases. The cubic type nonlinearity of the limiter was achieved by connection of two back-to-back point contact diodes in conjunction with the tank circuit of the oscillator. The oscillator had a frequency of about 5 KHz and was of high frequency stability. The use of a noise generator marks the difference of the present experimental structure from the previous ones. To generate noise, a break down voltage (-120 volts) was switched on to the collector junction of the transistor AF 116. Subsequent amplifying stages, made of OPAMP 741, amplified the noise voltage to the required level. The noise generator circuit arrangement is shown in Fig-4.5. The bandwidth of the noise process was more than 10 KHz and can be easily assumed to be a white noise for an oscillator having a baseband bandwidth as low as 100 Hz for ultraharmonic synchronization. For the realization of the noise process as in equation (4.1), an arrangement of two mixers and adders was used as shown in the block diagram of Fig-4.6. The mixers formed with the help of IC LM 1496 were sufficiently widebanded to accommodate the frequency band of the noise process. The output of Adder-1 of Fig-4.6 was found to attain a Gaussian type probabilistic character around the centre frequency of the CW signal.

To carry out the experiment on the capture capability of the injection locked ultraharmonic class-A oscillator in
Fig. 4.5. Circuit configuration of a noise-generator.
Fig. 4.6. Realization of the noise process as assumed in the theory.
response to a noisy CW signal, the strength of the CW signal was fixed at a definite level. The locking of the ISO was then achieved and the locking band measured. Now the noise part of the signal was applied in steps and the consequent changes in locking ranges were noted. Finally, a graph was plotted with the normalised locking band of the ISO on one side and the noise to signal power ratio on the other. The corresponding theoretical results were also indicated on the same plot. These are depicted in Fig. 4.2.

4.5. Conclusions:

A mathematical framework has been developed in this chapter to analyse the third order mode of ultraharmonic synchronization in the face of a wideband noisy CW signal. The principle of quasilinearization technique has been adopted in the deduction of the theory. Correctness of the theoretical predictions arising out of this analysis depends on two assumptions. First, the input-output phase error of the oscillator has been assumed to be a Gaussian variable and secondly, the noise process appearing in the oscillator-phase equation has been assumed to be Gaussian in nature. Both of these assumptions are only approximate in nature.

Theoretical and experimental findings in this regard are:

1) The locking range for ultraharmonic synchronization of an ISO falls with the increase of the noise-to-signal power ratio.
ii) There is a fair agreement between the theoretical and experimental findings in respect of reduction of locking bandwidth with the noise-to-signal power ratio (Fig. 4.2). It has, however, been observed that after a certain value of the noise-to-signal power ratio the locking range diminishes at a faster rate compared to the theoretical one. This is due to the occurrence of phase jumping phenomenon in an oscillator. The rate of cycle slipping increases with the increase of the noise power and this is equivalent to the generation of extra noise - not considered in the present theory.

iii) It is clear from Fig. 4.3 that up to an appreciable range of the noise-to-signal power ratio, the phase noise variance of an ultraharmonic oscillator remains lower than that of a frequency multiplier, both being of the same order. This suggests that an ultraharmonically synchronized oscillator can be used in some applications of frequency multiplication.
Determination of the Spectral Power Density of the Net Output Noise $N_2(t)$ in Terms of that of the Input Noise $n(t)$:

The noise term $N_2(t)$ appearing in the phase equation (4.11) is related to the output noise $n'(t)$ of the oscillator by the equation,

$$
N_2(t) = n_c^3 \cos \varphi + n_s^3 \sin \varphi - 3n_c' n_s' (n_c' \cos \varphi + n_s' \sin \varphi) + 3A(n_c^2 - n_s^2) \sin \varphi - 3A^2 (n_c' \cos \varphi + n_s' \sin \varphi) + 6A n_c' n_s' \cos \varphi
$$

