CHAPTER-II

ASPECTS OF ULTRAHARMONIC SYNCHRONIZATION IN
CLASS-A OSCILLATORS

2.1. Introduction:

Since the innovation of nonlinear oscillations, an appreciable amount of work II-1, II-2, II-3, II-4, II-5, II-6 has been performed during the last three decades on the effect of forcing an electronic oscillator by a periodic signal. Synchronization of an oscillator, under certain favourable conditions, occurs not only when the forcing signal is close to the oscillating free-running frequency but also for frequencies multiple or submultiple of that of the free-running oscillator. Ultraharmonic entrainment of an oscillator takes place when, under suitable conditions, the instantaneous phase or frequency of the oscillator is entrained by a forcing signal having frequencies submultiple to that of the oscillator. Sometimes this mode of synchronization is designated as subharmonic entrainment.

The phenomenon of ultraharmonic synchronization has been investigated by many authors. Compared to the same frequency synchronization, it has received much less attention. Scientists and engineers have discussed the problem mainly on the basis of the van der Pol theory. Papers by Mag II-7, Hayashi II-8 et al are good examples of a purely mathematical exercise of the problem. Diakoku and Mizushima II-9 have analysed
different orders of synchronization with the help of a theory based on a simple model of a parallel equivalent circuit of the oscillator which includes the nonlinear negative conductance and nonlinear susceptance. Recently, Gustafsson et al. [11-10] applied the technique of describing function quite successfully in the study of synchronization phenomenon in microwave oscillators. The first attempt to throw some light on the physical picture of the locking mechanism of fundamental, subharmonic and ultraharmonic synchronization was possibly made by M T Jezewski [11-11]. However, the simplified analysis of that work resulted in agreement between the predicted theory and experimental measurement only for higher circuit Q-factors.

The present author feels that the interest of these workers were centered mainly in achieving the solution of nonlinear differential equations describing the system behaviour without going into the actual details of the physics of the phenomenon of ultraharmonic synchronization. This resulted in lack of perfect agreement between the theoretical and experimental data as can be observed in the works of Hayashi, Jezewski etc. The present author, on the other hand, has tried to build up a theory, based completely on the physical reasonings of the problem. Fortunately, the analytical approach developed gives a fairly good agreement with the experimental findings.
The phenomenon of entrainment of oscillators has a lot of applications in the different fields of communication. Injection locked oscillators utilizing harmonic synchronization find applications for the detection of AM, FM and PM signals, for generating the reference carrier in communication systems, for reducing the depth of modulation of AM signals etc. [II-12]. Injection locking in the harmonic mode is also considered to be one of the effective methods of stabilizing the frequency of solid state oscillators. In fact, microwave or millimeter-wave oscillators utilizing semiconductor bulk effects and the avalanche transit time of carriers create high-level noise, producing instability in the operation of communication systems which require high stability. Injection locking has been proposed to overcome such difficulties with which the frequency is stabilized by externally applied signals [II-13, II-14, II-15]. The field of application of ultraharmonic synchronization is also broad. One of the most important uses of an ultraharmonically locked oscillator is in frequency multiplication [II-16]. An ultraharmonically locked oscillator also finds applications in FM or PM demodulation, AM/PM conversion etc. [II-17]. Different applications of oscillators locked in ultraharmonic mode have been described in details in a separate Chapter of the thesis.
2.2. Physics of the Phenomenon:

To develop a physical picture in the case of ultraharmonic synchronization, let us first see what happens in the comparatively easier case of harmonic synchronization of an oscillator. Let us consider Fig. 2.1 which is an equivalent functional representation of an oscillator consisting of a linear amplifier and a nonlinear feedback device. If a CW signal of frequency \( w \) is injected externally into the oscillator of frequency \( w_0 \), beats will be observed at the output of the oscillator. The variation of the beat frequency with the frequency of the injecting signal is as shown in Fig. 2.2. It is seen that the beats disappear suddenly and only the external frequency \( w \) prevails, i.e., the free-running frequency \( w_0 \) is entrained by the external frequency \( w \). If \( w \) is varied beyond a certain limit, the oscillator will reach an unlocked state and the beats reappear. The zone of frequency \( (w_1 - w_f) \) over which entrainment occurs is called the synchronization range of the oscillator. Referring to Fig. 2.2 one observes that \( (w_f - w_0) \) is greater than \( (w_0 - w_1) \) and this is called the asymmetric locking characteristics of the oscillator. A simple explanation of this matter may be obtained by considering the mixing process provided by the inherent nonlinearity of the oscillator circuit. Outside the synchronization band one may assume that the input to the nonlinear element \( N(e_0) \) consists of the oscillator output \( E_0 \cos(wt + \phi(t)) \) and the external signal \( E_1 \cos(wt + \phi_1(t)) \);
Fig. 2.1. An equivalent analytical representation of an injection synchronized class-A oscillator.
Fig. 2.2. Variation of beat-frequency of an IS0 with the synchronizing signal frequency.
the phase difference between the two signals being $\phi$. If the nonlinear element of the oscillator is assumed to be a cubic type and given by $N(e_0) = a_1 e_0 - a_3 e_0^3$, $a_1$ and $a_3$ being two constants of nonlinearity then the equivalent gain parameter of $N(e_0)$ outside the locking range has an inphase component and a quadrature component expressed by,

Inphase component $= a_1 - \frac{3}{4} a_3 (E_0^2 + 2E_1^2) - \frac{3}{4} a_3 E_0 E_1 \cos \phi$  \hspace{1cm} (2.1)

Quadrature component $= \frac{3}{4} a_3 E_0 E_1 \sin \phi$  \hspace{1cm} (2.2)

The phase difference $\phi$ for a single tuned oscillator, is given by,

$$\phi = \arcsin \frac{E_0}{E_1} Q \left( \frac{w}{w_0} - \frac{w_1}{w} \right)$$  \hspace{1cm} (2.3)

where $Q$ is the quality factor of the oscillator tank circuit.

