CHAPTER-VII

PHASE NOISE IN OSCILLATORS

7.1. Introduction:

In all the chapters discussed so far, the oscillator considered has been assumed to be an ideal one. That is, the effect of internal noise of oscillators has been ignored. But, in reality, the output phase of an oscillator has been observed to become noisy due to the presence of various noise sources within the oscillator circuit. Actually, the phase noise is an outcome of causally generated signals and nondeterministic random noise being introduced in the system. The causally generated effects are produced by supply voltage fluctuations, oscillator temperature variations, humidity, magnetic field, output load impedance etc. The phase noise in an oscillator leads to spectral broadening of the output waveform. Spectral purification of the output waveform is, however, needed for many applications related to generation of secondary time and frequency standards.

The theory of noise in oscillators was initiated by Berstein in 1938. Since then a considerable amount of work has been carried out on the noise characteristics of the free-running and synchronized oscillators. Of the many publications we can mention concerning
the phase noise of oscillators. Many of these published papers have taken up for discussion the FM white noise close to the carrier. Theoretical models on the spectral characteristics of a free-running oscillator have been heuristically suggested by Lesson [VII-4]. Kroupa [VII-5], on the other hand, has started from actual experimental noise characteristics and developed a theoretical model. In our analysis of oscillator noise two things have been highlighted. Firstly, the spectral characteristics of both the feedback and negative resistance oscillators (like Gunn, IMPATT oscillators) have been derived starting from the fundamental oscillator equation. Secondly, the noise spectra have been calculated for oscillators synchronized in harmonic as well as in ultraharmonic modes.

7.2. Theory:

Let us consider a typical arrangement of a Gunn oscillator with the provision of injection locking as shown in Fig. 7.1a. The analytical equivalent representation of this circuit may be drawn as shown in Fig. 7.1b. Here, \( i_s \) denotes the current from the synchronizing signal source and \( v \) is the voltage drop across the parallel combination of capacitance \( (C(v)) \), conductance \( (G(v)) \) and the inductance \( L \) representing the oscillator tank circuit. It is then easy to write,

\[
i_s = \frac{d}{dt} \sqrt{C(v) v^2 + G(v) v^2} \int \frac{v}{L} dt \quad \cdots (7.1)
\]
Fig. 7.1(a). Arrangement of injection locking in a Gunn oscillator.
(b) Equivalent representation of the oscillator in Fig. 7.1(a).
In differentiating both sides,

\[ \frac{d^2 i_s}{dt^2} = \frac{d^2}{dt^2} \sqrt{C(v)} \frac{d}{dt} + \frac{d}{dt} \sqrt{G(v)} \frac{d}{dt} + \frac{v}{L} \] \hspace{1cm} \cdots (7.2)

Now, expressing the nonlinear device capacitance and the nonlinear device conductance in the forms given by,

\[ C(v) = C(1 + \beta v^2) \] \hspace{1cm} \cdots (7.3)

and

\[ G(v) = G(-T_1 + T_3 v^2) \] \hspace{1cm} \cdots (7.4)

One can rewrite the equation (7.2), in the free-running condition of the oscillator (i.e., \( i_s = 0 \)), in the following differential form,

\[ \frac{d^2 v}{dt^2} + \frac{w_0}{Q} \frac{d}{dt} \sqrt{T_1 v} + T_3 v^3 + \frac{Q}{w_0} \beta \frac{d}{dt} v^2 \sqrt{w_0} v = 0 \] \hspace{1cm} \cdots (7.5)

In order to compare this equation with that of a feedback oscillator, we refer to Fig.7.2a which is the equivalent nonlinear model of such an oscillator (cf. Chapter-II). The loop equation of the oscillator may be expressed as,

\[ Y(p) N(v) = v \] \hspace{1cm} \cdots (7.6)

\( p \) is the Heaviside operator and \( Y(p) \) is the transfer function of a single tuned resonant circuit with the quality factor \( Q \) and the resonant frequency \( w_0 \). And, \( v \) is the oscillator output and \( N(v) \) is the transmittance function of the nonlinear element.

On substituting the familiar relations for \( Y(p) \) and \( N(p) \) given by,
Fig. 7.2(a). Block diagrammatic representation of an ordinary feedback oscillator.

