CHAPTER - VI

D-SPACES AND THE ASSOCIATED TOPOLOGIES

As we have seen that in [20], Crossley and Hildebrand have made an operator approach in defining the $F(\tau)$ topology which is in fact, equivalent to the topology $\tau^\alpha$ owing to Njastad [49]. Rose and Hamlett in their paper [61] studied $\alpha$-topology using the ideal of the collection of nowhere dense subsets. We, in this chapter, shall use a different technique in arriving at the $\alpha$-topology by means of a binary relation $\theta$ defined on the power set $\mathcal{P}(X)$ of a set $X$. Moreover, given a topological space $(X, \tau)$, it will happen that such a relation $\theta$ can be defined on $\mathcal{P}(X)$ so that $\tau^*(\theta) = \tau^\alpha$ where $\tau^*(\theta)$ is the topology induced by $\theta$. Now note that topological spaces $(X, \tau)$ and $(X, \tau^\alpha)$ possess the same collection of dense subsets and thereby satisfy the property that if one of the spaces has the property $P$, the other also has the property $P$, where $P$ stands for resolvability, irresolvability, open hereditary irresolvability, hyperconnectedness and extremally disconnectedness. These observations enable us to finding conditions on the relation $\theta$ by which resolvable, irresolvable, o.h.i., hyperconnected and extremally disconnected spaces can be described completely.
6.1. D-SPACE AND THE ASSOCIATED $\alpha$-TOPOLOGY

Let $X$ be a nonempty set and $\mathcal{P}(X)$ be its power set.

**Definition 6.1.1.** A binary relation $\theta$ on $\mathcal{P}(X)$ will be called a D-relation if $\theta$ satisfies the following axioms:

1. **D$_1$.** $A \theta B \iff A \cap B \neq \emptyset$,
2. **D$_2$.** $A \cup \forall \alpha \in \Lambda \theta (\cup A \alpha) \theta B$,
3. **D$_3$.** $A \theta B$ and $C \theta D \implies A \theta D$ and $C \theta B$,
4. **D$_4$.** $A \theta X \implies$ there exists a nonempty set $C \subseteq A$ satisfying the condition that $(P \cap C) \in E$ whenever $P \in E$ and $P \cap C \neq \emptyset$,
5. **D$_5$.** $A \neq \emptyset$, $A \neq X \implies$ there exists $0 \subseteq X$ satisfying the following conditions:
   (i) $A \cap 0 \neq \emptyset$ and if $P \in E$, $P \cap 0 \neq \emptyset$ then $(P \cap 0) \in E$,
   (ii) If $\emptyset \neq B \subseteq A \cap 0$ then there exist subsets $P$, $E$ such that $P \in E$, $P \cap B \neq \emptyset$, but $(P \cap B) \notin E$,
6. **D$_6$.** $X \neq Y \implies$ there exists $P \subseteq X - Y$ such that $P \in E$.

If $\theta$ is a D-relation on $\mathcal{P}(X)$ then $(X, \theta)$ will be called a D-space.

**Theorem 6.1.1.** If $(X, \theta)$ is a D-space then

(a) $X \theta X$
(b) $A \theta B$ and $B \subseteq D \implies A \theta D$.
Proof. (a) If \( X \not\in X \) then by axiom \( D_6 \) it follows that \( \emptyset \in X \) which contradicts axiom \( D_1 \).

(b). Let \( A \not\subseteq D \). Then \( X \not\subseteq D \); otherwise by axiom \( D_3 \), \( A \in B \) and \( X \in D \) would imply \( A \in D \). Hence by axiom \( D_6 \), there exists \( C \subseteq X - D \) such that \( C \not\subseteq X \). Thus by axiom \( D_3 \), \( C \not\subseteq B \), which by axiom \( D_1 \) implies \( C \cap B \not\in \emptyset \). But \( C \subseteq X - D \) whereas \( B \subseteq D \); a contradiction. This completes the proof.