To obtain the spectral power density of a noise process the determination of its autocorrelation function is necessary. Following the method of Lindsay and Simon [IV.2], we calculate the autocorrelation function $R_2(\tau)$ of the noise process $N_2(t)$ in terms of the autocorrelation function $R(\tau)$ of $n_c'(t)$ (or $n_s'(t)$) and obtain,

$$
R_2(\tau) = \frac{N_2(t) N_2(t+\tau)}{N_2^2}
$$

$$
= \int 2R^2(0) + 9A^2 + 12A^2R(0) \exp(-2 \delta \varphi) R(\tau) + 36A^2R^2(\tau) + 24R^3(\tau)
$$

In order to relate $R(\tau)$ with the input noise ($n(t)$) parameters, we first equate the noise parts from the oscillator loop-equation (2.41) which results in,
\[ N(e_0)_n + (e_i)_n = \frac{1}{Y(p)} (e_0)_n \quad \ldots (4A-3) \]

the subscript \( n \) denoting the corresponding noise terms. If we can assume that \( N(e_0)_n \ll (e_i)_n \), which is not an unreasonable assumption, we get from \( (4A-3) \) the relation,

\[ (e_i)_n = \frac{1}{Y(p)} (e_0)_n \quad \ldots (4A-4) \]

Let us assume the input noise spectrum to be given by,

\[ S(w) = \frac{N_0}{1+(\frac{w}{B_1})^2} \quad \ldots (4A-5) \]

where \( N_0 \) represents the one-sided spectral power density (expressed in Watts/Hz) and \( B_1 \) the single-sided angular bandwidth of the noise \( n(t) \).

Expressing the low-pass version of the oscillator tuned circuit as,

\[ Y_{Lp}(p) \mid^2 = \frac{(w_o/2Q)^2}{w^2 + (\frac{w_o}{2Q})^2} \quad \ldots (4A-6) \]

one can easily write for \( R(\tau) \) the relation given by,

\[ R(\tau) = \frac{N_0}{2\pi} \int_{0}^{\infty} \frac{1}{1+(\frac{w}{B_1})^2} \frac{(w_o/2Q)^2}{w^2 + (\frac{w_o}{2Q})^2} \cos \omega \tau \, dw \quad \ldots (4A-7) \]

where we have made use of \( (4A-4) \), \( (4A-5) \) and \( (4A-6) \).

The R.H.S. of \( (4A-7) \) may be written as,
\[ \frac{N_0}{2\pi} \frac{1}{w_1} \int_0^{\infty} \left( \frac{w_0}{2Q} \right)^2 \frac{1}{1+(\frac{w}{B})^2} - \frac{R_1^2}{1+(\frac{w}{W})^2} \cos \theta \, dw \]

Now on performing the integration we obtain,

\[ R(\tau) = \frac{N_0}{4} \frac{w_0 - B_1}{w_1} \int \left( \frac{w_0}{2Q} \right)^2 e^{-B_1 \tau} \left( \frac{w_0}{2Q} \right)^2 e^{-B_1 \tau} \]  

\( \ldots (4A-8) \)

It is easy to write \( R(0) \) from (4A-8) and is given by,

\[ R(0) = \frac{N_0 w_0 B_1}{8Q(w_0 + B_1)} \]  

\( \ldots (4A-9) \)

which is actually the noise power of the oscillator output noise \( n'(t) \).

Now when the autocorrelation function \( R(\tau) \) has been obtained in terms of the input noise characteristics \( N_0 \) and \( B_1 \), it is easy to find out the spectral power density \( N_{o2} \) of the noise process \( N_2(t) \) by taking the Fourier transform of \( R_2(\tau) \). Thus,

\[ N_{o2} = \int_{-\infty}^{+\infty} R_2(\tau) e^{-j\omega \tau} \, d\tau \]  

\( \ldots (4A-10) \)