Fig. 7.2(b). Block diagrammatic representation of negative resistance oscillator.
\[ Y(p) = \frac{pw_0}{pw_0 + p^2 Q + Qw_0^2} \]  

\[ N(v) = a_1 v - a_3 v^3 \]

\( a_1 \) and \( a_3 \) being the constants of nonlinearity.

we can write from (7.6),

\[
\frac{d^2 v}{dt^2} + \frac{w_0}{Q} \frac{d}{dt} \sum \left[ (a_1 - 1) v + a_3 v^3 \right] + w_0^2 v = 0
\]

Comparison of equation (7.5) with this equation reveals that the nonlinearity parameter \( (N(v)) \) for the negative resistance oscillator has a value given by,

\[
N(v) = (1+\gamma_1) v - \gamma_3 v^3 - \frac{Q}{w_0} \beta \frac{d}{dt} v^3
\]

This relation transforms to that in equation (7.8) by replacing the parameters \( (1+\gamma_1) \) by \( a_1 \) and \( (\gamma_3 + \frac{Q}{w_0} \beta \frac{d}{dt}) \) by \( a_3 \); the only difference being that the nonlinearity in this case is time dependent.

It is now obvious, that just like feedback oscillators, the negative resistance oscillators can also be represented by a combination of a resonator and a nonlinear element. This is shown in Fig.7.2b.

For an ideal oscillator, the output observed across the resonator, would have been a pure sinusoidal wave of the form \( Asin \omega t \). But because of the noise in the limiter and the resonant amplifier, the output, in practice, will appear as
\[ A(t) \sin(w_1 t + \phi(t)) \]. Neglecting the amplitude fluctuation, as can be done in most of the oscillators, the output may be represented as \( A \sin(w_1 t + \phi(t)) \).

Let us now break the oscillator loop at a point, say \( M \) (cf. Fig. 7.3a) and observe the output by feeding a pure signal \( A \sin w_1 t \) to the input of the limiter type nonlinear element. Since the origin of the phase noise of the oscillator is associated with the noise characteristics of the limiter and the resonant amplifier, the output may be written as \( A \sin(w_1 t + \alpha(t)) \), where \( \alpha(t) \) represents the phase noise arising out of the noisy nature of the two devices. The spectral characteristics of \( \alpha(t) \) is known to be given by,

\[ S_\alpha(f) = \frac{a_{-1}}{f^2} + a_0 \]  \hspace{1cm} (7.11)

where \( a_{-1} \) is the flicker noise constant and \( a_0 \) is the white noise constant.

Now since \( \alpha \) is small, we can approximate the output as,

\[ A \sin(w_1 t + \alpha(t)) \approx A \sin w_1 t + A \cos w_1 t \]

The appearance of the second term viz., \( A \cos w_1 t \) suggests that the loop equation of the oscillator, with due consideration of internal noise, can be expressed as,

\[ Y(p) = N(v) + A \cos w_1 t = A \sin(w_1 t + \phi(t)) \]  \hspace{1cm} (7.12)

This is represented diagrammatically in Fig. 7.3b.
Fig. 7.1(a). Open loop representation of an oscillator.

Fig. 7.1(b). Open loop representation of an oscillator with consideration of internal noise.
On substituting the values of $Y(p)$ and $W(v)$ from (7.7) and (7.10) in this relation we can write,

$$p \omega_0 \left[(1+\gamma_1)w^3 - \frac{Q}{\omega_0} \frac{d}{dt}(v^3)\right] = \frac{\omega_0}{Q} (w_0^2 p + p^2 w_0).$$

$$\left[\sin(w_1 t + \phi) - A \cos w_1 t\right]$$

Replacing $v$ by the oscillator output viz., $A \sin(w_1 t + \phi)$ and equating the coefficients of $\sin(w_1 t + \phi)$ from both sides, we obtain

$$\frac{3}{4} QA^3 (w_1^2 + 2w_1 \phi) = -\omega_0 \sin \phi \dot{\phi} + A \omega_1 \cos \phi - QA(w_1^2 + 2w_1 \phi)$$

$$+ 2QA \omega_1 \dot{\phi} \cos \phi + QA \omega_1^2 \sin \phi + Q \omega_0^2 A - QA \omega_0^2$$