The inphase component modulates the amplitude and the quadrature component controls the frequency of the oscillator.

Now if the signal frequency is not too distant from the oscillator frequency and the signal strength is adequate it is expected that this phase difference attains a steady state value $\phi_s$ and the oscillator falls in synchronism with the external signal. Again, the maximum and minimum permissible values of the phase difference being $\pi/2$ and $-\pi/2$ respectively, relation (2.3) above indicates that the frequency difference $(w_0 - w_1)$ for which $\phi_s$ becomes $-\pi/2$ on the lower side of the
oscillator frequency \( (w_0) \) is definitely lower than what is necessary to make \( \phi_s = \pi/2 \) on the upper side \( (i.e., w_f - w_0) \). This provides a clear explanation of the asymmetric locking characteristics of the oscillator.

2.2.1. Ultraharmonic Synchronization:

With the background of the idea introduced in the case of harmonic entrainment let us now consider the phenomenon of ultraharmonic entrainment of oscillators. It has been well established that for achieving locking in the ultraharmonic mode the strength of the synchronizing signal is required to be large compared to what is needed for harmonic synchronization. Under the influence of such a strong signal power, the inherent nonlinear element of the oscillator circuit spontaneously generates ultraharmonics of the signal frequency. It is quite likely that depending upon the type of the nonlinearity and the signal frequency, one of the generated ultraharmonics will have a frequency close to that of the local oscillator. If the conditions are favourable, this generated signal will initiate locking phenomenon in the oscillator in the ultraharmonic mode.

Extension of the ideas developed for explaining the harmonic entrainment of an oscillator, i.e., introducing the existence of inphase and quadrature components of the
equivalent gain, can lead to a clarification of the physics involved in the case of ultraharmonic synchronization, but with a slight difference. Unlike in the case of harmonic synchronization, here, a component at the frequency of the forcing signal (which is quite apart from the oscillator free running frequency) is always present at the input to the oscillator and this exerts a pull on the frequency of the local oscillator. This pull results in a shift of the oscillator frequency in the direction of the forcing function. The shift in the frequency of the oscillator depends, in general, on the following three factors: (i) the difference of frequency between the oscillator and the external input, (ii) the selectivity of the oscillator tank circuit and (iii) the relative amplitudes of the oscillator and the synchronizing signal. As a result, the centre frequency of the oscillator for ultraharmonic case is shifted from $w_0$ to some other value, say, $w'_o$. Then the equivalent inphase and quadrature components of the nonlinear gain for this case may be represented by exactly the same relations (2.1) and (2.2), keeping in mind, that here $E_1$ corresponds to the generated ultraharmonic component and the phase difference $\phi$ is given by,

$$
\phi = \arcsin \frac{E_0}{E_1} Q \left( \frac{nw}{w_0} - \frac{w'_o}{nw} \right)
$$

Here $n$ is an integer, that determines the order of ultraharmonic
entrainment. A look at relation (2.4) suggests that around the shifted oscillator frequency \( w'_0 \), the lower side locking range of the oscillator will be smaller than the upper side one. But the entire fact about ultraharmonic synchronization will not be revealed with mere consideration of this relation (2.4). To appreciate fully the reasons behind the statement we have to refer to the spectral representation of the phenomenon of ultraharmonic locking as depicted in Fig. 2.3(a) and Fig. 2.3(b). In the first figure, the generated ultraharmonic component lies on the lower side of the shifted frequency \( w'_0 \) of the oscillator. Here the pulls on the oscillator frequency exerted by the forcing signal and the created ultraharmonic are in the same direction. Obviously, this has an effect of increasing the locking range on the lower side of oscillator frequency. The situation is just the reverse in Fig. 2.3(b). The location of the generated ultraharmonic being on the upper side of the shifted oscillator frequency, the pulls due to the forcing signal and the ultraharmonic component are in the opposite direction. This explains the possibility of upper side locking range to be smaller than that of the lower side in the case of ultraharmonic synchronization. This possibility of asymmetric locking ranges on the two sides of the oscillator frequency is just opposite to that in the case of harmonic synchronization. If one proceeds in exactly the same way to deal with the case of subharmonic entrainment one arrives at a conclusion which is in line with the observations.
Fig. 2.3. Spectral representation of the phenomenon of ultraharmonic entrainment with
(a) Generated ultraharmonic component on the lower-side of oscillator frequency ($w'_0$).
(b) Generated ultraharmonic component on the upper-side of oscillator frequency ($w'_0$).
for same frequency synchronization. It may appear at the first instance, that the phenomenon of ultraharmonic synchronization is basically the same as that of subharmonic synchronization. But the latter is different in some fundamental respects from the former. Firstly, the amplitude of the equivalent synchronizing signal for subharmonic case is not only dependent on the amplitude of the forcing signal but also on the strength of the oscillator output, which in turn is a function of amplitude of the injected input. Secondly, the injected synchronizing signal lies far off from the oscillator frequency compared to the case of ultraharmonic synchronization. This makes the pull due to the incoming signal weaker and as a result the shift in the oscillator frequency becomes smaller than that in the case of ultraharmonic synchronization.

In fact, there is close similarity in the behaviour of ultraharmonic synchronization and harmonic synchronization. This is because in both these cases the equivalent synchronizing amplitude is proportional to the forcing function only. Actually, the pull of the oscillator free-running frequency in the direction of the forcing signal results in an asymmetry of locking ranges for ultraharmonic synchronization, which is just contrary to that observed in the case of harmonic synchronization. Had it been possible to synchronize an oscillator in the ultraharmonic mode with a very small
amplitude signal, the phenomenon of ultraharmonic entrainment would have been identical to that of fundamental frequency entrainment. Incidentally, it may be mentioned that the subharmonic mode of synchronization is not at all similar to the harmonic mode of synchronization.