Using the relations (4A-2) and (4A-9) in (4A-10) one arrives at the following expression for \( N_{o2} \),
\[ W_{02} = A_1 \left( \frac{1}{B_1 + w^2} - \frac{1}{w_0^2} \right) + A_2 \left( \frac{w_0}{Q(w^2 + 4B_1^2)} + \frac{2B_1}{w_0} \right) \]

\[ + \frac{2(B_1 + \frac{w_0}{2Q})}{w^2 + (B_1 + \frac{w_0}{2Q})^2} J^A + A_3 \left( \frac{1}{Q^2(w^2 + 9B_1^2)} \right) - \frac{B_1^2}{w^2 + (\frac{2B_1 + w_0}{2Q})^2} \]

\[ - \frac{w_0}{2Q} \left( \frac{2B_1 + w_0}{2Q} \right) + \frac{B_1(B_1 + \frac{w_0}{Q})}{w^2 + (B_1 + \frac{w_0}{Q})^2} \]

\[ \left( \frac{\frac{N_0 w^2 B_1}{8Q (\frac{w_0}{4Q} - B_1^2)} \right) \]  

where,

\[ A_1 = \left\{ 12R^2(0) + 9A^4 + 12A^2 R(0) \exp(-2 \sigma^2) \right\} \]

\[ A_2 = \frac{9}{16} A^2 \frac{N_0^2 w B_1^3}{Q^2 (\frac{w_0}{4Q} - B_1^2)^2} \]

\[ A_3 = \frac{9}{64} \frac{N_0^3 w B_1^4}{Q^4 (\frac{w_0}{4Q} - B_1^2)^3} \]
Approximation of $\sin \varphi_n$ and $\cos \varphi_n$ by Applying Statistical Linearization Technique:

The principle of statistical linearization method suggests that any nonlinear function $f(x)$, without memory, can be replaced by some linear function of the form,

$$f(x) = c_1 + c_2 x \quad \ldots \quad (4B-1)$$

where, $c_1$ and $c_2$ are given by,

$$c_1 = \left. \frac{\int_{-\infty}^{+\infty} f(x)p(x)dx}{\int_{-\infty}^{+\infty} p(x)dx} \right. \ldots \quad (4B-2)$$

and

$$c_2 = \left. \frac{-\int_{-\infty}^{+\infty} xf(x)p(x)dx}{\int_{-\infty}^{+\infty} x^2p(x)dx} \right. \ldots \quad (4B-3)$$

$p(x)$ denoting the probability density distribution function of the variable 'X'.

Using the same logic, let us write,

$$\sin \varphi_n = c_1 + c_2 \varphi_n \quad \ldots \quad (4B-4)$$

with,

$$c_1 = \left. \frac{\int_{-\infty}^{+\infty} \sin \varphi_n p(\varphi_n) d\varphi_n}{\int_{-\infty}^{+\infty} p(\varphi_n) d\varphi_n} \right. \ldots \quad (4B-5)$$

and

$$c_2 = \left. \frac{-\int_{-\infty}^{+\infty} \varphi_n \sin \varphi_n p(\varphi_n) d\varphi_n}{\int_{-\infty}^{+\infty} \varphi_n^2 p(\varphi_n) d\varphi_n} \right. \ldots \quad (4B-6)$$

Assuming $\varphi_n$ to be a Gaussian process with variance $\sigma_\varphi^2$, $p(\varphi_n)$ has the expression,
\[ p(\phi_n) = \frac{1}{\sqrt{2\pi} \sigma_{\phi}^2} \exp\left(-\frac{\phi_n^2}{2\sigma_{\phi}^2}\right) \quad \ldots \quad (4B-7) \]

With this value of \( p(\phi_n) \), one can easily find,

\[ C_1 = 0 \]

and

\[ C_2 = \exp(-\sigma_{\phi}^2/2) \quad \ldots \quad (4B-8) \]

Substitution of (4B-8) in (4B-4) expresses \( \sin \phi_n \) in the form,

\[ \sin \phi_n = \phi_n \exp(-\sigma_{\phi}^2/2) \quad \ldots \quad (4B-9) \]

Proceeding exactly in this way, one arrives at the approximate value of \( \cos \phi_n \), given by,

\[ \cos \phi_n = \exp(-\sigma_{\phi}^2/2) \quad \ldots \quad (4B-10) \]
REFERENCES


IV-8 K Pramanik, 'Studies on Unilaterally and Bilaterally Coupled Oscillators', Ph D Dissertation, Burdwan University, 1981.