Assuming, $\phi$ and $\alpha$ to be quite small, equation (7.14) reduces to the following relation given by,

$$\left(\frac{3}{4} QA^2 + 1\right) \dot{\phi} = \dot{\alpha} + \frac{\omega_0}{2Q} \alpha + \frac{1}{2w_1} (w_0^2 - w_1^2 - \frac{3}{4} \omega^2 A^2) - \frac{\omega_0}{2w_1 Q} \dot{\phi}$$

On substituting $\dot{\phi} = \dot{\alpha} = 0$ and neglecting the rather small term containing $\alpha$ in (7.15) we can arrive at a relation connecting the free-running frequency of oscillator ($w_1$) and the resonator frequency ($\omega_0$). This gives,

$$\omega_0^2 = w_1^2 \left[1 + \frac{3}{4} \beta A^2\right]$$

Using this relation in (7.15) and omitting the small term involving $\dot{\phi}$ we can easily write,
\[ (1 + \frac{3}{4} \beta A^2) \frac{d\phi}{dt} = \frac{da}{dt} + \frac{w_0}{2Q} \alpha \quad \text{..(7.17)} \]

The phase spectral density of the free-running oscillator then assumes the form,

\[ S_\phi(f) = \frac{1}{(1 + \frac{1}{4\beta A^2})^2} \left( 1 + \frac{f_0}{2fQ} \right)^2 S_\alpha(f) \quad \text{..(7.18)} \]

Equation (7.18) suggests that high power oscillators lower the phase noise variance. This is a characteristic of microwave oscillators and is not observable in the case of ordinary oscillators.

Using the known value of \( S_\alpha(f) \) from (7.11), one can rewrite (7.18) in the form,

\[ S_\phi(f) = \frac{a_1 f_1^2}{4Q^2 f^3 (1 + \frac{3}{4\beta A^2})} + \frac{a_0 f_1^2}{4Q^2 f^2 (1 + \frac{3}{4\beta A^2})} + \frac{a_{-1}}{f(1 + \frac{3}{4\beta A^2})^2} + \frac{a_0}{(1 + \frac{1}{4\beta A^2})^2} \quad \text{..(7.19)} \]

where, \( f_1 \) represents the free-running frequency of the oscillator.

On putting \( \beta = 0 \) in (7.18) and (7.19) we arrive at the results that coincide exactly with those derived heuristically by Lesson [VII-4]. Equation (7.19) reveals an interesting fact that unlike in ordinary oscillators, the phase spectral density of the output waveforms of the Gunn and IMPATT oscillators depends on the amplitude of the free-running oscillator.
7.2.1. Consideration of Frequency Stability of Oscillators:

Let us calculate the frequency instabilities arising out of the phase noise in oscillators, both feedback and negative-resistance type. In order to do this, we express the phase noise power spectral density (one-sided) in the following form (cf. equation (7.19))

\[ S_\phi(f) = \frac{K_a}{f^2} + \frac{K_b}{f^2} + \frac{K_c}{f^2} + N_o \]  ...(7.20)

where,

\[ K_a = \frac{a_{-1}^2 f_1^2}{4Q^2(1 + \frac{3}{4\beta A^2})} \]  ...(7.21)

\[ K_b = \frac{a_0^2 f_1^2}{4Q^2(1 + \frac{3}{4\beta A^2})} \]  ...(7.22)

\[ K_c = \frac{a_{-1}}{(1 + \frac{3}{4\beta A^2})^2} \]  ...(7.23)

and

\[ N_o = \frac{a_0}{(1 + \frac{3}{4\beta A^2})^2} \]  ...(7.24)

Here, the parameters \( K_a, K_b, K_c \) and \( N_o \) stand for a measure of the strengths of the flicker-frequency noise, white-frequency noise, flicker-phase noise and white-phase noise respectively.