2.3. Analytical Representation of Class-A Oscillator:

To build up a sound mathematical framework for studying ultraharmonic synchronization, let us first develop an analytical representation of a Class-A oscillator. An oscillator, in general, is a regenerative feedback device that derives its input from its own output. Negative resistance oscillators and feedback oscillators utilizing the mechanisms of internal feedback and external feedback respectively form the two broad classifications of oscillators. Whatever be the type of oscillator, it is nothing but a positive feedback amplifier, the feedback factor being equal to or greater than unity. Two things are essential in an oscillator, one is a frequency selective network and the other is a limiter type nonlinearity needed for checking the growth of oscillation to a certain limit. Whether an oscillator will be called a Class-A type or a Class-C type is determined by the form of limiting, it is provided with. When the limiting process is instantaneous, it indicates Class-A oscillators. And the non-instantaneous limiting characterizes Class-C oscillators.
We however, restrict our discussions to Class-A oscillators only in this Chapter.

Let us start our analysis with the most common type of tube oscillator shown in Fig. 2.4. Here $i_p$, the total current is the sum of $i_L$ and $i_C$, the instantaneous values of the currents through the inductive branch and the capacitive branch of the tank circuit of the oscillator. Thus,

$$i_p = i_L + i_C$$

If $V_P$ and $V_G$ indicate the total plate voltage and total grid voltage respectively (including dc and ac parts) and $\mu$ be the amplification factor of the tube, then one can write,

$$i_p = f\left(\frac{V_P}{\mu} + V_G\right)$$

where $f$ denotes a function.

Putting $V_{PP}$ and $V_{GG}$ for the dc voltages in the plate and grid circuits respectively and referring to Fig. 2.4, one obtains the relations,

$$V_P = V_{PP} - L \frac{di_L}{dt} - R_i L$$

$$V_G = V_{GG} + M \frac{di_L}{dt}$$

and

$$\frac{1}{C} \int i_C dt = L \frac{di_L}{dt} + R_i L$$

where $M$ is the mutual inductance of the coils.

Consideration of equations (2.5) and (2.9) gives,

$$i_p = LC \frac{d^2 i_L}{dt^2} + RC \frac{di_L}{dt} + i_L$$
Fig. 2.4. Circuit diagram of a class-A tube oscillator.
From equations \((2.7)\), \((2.8)\) and \((2.10)\) the following differential equation in \(v_G\) results:

\[
\frac{d^2v_G}{dt^2} + \frac{R}{L} \frac{dv_G}{dt} + w_R^2 (v_G - v_{GG}) = Mw_R^2 \frac{di_p}{dt} \tag{2.11}
\]

where \(w_R (= \frac{1}{\sqrt{LC}})\) is the resonant frequency of the tank circuit of the oscillator.

If we consider only the varying component of the grid voltage \((v_G)\), equation \((2.11)\) reduces to the following relation,

\[
\frac{d^2v_G}{dt^2} + \frac{R}{L} \frac{dv_G}{dt} + w_R^2 v_G = Mw_R^2 \frac{di_p}{dt} \tag{2.12}
\]

Let us now express \(i_p\) in a different form. To do this, we express the factor \(\frac{v_p}{u} + v_G\) in equation \((2.6)\) in the following form:

\[
\frac{v_p}{u} + v_G = \frac{v_{pp} - I_{L} R}{u} - \frac{1}{u} (i_{L} R + \frac{L}{M} v_G) + (v_{GG} + v_G) \tag{2.13}
\]

where we have made use of the relations \((2.7)\) and \((2.8)\) and the current \(i_L\) through the inductance has been written as the sum of a direct current \(I_L\) and an alternating component \(i_L\). If we substitute,

\[
V_o = \frac{v_{pp} - I_{L} R}{u} + v_{GG}, \text{ which actually defines the position of the operating point, then } i_p \text{ can be written as (from } (2.6) \text{ and } (2.13))
\]

\[
i_p = f \int V_o - \left( \frac{i_{L} R}{u} + \frac{L v_G}{u M} \right) + v_G \tag{2.14}
\]
The factor $R$, which is the resistance associated with the inductive element, is actually small so that $i_p$ can be written as the following Taylor's series expansion:

$$i_p = f(V_o + \frac{\mu M - L}{\mu M} v_g) - \frac{R}{\mu} i \frac{\partial f}{\partial v_g}$$

(2.15)

The factor $\frac{1}{\mu} \frac{\partial f}{\partial v_g}$ in the above equation represents inverse of $r_p$, the ac plate resistance of the tube since $\frac{\partial f}{\partial v_g}$ is the ac transconductance of the tube at the operating point $(V_o)$. Now using this value of $i_p$ in equation (2.12) and noting that $M \frac{dv}{dt} = v_g$, one can write,

$$\frac{d^2 v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \omega_R^2 (1 + \frac{R}{r_p}) v_g = \frac{M}{R} \frac{df}{dt} (V_o + \frac{\mu M - L}{\mu M} v_g)$$

(2.16)

Equation (2.16) establishes a relation between the oscillator free running frequency ($w_o$) and the resonant frequency of the tank circuit ($w_R$) viz.,

$$w_o = \omega_R (1 + \frac{R}{r_p})^{\frac{1}{2}}$$

(2.17)

If we replace, for the time being, $w_R$ by $w_o$, (the factor $R/r_p$ being very small) and put,

$$Q = \frac{1}{w_o CR}$$

(2.18)

and

$$F(v_g) = \frac{M}{CR} f(V_o + \frac{\mu M - L}{\mu M} v_g)$$

(2.18b)

then the equation (2.16) assumes the form:
\[
\frac{d^2 v}{dt^2} + \frac{w_0}{q} \left( \frac{dv}{dt} - \frac{df(v)}{dt} \right) + w_0^2 v = 0 \tag{2.19}
\]

Let us now introduce the variable \( I \), related to \( v_g \), by the expression,

\[
I = \frac{nN-I}{nN} v_g \tag{2.20}
\]

and substitute,

\[
N(I) = \frac{nN-I}{nC} f(v_0 + I) \tag{2.21}
\]

Equation (2.19) then looks like,

\[
p^2 I + \frac{w_0}{q} p(I-N(I)) + w_0^2 I = 0 \tag{2.22}
\]

where \( p(= \frac{d}{dt}) \) stands for the Heaviside operator.

Equation (2.22) can be expressed in a more compact form, viz.,

\[
I = Y(p) N(I) \tag{2.23}
\]

where,

\[
\frac{1}{X(p)} = 1 + Q \left( \frac{w_0}{p} + \frac{p}{w_0} \right) \tag{2.24}
\]

and is the transfer function of the frequency selective amplifier. The functional relation (2.23) may be represented block diagrammatically as in Fig. 2.5. The figure actually represents a nonlinear positive feedback device and stands for an equivalent analytical representation of a class-A oscillator.

To make the oscillator analysis more general, let us now develop an equation for a transistorized oscillator which will be exactly like that of equation (2.22), derived for a vacuum
Fig. 2.5. Equivalent analytical configuration of a class-A oscillator.
tube oscillator. The collector current \( i_C \) in the transistor oscillator, shown in Fig. 2.6 may be easily written as,

\[
\frac{d^2 i_C}{dt^2} + \frac{RC}{L} \frac{di_C}{dt} + i_L = 0 \quad \ldots (2.25)
\]

If, as in the earlier case, we denote by \( V_{BB}, V_B \) and \( v_b \) the base to emitter DC supply, the total instantaneous voltage drop across the base and emitter (including ac and dc parts) and the ac part of \( V_B \) respectively, we can write,

\[
V_{BB} - V_B = M \frac{di_L}{dt} - L \frac{di_B}{dt} \quad \ldots (2.26)
\]

\( i_B \) denoting the base current in the circuit and \( M \) being the mutual inductance of the coils.

If we assume \( L \ll M \), we may put,

\[
V_{BB} - V_B = M \frac{di_L}{dt} \quad \ldots (2.27)
\]

The LHS of this equation is again \(-v_b\).

Using equations (2.25) and (2.27) we can arrive at the following differential equation for \( v_b \),

\[
\frac{d^2 v_b}{dt^2} + \frac{R}{L} \frac{dv_b}{dt} + \frac{1}{LC} v_b = -\frac{M}{LC} \frac{di_C}{dt} \quad \ldots (2.28)
\]

The collector current \( i_C \) is related to the emitter current \( i_E \) by the forward current transfer ratio \( \alpha \) and \( i_E \), in turn, has a functional relation with the base voltage.

Thus,

\[
i_E = f(V_{BB} + v_b) \quad \ldots (2.29)
\]
Fig. 2.6. Circuit diagram of a class-A transistor oscillator.
Equation (2.28) automatically reduces to the form,

\[
\frac{d^2v_b}{dt^2} + \frac{w_0}{Q} \int \frac{dv_b}{dt} - \frac{aM}{RC} \frac{d}{dt} f(v_{BB} + v_b) + w_0^2 v_b = 0 \quad (2.30)
\]

where, \(w_0\) is, as usual, the resonant frequency of the tank circuit of the oscillator and \(Q = 1/w_0 CR\).

Introducing a function \(N(v_b)\) given by the relation,

\[
N(v_b) = \frac{aM}{RC} f(v_{BB} + v_b) \quad (2.31)
\]

in equation (2.30) we get the familiar oscillator equation,

\[
p^2 v_b + \frac{w_0}{Q} p(v_b - N(v_b)) + w_0^2 v_b = 0 \quad (2.32)
\]

which is exactly similar to the equation (2.22) deduced for a tube oscillator. This establishes the validity of the oscillator loop equation (2.32) for an oscillator incorporating a semiconductor device too.

\(N(X)\) in equation (2.21) provides the nonlinearity in the oscillator. It is actually the equivalent transfer characteristics of the limiter type nonlinear element incorporated in the oscillator. Usually, the mutual characteristics of the tube, i.e., the variation of the plate current with the grid voltage, represents the nonlinear characteristics \(N(X)\) of the oscillator. The variation of \(N(X)\) with \(X\) is of the form shown in Fig. 2.7. Let us denote a parameter \(\eta\) given by,

\[
\eta = \frac{aM-L}{\mu CR} \quad (2.33)
\]
Fig. 2.7. Mutual characteristic of a typical triode valve.
which indicates the strength of coupling between the output and the input port of an oscillator. 

On expanding the function \( f(V_0 + X) \) around \( V_0 \) in equation (2.21) one can write,

\[
N(X) = \eta \sqrt{f(V_0) + xf'(V_0) + \frac{x^2}{2} f''(V_0) + \ldots} \quad \text{(2.34)}
\]

where

\[
f'(V_0) = \frac{df}{dX} \bigg|_{V_0}, \quad f''(V_0) = \frac{d^2f}{dx^2} \bigg|_{V_0}, \quad \text{etc.}
\]

With the help of equations (2.22) and (2.34) one can easily write the nonlinear differential equation of the oscillator in the form:

\[
\frac{d^2x}{dt^2} - \epsilon \frac{dx}{dt} \sum_{n=0}^{\infty} C_n x^n = 0
\]

where

\[
\epsilon = \frac{\omega_0}{Q} \quad \text{(2.36)}
\]

\[
C_0 = f(V_0) \quad \text{(2.37)}
\]

\[
C_1 = f'(V_0) - \frac{1}{\eta} \quad \text{(2.38)}
\]

\[
C_n = \frac{1}{n!} \frac{d^n}{dx^n} f(x) \bigg|_{x=V_0} \quad \text{(2.39)}
\]

In the problems of oscillator, a power series representation of the nonlinearity is generally accepted. An approximation of the series up to the third order of nonlinear terms yields a sufficiently accurate picture of the problems and also makes the analysis rather simplified. It has been observed that the
second order nonlinearity is much less than the third order term and therefore it can be neglected.