Let us now find the individual contribution of these noise components towards the noise frequency variance and then add them to evaluate the total contribution.
The variance of frequency depends on the autocorrelation \( R(T) \) of the phase jitter \( \phi(t) \) and is expressed by \( \sqrt{\text{II-17}} \),

\[
\sigma_T^2 \equiv \frac{1}{T^2} E \left( \frac{(\phi(t) - \phi(0))^2}{(2\pi)^2} \right) = \frac{2}{T^2} \frac{R(0) - R(T)}{(2\pi)^2} \text{(rad/sec)}^2
\]

\[
= \frac{2}{T^2} \int_0^\infty S_\phi(w) \frac{1-\cos(wT)}{(2\pi)^2} \, dw = \frac{2}{2\pi} \int_0^\infty S_\phi(w) \frac{1-\cos(wT)}{(2\pi)^2} \, dw
\]

\[\cdots (7.25)\]

where \( S_\phi(w) \) is the power spectral density of \( \phi(t) \).

\( \text{(a) Flicker-frequency noise contribution:} \)

The flicker-frequency noise spectrum may be conveniently expressed as a combination of two parts, viz.,

\[
S_{\phi_F}(f) = \frac{K_a}{f^3}, \text{ for } f_1 < |f| < \frac{1}{T}
\]

and

\[
S_{\phi_F}(f) = \frac{K_a}{f_1 f^2}, \text{ for } |f| < f_1
\]

\[\cdots (7.26)\]

Here \( f_1 \) denotes the low-frequency cut-off and is necessary if \( S_\phi(f) \) is to have finite power. In the limiting case, however, \( f_1 \to 0 \). Inserting these values of \( S_{\phi_F}(f) \) in (7.25), the frequency noise variance becomes:

\[
\sigma_{TF}^2 = 2 \int_0^{w_1} \frac{K_a}{w_1 w^2} \frac{1-\cos(w)}{T^2} \, dw + \int_{w_1}^{W} \frac{K_a}{w^3} \frac{1-\cos(wT)}{T^2} \, dw
\]

\[\cdots (7.27)\]
where, \( w_1 = 2\pi f_1 \) and \( w = 2\pi f \)

Let us denote \( x = wT, x_1 = w_1 T \) and \( x_0 = \eta T \) and assume that \( x_1 \ll 1 \) and \( x_0 \gg 1 \). One can then write from (7.27),

\[
\sigma_{TF}^2 \simeq 2K_a + 2K_a \left[ -\frac{(1-\cos x)}{x^2} \right]_x^{x_1} + \int_{x_1}^{x_0} \frac{\sin x}{x^2} \, dx \\
\simeq 2K_a \left[ -\frac{5}{2} \right] = C_1(x_1)^\gamma 
\]

where, \( C_1(x_1) \) is the cosine integral, given by,

\[
C_1(x_1) = -\int_{x_1}^{\infty} \frac{\cos t}{t} \, dt = C + \ln x_1 + \sum_{k=1}^{\infty} \frac{(-1)^k x_1^{2k}}{2k(2k)} 
\]

with, \( C = 0.577215 \)

Now, \( x_1 \) being small, one can approximately write,

\[
\sigma_{TF}^2 \simeq 2K_a \left( 1 - 0.92 + \ln \frac{1}{x_1} \right) \quad \ldots (7.30)
\]

(b) White-frequency noise contribution:

The oscillator phase-noise spectral density created by white frequency noise may be represented as,

\[
S_{\phi_b}(f) = \frac{K_b}{f^2}, \quad f > 0 \quad \ldots (7.31)
\]

Using this value of spectral density in (7.25) we can write for the frequency-noise variance \( \sigma_{TFb}^2 \) for an averaging time \( T \) in the form,
The phase-noise spectral density in this case, has the form,

\[ S_{\phi p}(f) = \frac{K_c}{f}, \quad f_1 < |f| < f_h \]  

\[ \sigma^2_{T_b} = \frac{2 K_c}{(2\pi)^2} f_1^h \int_{f_1}^{f_h} \frac{1 - \cos\omega T}{f} df \]

\[ = \frac{K_c}{2\pi^2 T^2} \int_{x_1}^{x_h} \left( \frac{dx}{x} - \frac{\cos x}{x} \right) dx \]

\[ \approx \frac{K_c}{2\pi^2 T^2} \left( \ln x_h + C \right) \]

where, \( x = wT, \ x_1 = w_1 T, \ x_h = w_h T \) and \( C = 0.577215 \).