Before putting an end to our mathematical representation of a self excited oscillator, we spend a few steps on the form of the transfer function of the frequency selective amplifier \( Y(p) \) as given in (2.24). The Heaviside operator \( p \) has always a fast varying part \( p_1 \) and a slowly varying part \( p_2 \). Thus

\[
p = p_1 + p_2
\]

Taking the eigenvalues of the operators \( p_1 \) and \( p_2 \) to be \( jw \) and \( D \) respectively, the transfer function \( Y(p) \) can be written with some simplification, in another convenient form, viz.,

\[
\frac{1}{Y(p)} = 1 + \frac{2Q}{w} D + jQ \frac{w^2 - w_0^2}{w w_0}
\]

Obviously, \( D \) operates on slowly varying quantities e.g. \( A(t) \), \( \phi(t) \) etc. With all these background we are now in a position to examine the ultraharmonic synchronization of a Class-A oscillator. It has been observed that the use of the transfer function \( Y(p) \) in the form (2.40) instead of (2.24) makes the deductions of governing amplitude and phase equations simpler indeed.

2.3.1. Analytical Treatment of the Phenomenon/Ultraharmonic Synchronization:

(1) Asymmetry in Locking-range:

The functional representation of an oscillator under the
influence of an external signal is almost the same as that for the self-excited oscillator, since the variation of the output amplitude in a practical oscillator is very small compared to the variation of the output phase.

Thus for a class-A oscillator with an output voltage \( e_o \) and forced by an injected voltage \( e_i \), the oscillator loop equation may be written as,

\[
N(e_o) + e_i = \frac{1}{V(p)} e_o
\]

...(2.41)

This is represented in the block diagram of Fig.2.1.

For a CW signal injection, \( e_i \) may be written as,

\[
e_i = B \sin wt
\]

...(2.42)

The expression for \( e_o \) in the ultraharmonic synchronization must contain two frequencies, one is the directly transmitted component, say, \( A \cos (wt + \theta_1) \) and the other is an ultraharmonic of the synchronizing signal, close to the oscillator free-running frequency \( w_0 \). Thus,

\[
e_o = A \sin (wt + \theta_1) + B \sin (nwt + \theta_n)
\]

...(2.43)

Here 'n' is an integer that determines the order of ultraharmonic synchronization. In our present analysis of ultraharmonic entrainment we restrict our discussion to third order mode of synchronization. Thus for our case, \( n = 3 \).

Although, there is now restriction on the injection signal strength to fit into the oscillator loop equation (2.41), the
analysis becomes quite cumbersome with a larger strength of
the injecting signal. Moreover, in practice, the strength of
an injecting signal is always small compared to the oscilla-
tor output. This is why we deal our oscillator synchronization
with a low level of injection, i.e., in the so-called under-
driven operating condition of the oscillator. The same
arguments hold good for an arbitrary nonlinear feedback
coupling and we choose the nonlinearity to be a combination
of first and third order nonlinear terms. We thus write,

\[ N(e_0) = a_1 e_0 - a_3 e_0^3 \]  \hspace{1cm} (2.44)

where \( a_1 \) and \( a_3 \) are the constants of non-linearity. Inciden-
tally, it may be mentioned that the nonlinear characteristic
of van der Pol oscillator has exactly this form.

Using the form of transfer function given by equation
(2.40) in the oscillator loop equation (2.41) and adopting
the values of \( e_1 \) and \( e_0 \) from (2.42) & (2.43) one can easily
obtain the following relations for the variations of ampli-
tudes A and B and the phases \( \Theta_1 \) and \( \Theta_3 \) by the method of
harmonic balance:

\[ a_1 A - a_3 \left[ \frac{3}{4} A^3 + \frac{1}{2} AB^2 \right] + \frac{3}{4} a_3 A^2 B \cos \Theta + E \cos \Theta_1 = A + \frac{2q}{w_0} \frac{dA}{dt} \]  \hspace{1cm} (2.45)

\[ - \frac{3}{4} a_3 A^2 B \sin \Theta - E \sin \Theta_1 = \frac{w^2 - w_0^2}{w w_0} QA + \frac{2q}{w_0} A \frac{d\Theta_1}{dt} \]  \hspace{1cm} (2.46)
An approximate relation between the amplitude $A$ of the input frequency component and the amplitude $E$ of the synchronizing signal can be written from (2.46), on putting $\theta_1 \approx -\pi/2$ and neglecting the rather small term containing $a_3$. This gives,

$$A = \frac{E}{Q} \frac{w w_0}{w^2 - w_0^2} \quad \text{(2.50)}$$

The oscillator free-running amplitude $B_0$ can be evaluated from (2.47) on putting $A = 0$ and $\frac{dB}{dt} = 0$. This gives,

$$B_0 = \sqrt{\frac{4}{3} \frac{a_{1+1}}{a_3}} \quad \text{(2.51)}$$

Henceforth, we shall write $A$ and $B$ normalized by $B_0$ and put,$$
a = \frac{A}{B_0} \quad \text{and} \quad b = \frac{B}{B_0}
$$