It has been assumed in the above deduction that \( x_1 \ll 1 \) and \( x_h \rightarrow \infty \).
(d) White-phase noise contribution:

The white-phase noise contributes an amount of frequency-variance given by

\[ \delta_{TW}^2 = \int_0^{W_h} \frac{N_o T^2}{2} \frac{(1-\cos wT)}{(2\pi)^2} dw \]  \hspace{1cm} \text{(7.35)}

since, \( S_{\phi W}(f) = N_o \) for \( f < f_h \)  \hspace{1cm} \text{(7.36)}

Solution of (7.33) gives,

\[ \delta_{TW}^2 = \frac{2f_h N_o}{(2\pi T)^2} \left(1 - \frac{\sin w_h T}{w_h T}\right) \]

\[ = \frac{2P_n}{(2\pi T)^2} \left(1 - \text{sinc} w_h T\right) \]  \hspace{1cm} \text{(7.37)}

where, \( P_n = f_h N_o \) is the phase-noise power.

At this stage we can calculate the oscillator frequency stability (\( \delta \)) using the relations,

\[ \delta = \frac{\Delta f}{f_1} \]  \hspace{1cm} \text{(7.38)}

and \( (\Delta f)^2 = \delta_{TF}^2 + \delta_{TB}^2 + \delta_{TP}^2 + \delta_{TW}^2 \)  \hspace{1cm} \text{(7.39)}

where, \( f_1 \) is the oscillator centre-frequency.

Example:

Let us take a crystal oscillator of 10MHz frequency. The phase-noise for such an oscillator can be expressed as a function of the oscillator frequency \( f_1 \) by the following semi-theoretical and semi-experimental relation \( \sqrt{\text{VII-6}} \),
\[ S_\phi(f) = \frac{10^{-37.25}}{f^3} f_1^4 + \frac{10^{-39.4}}{f^2} f_1^4 + \frac{10^{-12.15}}{f} + 10^{-15} \]

On comparing this relation with (7.20) we can put,

\[ K_a = 10^{-37.25} \times 10^{28} = 10^{-9.25} \]

\[ K_b = 10^{-39.4} \times 10^{28} = 10^{-11.4} \]

\[ K_c = 10^{-12.15} \]

and \[ N_0 = 10^{-15} \]

Utilizing these values and assuming the frequency limits \( w_1 = 10^{-4} \) and \( w_2 = 2\pi \) and the averaging time \( T = 0.1 \) sec, we can calculate the frequency variance contribution by the noise components. This yields,

\[ \Delta F^2 = \delta_{TP}^2 + \delta_{Tb}^2 + \delta_{Tp}^2 + \delta_{TW}^2 \]

\[ = (1.229 \times 10^{-4})^2 \]

i.e., \[ \Delta F = 1.229 \times 10^{-4} \]

Then the frequency stability of the crystal oscillator has the magnitude given by,

\[ \delta = \frac{\Delta F}{f_1} = 1.229 \times 10^{-11} \]

Let us now calculate this parameter for negative resistance oscillators like Gunn and IMPATT oscillators. We assume
these oscillators to have a free-running frequency of 10GHz and a Q-value of 3000 (say). Evaluation of the factors $K_a$, $K_b$, $K_c$ and $\Pi_0$ in these cases may be done by utilizing the equations (7.21) to (7.24), provided the constants $a_{-1}$ and $a_0$ and the oscillator amplitude are known. If we consider the expression $(1 + \frac{3}{4} \beta A^2)$ to be equal to 1.1, then as has been shown in the subsequent section 7.2.4, the constants $a_{-1}$ and $a_0$ have the following values:

\[
\begin{align*}
a_{-1} &= 2.42 \times 10^{-13} \quad \text{for the Gunn oscillator} \\
a_0 &= 1.21 \times 10^{-17} \quad \text{for the IMPATT oscillator}
\end{align*}
\]

We can then easily write,

\[
\begin{align*}
K_a &= 0.6111 \\
K_b &= 3.055 \times 10^{-5} \\
K_c &= 2 \times 10^{-13} \\
\Pi_0 &= 10^{-17}
\end{align*}
\]

for the Gunn oscillator

and

\[
\begin{align*}
K_a &= 30.55 \\
K_b &= 7.7 \times 10^{-4} \\
K_c &= 10^{-11} \\
\Pi_0 &= 2.52 \times 10^{-16}
\end{align*}
\]

for the IMPATT oscillator.
Now making use of the relations (7.30), (7.32), (7.34) and (7.37) and using the same values of \(T\), \(w_1\) and \(w_h\), we obtain the following values for the frequency deviation and hence the frequency stability for the two types of oscillators:

**Gunn oscillator**

\[
\Delta F = 4.051 \\
\delta = 4.051 \times 10^{-10}
\]

**IMPATT oscillator**

\[
\Delta F = 28.62 \\
\delta = 28.62 \times 10^{-10}
\]

It is clear from this example that the crystal oscillator is more stable in frequency compared to the negative resistance microwave oscillators like Gunn, IMPATT etc. Again, compared with the IMPATT oscillator, the Gunn oscillator is better than it in respect of frequency-stability.

The frequency stabilities calculated here are all for the free-running condition of the oscillators. It may be shown that the frequency-stability improves a lot for an IMPATT oscillator when synchronized by a Gunn oscillator than when it is in the free-running state.

7.2.2. Harmonic Synchronization:

After evaluating the phase spectral density for a
free-running oscillator, let us explore how it is modified in the presence of a harmonic synchronizing signal, i.e., for same frequency synchronization. Let us assume the oscillator output, corresponding to an input signal of $E \sin(wt+\phi_1)$, to be $A \sin(wt+\phi)$. Then following the arguments, stated earlier, the oscillator loop equation may be written as,

$$Y(p)E \sin(wt+\phi_1) + N(v)\dot{v} + A \cos wt = A \sin(wt+\phi)$$

...(7.44)

Making use of (7.7) and (7.10) this equation looks like,

$$pw_0 E \sin(wt+\phi_1) + (1+\gamma_1)v - \gamma_1 v^3 - \frac{Q_0}{w_0} \frac{d}{dt}(v^3)\dot{v} =$$

$$\sqrt{p^2 w_0 + Q(p^2 + p^2 w_0)} \sqrt{A \sin(wt+\phi) - A \cos wt}$$

...(7.45)

Substituting $v = A \sin(wt+\phi)$ in this expression and applying the method of harmonic balance, one obtains the following phase equation from (7.45)

$$(1+\frac{3}{4}a^2) \frac{d\phi}{dt} = \frac{w_0}{2} \left( \frac{w_1}{w_0} - \frac{w_0}{w_1} \right) + \frac{\alpha}{2Q} \frac{d\alpha}{dt} + \frac{w_0}{2Q} a + X_0 \dot{\phi}$$

...(7.46)

where we have assumed $\phi$ and $a$ to be small and $K(= \frac{w_0^E}{2Q\alpha})$ represents the one-sided locking band of the harmonic oscillator. For the in-tune case (i.e., $w_1 = w_0$), equation (7.46) reduces to the following expression for the oscillator phase:

$$\phi(p) = \frac{K}{p(1+\frac{3}{4}a^2)+K} \cdot \frac{\frac{w_0^2}{2Q} + p\alpha}{p(1+\frac{3}{4}a^2)+K}$$

...(7.47)
Denoting the phase noise spectral densities for the synchronizing signal (i.e., \( \theta_1 \) and \( a(t) \) by \( S_{\phi_1}(f) \) and \( S_a(f) \) respectively, one can easily write for the noise spectral density of the phase \( \phi \) with the following relation,

\[
S_{\phi}(f) = \frac{K^2}{K^2 + (1 + \frac{3}{4}A^2) f^2} S_{\phi_1}(f) + \frac{f^2(1 + \frac{f_0^2}{4Q^2 f^2})}{K^2 + (1 + \frac{3}{4}A^2) f^2} S_a(f)
\]

\[\text{(7.48)}\]

At this stage we make one pertinent assumption. We consider the case of spectral purification of an IMPATT oscillator through injection synchronization with the help of a Gunn oscillator. The reason for such a selection lies in the fact that a Gunn oscillator, although its output power is low compared to an IMPATT oscillator, has a purer spectral character.