An approximate relation between $a$ and $b$ can be obtained from (2.47), putting $\frac{dB}{dt} = 0$ in the steady state and neglecting the term containing $a_3$. This results in,

$$b = \sqrt{1 - 2a^2} \quad \text{(2.52)}$$
Written in terms of 'a' and 'b' the governing amplitude and phase equations (2.47) and (2.48) look like,

\[
\frac{q}{w_0} \frac{db}{dt} = b(1-b^2-2a^2) + \frac{1}{3} a^3 \cos \varphi \\
\text{and}
\frac{dw}{dt} = \frac{w_0}{2} \left( \frac{3w}{w_0} - \frac{w'}{w_0} \right) - \frac{a^3}{3bq} w'_0 \sin \varphi
\]

where \( q = \frac{2Q}{a_1^{-1}} \) and the rather small term \( \frac{dc_1}{dt} \) has been omitted in (2.54). The oscillator free-running frequency \( w_0 \) has been replaced by \( w'_0 \) in the phase equation. This is in accordance with our reasoning of shift of oscillator frequency by the pull of the forcing signal, stated in the section 2.2.1 of "physics of ultraharmonic synchronization". To calculate this shift of the oscillator free-running frequency one should start from the instantaneous phase equation (2.54) of the oscillator in the unlocked state. The case of ultraharmonic locking resembles exactly the corresponding case of harmonic synchronization, the detuning of the injecting signal being much larger than the lock band of the oscillator. It is obvious that the oscillator tends to remain in such a case at its free-running value and one may easily neglect the amplitude perturbation of the oscillator and assume the phase process \( \varphi(t) \) to be a slowly varying function of time. We can express the output of the oscillator under such a situation, in the form,

\[
e_0 = \Re \sqrt{B(t)} e^{j(w_0 t + \varphi(t))}
\]

\( \text{...(2.55)} \)
where the phase process \( \Psi(t) \) is a slowly varying function of time.

Let us write the injecting signal \( e_1 \) as,

\[
e_1 = E \cos \omega t = Re \left[ E e^{j(w_0 t + \psi)} \right]
\]  

where,

\[
\psi(t) = (w-w_0) t - \Psi(t)
\]  

Obviously, the instantaneous phase error \( \psi(t) \) between the injecting signal and the oscillator is fairly large.

On using these \( e_1 \) and \( e_0 \) values in the oscillator loop equation (2.41) and with some simplifying assumptions we arrive at the following phase equation of the oscillator,

\[
\frac{d\psi}{dt} = \Delta w - \frac{w-w_0}{w_0} K \sin \psi
\]  

where,

\[
\Delta w = w-w_0
\]  

and

\[
K = \frac{w_0 E}{2Q\delta_0}
\]  

It is now easy to express the Fourier component of the angular beat frequency of the oscillator in the form,

\[
\frac{d\psi}{dt} = \mp (\Delta w)^2 - \left( \frac{w-w_0}{w_0} \right)^2 \sum_{n=1}^{\infty} \left[ \frac{(-1)^n r_n^2 \cos(2\beta' - \beta_0)}{n} \right]
\]  

where,

\[
r' = x' - (x'^2 - 1)^{1/2}
\]
\[ 2\beta' = \frac{w}{w_0} K (\chi' - 1)^{3/2} (t + t'_0) \]

\[ \beta'_0 = \arctan (\chi'^2 - 1)^{3/2} \]

\[ \chi' = \frac{w_0}{w} \Delta w \frac{K}{K} \]

and \( t'_0 \) is a constant of integration. Thus the average beat frequency is given by

\[ \frac{d\phi}{dt} = \mp \left( (\Delta w)^2 - \left( \frac{w}{w_0} K \right)^2 \right)^{3/2} \]  \( \cdots (2.62) \)

the minus and plus signs correspond to cases for \( w > w_0 \) or \( w < w_0 \). For ultraharmonic synchronization, obviously, \( w < w_0 \) and the corresponding instantaneous frequency of the oscillator in the presence of the external signal is given by,

\[ w'_o = w_o + (w - w_0) + \mp \left( (w - w_0)^2 - \left( \frac{w}{w_0} K \right)^2 \right)^{3/2} \]  \( \cdots (2.63) \)

Once the relation between the shifted oscillator frequency \( (w'_o) \) and the original free-running frequency of the oscillator \( (w_o) \) is known by the equation \( (2.63) \), it is not difficult to find the upper and lower side locking ranges of the oscillator.

At the steady state as the phase difference \( \phi \) attains a steady state value \( \phi_s \), the phase equation \( (2.54) \) assumes the form:

\[ \frac{w'_o}{2} \left( \frac{3w'}{w'_o} - \frac{w'_o}{3w} \right) = \frac{s_s}{3b_s} \frac{w'_o}{q} \sin \phi_s \]  \( \cdots (2.64) \)

where \( a_s \) and \( b_s \) denote the corresponding values of 'a' and 'b' in the steady state. The extreme values of \( \phi_s \) at the two ends
of locking zone are $\pm \pi/2$ for the upper and lower bands respectively. Thus for obtaining the upper and lower locking points we proceed from equation (2.54) in the following manner.

Putting, $\sin \varphi = \pm 1$, we write,

$$\frac{3y}{z} - \frac{z}{3y} = \pm \frac{2}{3q} \cdot \frac{a_s^3}{\sqrt{1-2a_s^2}}$$

where $y = w/w_0$ and $z = w'/w_0$ and $b_s$ has been replaced by $\sqrt{1-2a_s^2}$.

Let,

$$K_1 = \frac{2}{3q} \cdot \frac{a_s^3}{\sqrt{1-2a_s^2}} \quad \cdots (2.66)$$

then from (2.65),

$$y/z = \pm \frac{K_1}{6} + \frac{1}{3} \sqrt{1+K_1^2/4} \quad \cdots (2.67)$$

the positive sign representing the locking point at the upper side and the negative sign corresponds to that for the lower side.

After arriving at the equation (2.67) it is easy to obtain the zones of frequency entrainment on the upper and lower sides of the oscillator frequency $w_0$. Although it is not at once clear from (2.67) why should the lower side lock-range for ultraharmonic synchronization be greater than that for upper-side one, consideration of the 'z' parameter giving the shift of oscillator frequency explains the phenomenon. The z-values
of an oscillator become smaller with lower values of Q-factor and higher values of synchronizing voltage as can be seen from (2.63). Fig. 2-8, represents the theoretical variation of the locking ranges around the oscillator free running frequency with the normalized input signal amplitude \( \frac{E}{B_0} \). It is shown there that for lower strengths of synchronizing voltage \( \frac{E}{B_0} \), the upperside locking points fall on the same side of the oscillator centre frequency as that of the lowerside ones.