Let us denote by \( S_g(f) \) and \( S_I(f) \) the phase noise spectral densities for a free-running Gunn and IMPATT oscillators respectively. Then in the present situation, \( S_g(f) \) corresponds to the spectral density \( S_{\phi_1}(f) \) of the synchronizing signal in (7.48). Again, \( S_I(f) \) is related to \( S_a(f) \) by the expression (cf. equation 7.18),

\[
S_I(f) = \frac{1}{(1 + \frac{3}{4}A^2)^2} \left( 1 + \frac{f_0^2}{4Q^2 f^2} \right) S_a(f)
\]

\[\text{(7.49)}\]

Using this equation (7.48), may then be written in following form
\[ S_\varphi(f) = \frac{K_1^2}{K_1^2 + f^2} S_G(f) + \frac{f^2}{K_1^2 + f^2} S_1(f) \quad \text{...(7.50)} \]

where,
\[ K_1 = \frac{K}{1 + \frac{3}{4} SA^2} \quad \text{...(7.51)} \]

This equation (7.50) indicates that for smaller values of \( f \) (i.e., close to the free-running frequency) the spectral character of the synchronized IMPATT follows closely to that of the free-running Gumm oscillator. At very high values of \( f \) (i.e., far from the free-running frequency), however, the phase noise spectral densities of the free-running and synchronized IMPATT merge together. These have been shown in Fig. 7.4.

### 7.2.3. Ultraharmonic Synchronization:

Let us now consider the spectral purification achieved through ultraharmonic synchronization of an oscillator. We take up ultraharmonic synchronization of order three, i.e., when the oscillator frequency is three times the synchronizing signal frequency. Assuming the incoming signal to be \( E\sin(wt+\vartheta_1) \), the oscillator output \( v \) in this case may be written as,

\[ v = E_1\sin(wt+\vartheta_1+\vartheta_2)+A\sin(3wt+\varphi) \quad \text{...(7.52)} \]

As in (7.44) the loop equation for the ultraharmonic oscillator assumes the form,

\[ Y(p)\sqrt{E\sin(wt+\vartheta_1)+N(v)}7+A\cos wt = E_1\sin(wt+\vartheta_1+\vartheta_2)+A\sin(3wt+\varphi) \quad \text{...(7.53)} \]
Fig. 7-4. Plot of phase spectral densities \( S_\phi(f) \) for free-running and synchronized IMPATT and Gunn oscillators.
With the help of the relations (7.7), (7.10) and (7.52) and assuming $\phi$ and $a(t)$ to be small, the oscillator phase equation may easily be derived from (7.53) and is given by,

$$(1+\frac{3\beta A^2}{4}) \frac{d\phi}{dt} = \Omega + \frac{da}{dt} + \frac{w_0}{2Q} a + 3K_LQ_1\cdot K_L \phi \quad \ldots (7.54)$$

Here,

$$\Omega = \frac{w_0}{2} \left(\frac{3w}{w_0} - \frac{w_0}{3w}\right)$$

and denotes the detuning of oscillator frequency ($w_0$) from three times the synchronizing frequency ($w$).

And,

$$K_L = \gamma_3 \frac{E_1 w_0}{24Q} \quad \ldots (7.55)$$

which gives the zone of entrainment for ultraharmonic synchronization.

If we consider the in-tune case, as in harmonic synchronization, we can express the oscillator phase from (7.54) in the form,

$$\varphi(p) = \frac{3K_L}{p+K_L} \theta_1 + \frac{1}{(1+\frac{3\beta A^2}{4})} \frac{w_0}{2Q+p} \frac{w_0}{p+K_L} a \quad \ldots (7.56)$$

where,

$$K_{LL} = \frac{K_L}{(1+\frac{3\beta A^2}{4})} \quad \ldots (7.57)$$

Equation (7.56) leads to the phase noise spectral density for ultraharmonic oscillator and is given by,
\[ S_\varphi(f) = \frac{9K_{LL}^2}{K_{LL}^2 + f^2} S_{g1}(f) + \frac{\gamma}{(K_{LL}^2 + f^2)} \frac{1}{(1 + \frac{\beta^2}{4Q^2 f^2})} \]

\[ \text{(7.58)} \]

Assuming, as in the harmonic case, that an IMPATT oscillator is synchronized by injection from a Gunn oscillator, we can express (7.58) in the form,

\[ S_\varphi(f) = \frac{9K_{LL}^2}{K_{LL}^2 + f^2} S_g(f) + \frac{\gamma}{(K_{LL}^2 + f^2)} S_1(f) \]

\[ \text{(7.59)} \]

\( S_1(f) \) being expressed exactly as in (7.49).