(ii) Variation of Amplitude, Phase and Beat-Frequency:

In the unlocked state, the amplitude and phase variations of an ultraharmonic oscillator are given by equations (2.53) and (2.54). Introducing the parameter \( K_0 \) given by

\[
K_0 = \frac{A}{8} \frac{A_3}{B_0} w_0 = \frac{1}{3} \frac{w_0}{2Q} (a_1 - 1) a^3 
\]  

(2.68)

which is actually the one-sided locking band of the ISO, only with the oscillator amplitude \( B \) being replaced by its free-running value \( B_0 \), we can express the amplitude and phase relations in the following forms,

\[
\frac{d\phi}{d\theta} = \frac{3b - 3(2a^2b + b^3) + a^3 \cos \phi}{a^3 \sqrt{\frac{Q}{K_0} - \frac{\sin \phi}{b}}} 
\]  

(2.69)

and

\[
\frac{d\phi}{d\theta} = \frac{Q}{K_0} - \frac{\sin \phi}{b} 
\]  

(2.70)

where,

\[
Q = \frac{w_0}{2} \left( \frac{3w}{w_0} - \frac{w_0}{3w} \right) 
\]  

(2.71)
Fig. 2.8. Theoretically evaluated zone of frequency entrainment in the ultraharmonic mode of order three (for $Q = 0.5$ and $Q = 1$).
and

$$7 = \frac{w_0}{2q}(a_1-1) \frac{a_1^3}{3}t \quad \ldots (2.72)$$

In writing these equations we have assumed $w_0 = w'_0$, i.e., the shift of oscillator free-running frequency has been neglected. The nature of variation of beat-frequency, phase and amplitude of the oscillator with time can easily be determined from (2.69) and (2.70). To find the phase and frequency variations in a situation when the oscillator amplitude remains constant, one has simply to equate $da/d\phi$ in equation (2.69) to zero. The curves showing these variations are plotted in Figs. 2.9, 2.10 and 2.11. To make a comparative study the corresponding curves for harmonic mode of synchronization are also plotted side by side.

(iii) Locking Time:

Let us now extend our discussion to a situation where the strength of the incoming signal and its detuning from the oscillator frequency are suitable for the oscillator to fall in synchronism. The parameter that is important to consider in this case is the time taken by the system to lock-in. This locking-time ($T_L$) can be calculated using (2.70) and is given by,

$$T_L = \int \frac{d\phi}{\frac{\phi}{K_0} - \sin \phi} \quad \ldots (2.73)$$

where $\phi_i$ and $\phi_f$ are the initial and final values of the phase difference $\phi$. 
Fig. 2.9. Variation of beat-frequency with time for harmonic and ultraharmonic oscillators.
Fig. 2.10. Variation of output phase with time for harmonic and ultraharmonic oscillators.
Fig. 2.11. Variation of oscillator amplitude with time for harmonic and ultraharmonic oscillators.
\( \Phi_i \), for practical purposes, may be assumed to be zero and \( \Phi_f \) may be computed from (2.70) and has the value,

\[
\Phi_f = \sin^{-1} b \frac{Q}{K_0}
\]

(2.74)

But with this value of \( \Phi_f \), the locking-time becomes infinite. To obtain a realistic value of \( T_L \), \( \Phi_f \) is assumed to be 90% of the value of \( \Phi \), given by (2.74).

Thus the variation of locking-time \( T_L \) with frequency-detuning \((Q/K_0)\) can be evaluated from the phase-time plot of the oscillator for different detuning and is shown in Fig. 2.12.

This figure also depicts the locking-time-detuning plot for harmonic synchronization. Fig. 2.12 and the figures 2.9, 2.10 and 2.11 depicting the variations of different locking characteristics indicate clearly that the response time of an oscillator towards locking is quicker for ultraharmonic mode of synchronization than that for harmonic mode.

2.3.2. Frequency Response Characteristics for Ultraharmonic Synchronization:

To find out the frequency entrainment property of an ultraharmonically locked oscillator, the procedure is to equate \( \frac{db}{dt} \) and \( \frac{dw}{dt} \) to zero in (2.53) and (2.54) respectively. Thus in the steady state, one obtains,

\[
b_s (1 - b_s^2 - 2a_s^2) + \frac{1}{3} a_s^3 \cos \Phi_s = 0
\]

(2.75)

and

\[
Q - \frac{w_s^3}{q} a_s^3 \sin \Phi_s = 0
\]

(2.76)
Figure 2.12. Variation of locking time ($T_L$) with frequency-detuning ($\Omega/K$) for harmonic and ultraharmonic synchronization of oscillators.
where, \( \Omega = \frac{w'}{2} (\frac{3w_0}{w'} - \frac{w'}{3w}) \) and denotes the initial detuning of the external signal with respect to the shifted free-running frequency of the oscillator. The frequency response characteristic of the synchronized oscillator can be obtained from (2.75) and (2.76) as,

\[
\frac{d^2 \Omega^2}{d\omega_0^2} + (1 - \frac{b_s^2 - 2a_s^2}{2})^2 = 1 \quad \text{...(2.77)}
\]

To ascertain the conditions of entrainment, stability criteria must be found out. Following the method of Liapounov, we add small perturbations 'u' and 'v' to the steady state values \((b_s, \phi_s)\) of \(b\) and \(\phi\) respectively, i.e.,

\[
b = b_s + u \quad \text{...(2.78)}
\]

and

\[
\phi = \phi_s + v \quad \text{...(2.79)}
\]

The corresponding incremental equations can be written from (2.53) and (2.54) and are given by,

\[
\frac{d}{d\omega_0} Su = u - 3b_s^2u - 2a_s^2u - \frac{1}{3} a_s^3 \sin \phi_s \cdot y \quad \text{...(2.80)}
\]

and

\[
Sv = - \frac{w'}{3bq} a_s^3 \cos \phi_s \cdot y + \frac{w'a_s^3}{3q b_s^2} \sin \phi_s \cdot x \quad \text{...(2.81)}
\]

The characteristic equation then assumes the form,

\[
\frac{d}{d\omega_0} S^2 + S (\frac{3}{3b} a_s^3 \cos \phi_s + 3b_s^2 + 2a_s^2 - 1) - \frac{w'}{3bq} a_s^3 \cos \phi_s (1 - 2a_s^2 - 3b_s^2)
\]

\[
+ \frac{a_s^6}{9q} \frac{w'}{b_s^2} \sin^2 \phi_s = 0 \quad \text{...(2.82)}
\]

where

\[
S = \frac{d}{dt}
\]
On substituting the values of \( \cos \phi_s \) and \( \sin \phi_s \) from (2.75) and (2.76), the necessary and sufficient conditions for stability are given by,

\[
b_s^2 > 0.5 - a_s^2
\]  \( \cdots (2.83) \)

and

\[
\frac{q^2}{w_0^2} \Omega^2 (1 - 2a_s^2 - 3b_s^2) (2a_s^2 + b_s^2 - 1) > 0 \]  \( \cdots (2.84) \)

The second relation (2.84) represents an elliptical boundary. The steady state frequency response characteristics of an ultraharmonically locked oscillator together with the zones of stability have been depicted in Fig. 2.13 for different values of synchronizing voltage. The curves indicate that unlike in the case of same frequency synchronization the hysteresis or jump phenomena are not observable in ultraharmonic synchronization.

2.4. Experiment:

A transistorised oscillator using AF 115 was hooked up for the experimental work and is shown in Fig. 2.14. The oscillator was of frequency of about 5 KHz and was a collector tuned one. The cubic type nonlinearity was introduced in the oscillator by connecting two point contact diodes arranged back to back in the oscillator collector circuit. It can be easily shown from the current voltage relations of the
Fig. 2.13. Frequency response characteristics of an ultraharmonic oscillator.
Fig. 2.14. The experimental set-up.
diodes joined in this fashion, that for not too high a value of the oscillator amplitude, such a connection of the diodes provides a nonlinearity containing mainly the first and third order terms. The amplitude as well as the waveform of the oscillator were controlled by the potentiometer connected in the feedback path. The Q-value of the oscillator was about unity. Signal frequencies of nearly one-third frequency of the free-running frequency were injected from a stable oscillator and the frequencies indicating the locking zones were determined by using a frequency counter at the oscillator output. The experimental curve showing the variation of locking ranges of the ultraharmonic oscillator in the third order mode of synchronization with the injected signal strength (E) is shown in Fig. 2.15.

2.5. Conclusion:

The important feature of this chapter lies in the way that it offers an elaborate physical explanation of what happens in the case of ultraharmonic synchronization. The mathematics for the theory is not a very new one but the technique of developing the theory by way of unifying the previous analytical approaches with the physical arguments stated in the text is quite novel in nature. The interesting outcome of the analysis is that the experimental findings observed with the help of a hard-ware model of a class-A
Fig. 2.15. Experimental locking behaviour in the ultraharmonic of order three.
transistorised oscillator exhibit a good agreement with the theoretical results. The exact coincidence of the theoretical and experimental curves showing the variation of locking ranges with the signal strength could not be attained because of the difficulty in finding out precisely the exact values of the nonlinearity constants \(a_1\) and \(a_3\). A study of the theoretical and experimental results leads to the following observations:

a) The locking zone of the ultraharmonically synchronized class-A oscillator is asymmetric in nature. This is due to the asymmetric behaviour of the tank circuit as well as the centre frequency shift of the oscillator by the existence of the forcing signal at the oscillator output.

b) A close look at the theoretical curves of Fig. 2.8 indicates that for a fixed value of Q-factor, at lower values of the synchronizing signal strength, both the upper and lower locking points fall on one side of the oscillator centre frequency. Obviously, this is a result of the consideration of the shift of the oscillator frequency. At higher values of the signal strength, however, the lower side lock-ranges become greater than those for upper side. This observation is just the opposite from that in the case of harmonic synchronization. For a fixed value of the signal strength, the difference between upper and lower sided lock bands is
definitely greater for lower values of Q-factor. The reason for this has already been stated in the theory.

c) The phenomena of hysteresis and jump are not observable in the case of a low driven ultraharmonic class-A oscillator, as considered in our analysis.

d) The variation of locking time with frequency detuning (Fig.2-12) for ultraharmonic and harmonic oscillators suggests that in achieving locking for a particular frequency detuning the former is more prompt than the latter. This is also evident from the plots of amplitude, phase and beat-frequency of the two types of oscillators.

e) The zone of synchronization increases, in general, with the increase in the external signal amplitude. However, after a certain strength of injected signal the zone tends to be infinitely extended. It is quite likely that the zones in different modes of entrainment merge at that stage and consideration of the merging effect may lead to a new explanation of the so-called infinite locking range [II-11, II-17].

Before putting an end to the 'conclusion' part we should point out two minor discrepancies of the theory discussed here. Firstly, the calculation for the shift in the oscillator frequency as presented for ultraharmonic synchronization has overlooked a little modification indicated in the work of Paciorek [II-18]. Secondly, the analysis pays no consideration to the flow of base current in the oscillator to avoid complexities in the description of the physics of ultraharmonic entrainment.
REFERENCES


II-10 L Gustafsson, I Lundström, and B Hausson, 'Maximum Phase Locking Bandwidth Obtainable by Injection Locking', Technology, Division of Network Theory, Gothenburg, Sweden, Nov 1972.