Equation (7.59) furnishes the variation of the phase spectral density for an ultraharmonically locked IMPATT oscillator with the noise frequency \( f \). In order to compare this plot with that of harmonic synchronization, the output in (7.59) is frequency divided by three and the corresponding spectral density \( \frac{1}{3} S_\varphi(f) \) is plotted in Fig. 7.4.

7.2.4. Evaluation of the Spectral Densities \( S_g(f) \) and \( S_1(f) \):

It is evident from (7.50) & (7.59) that a knowledge of the phase spectral densities \( S_g(f) \) and \( S_1(f) \) for the free-running Gunn and IMPATT oscillators is required to determine the phase noise of the IMPATT oscillator synchronized either in harmonic mode or in ultraharmonic mode by the other oscillator. The expression for \( S_g(f) \) or \( S_1(f) \) has already been derived and can be written as (cf (7.16) & (7.19)).
\[ S_f(f) (\text{or } S_i(f)) = \frac{1}{(1 + \frac{3}{4}B^2)^2} \left( \frac{a_{-1} f^2}{4Q_1^2} + \frac{a_0^2 f^2}{4Q_2^2} \right) + \frac{a_{-1}}{1} + a_0 \cdots (7.60) \]

In order to obtain the values of the constants \(a_{-1}\) and \(a_0\), we refer to the work of Kroupa [VII-67]. In the said paper, the coefficients \(h_{-1}\) (related to \(a_{-1}\)) and \(h_0\) (related to \(a_0\)) have been plotted as functions of loaded \(Q\) from a wealth of published data for different types of oscillators. It has been concluded there that the mean values of the flicker noise constant \(a_{-1}\) and the white noise constant \(a_0\) are practically independent of the oscillator type starting from IC transistor oscillators to reflex klystrons in the frequency range from 5 MHz to 100 GHz. It is felt that for a simplified oscillator noise theory the assumption is quite reasonable. However, a close scrutiny of the experimental points for the Gunn and IMPATT type of oscillators in the said plot of normalised phase-noise coefficients in respect of loaded \(Q\) reveals that there is a difference in the mean values of \(a_{-1}\) and \(a_0\) for the two types of oscillators. In the present work the results for Gunn and IMPATT oscillators have been marked separately (cf. Fig. 7.5) and the mean values of \(a_{-1}\) and \(a_0\) for them have been calculated. The coefficients \(h_{-1}\) and \(h_0\) plotted in the graph [VII-67] bear the following relations with the constants \(a_{-1}\) and \(a_0\) of (7.60) viz.,
Fig. 7.5. Plot of normalised phase-noise coefficients ($h_{-1}$ and $h_0$) in respect of loaded $Q$ for Gunn and IMPATT oscillators.
\[
q_{-1} = \frac{1}{(1 + \frac{3}{4}A^2)^2} \frac{a_{-1}}{4Q^2}
\]
and
\[
h_0 = \frac{1}{(1 + \frac{3}{4}A^2)^2} \frac{a_0}{4Q^2}
\]

From the graphical plot in Fig. 7.5, the values of \(a_{-1}\) and \(a_0\) for the Gunn and IMPATT oscillators come out to be:

\[
a_{-1} = 2.42 \times 10^{-13}
\]
\[
a_0 = 1.21 \times 10^{-17}
\]

for the Gunn oscillator.

And

\[
a_{-1} = 1.21 \times 10^{-11}
\]
\[
a_0 = 3.05 \times 10^{-16}
\]

for the IMPATT oscillator.

In calculating these constants we have assumed, \((1 + \frac{3}{4}A^2) = 1.1\).

Once these constants are known it is easy to obtain the values of \(S_g(f)\) and \(S_I(f)\) using (7.60).

7.3. Conclusions:

The phase spectral density for a free-running oscillator, be it a feed-back type or a negative-resistance one, has been derived in a very simple and straightforward approach without any heuristic model as imagined in earlier works.
It has been shown that unlike in feedback type of oscillators, the phase spectral density of Gunn and IMPATT oscillators has a clear dependence on the free-running amplitude of the oscillator.

Synchronizing an oscillator improves the spectral purity compared to a free-running oscillator. But the harmonic mode of synchronization yields a better result compared to ultraharmonic mode of synchronization.
REFERENCES


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