Let \((X, \emptyset)\) be a \( D \)-space. If

\[
\tau^*(\emptyset) = \{ A \subseteq X : P \in E \text{ and } P \cap A \not\in \emptyset \Rightarrow (P \cap A) \in E \},
\]

then it can be verified that \( \tau^*(\emptyset) \) is a topology on \( X \).

Given any \( D \)-space \((X, \emptyset)\), \( \tau^*(\emptyset) \) has the following properties:

Theorem 6.1.2. (i) \( B \subseteq X \) is dense in \( \tau^*(\emptyset) \) if and only if \( A \in B \) for some \( A \subseteq X \).

(ii) \( A \not\subseteq \emptyset \subseteq X \) is semi-open in \( \tau^*(\emptyset) \) if and only if \( A \in \emptyset \).

Proof: (i). Suppose that \( B \) is dense in \( \tau^*(\emptyset) \). Then we claim that \( X \not\subseteq B \), otherwise by axiom \( D_6 \), there exists \( A \subseteq X - B \) such that \( A \not\subseteq X \). Hence by axiom \( D_4 \) and definition of \( \tau^*(\emptyset) \), we have \( \text{int}_{\tau^*(\emptyset)} A \not\subseteq \emptyset \). But \( B \) is dense in \( \tau^*(\emptyset) \) and \( A \subseteq X - B \). Thus a contradiction. Hence \( X \not\subseteq B \).
Conversely, suppose that \( A \varsubsetneq B \) for some \( A \subseteq X \). Let \( \emptyset \neq H \in \mathcal{T}^*(\emptyset) \). Now from Theorem 6.1.1 (a) and definition of \( \mathcal{T}^*(\emptyset) \), it follows that \( H \varsubsetneq X \). Therefore, by axiom D_3, we get \( H \varsubsetneq B \) and by axiom D_1, \( H \cap B \neq \emptyset \). Hence \( B \) is dense in \( \mathcal{T}^*(\emptyset) \).

(ii). Let \( A \) be a nonempty semi-open set in \( \mathcal{T}^*(\emptyset) \). Then \( A \varsubsetneq X \), otherwise by axiom D_5, there exists an open set \( 0 \) in \( \mathcal{T}^*(\emptyset) \) with \( A \cap 0 \neq \emptyset \) and \( \text{int}_{\mathcal{T}^*(\emptyset)} (A \cap 0) = \emptyset \). But this contradicts the fact that \( A \cap 0 \) is a nonempty semi-open set in \( \mathcal{T}^*(\emptyset) \).

Conversely, suppose that \( A \varsubsetneq X \). Let \( x \in A \). We show that \( x \in \text{cl}_{\mathcal{T}^*(\emptyset)} \text{int}_{\mathcal{T}^*(\emptyset)} A \). Let \( \emptyset \neq H \in \mathcal{T}^*(\emptyset) \) and \( x \in H \). Then from definition of \( \mathcal{T}^*(\emptyset) \), it follows that \( (A \cap H) \varsubsetneq X \) and by axiom D_4, we get \( \text{int}_{\mathcal{T}^*(\emptyset)} (A \cap H) \neq \emptyset \). Hence \( x \in \text{cl}_{\mathcal{T}(\emptyset)} \text{int}_{\mathcal{T}(\emptyset)} A \). Thus \( A \) is semi-open in \( \mathcal{T}^*(\emptyset) \). This completes the proof.

Let \( (X, \mathcal{T}) \) be a topological space and \( (X, \emptyset) \) be a D-space. Then

**Definition 6.1.2.** (i) \( \emptyset \) is compatible with \( \mathcal{T} \) if \( \mathcal{T}^*(\emptyset) = \mathcal{T} \),

(ii) \( \emptyset \) is weakly compatible with \( \mathcal{T} \) if \( \mathcal{T}^*(\emptyset) \) and \( \mathcal{T} \) are \( \alpha \)-equivalent i.e., \( (\mathcal{T}^*(\emptyset))^{\alpha} = \mathcal{T}^{\alpha} \).

**Definition 6.1.3.** [49] A topology \( \mathcal{T} \) on \( X \) is said to be an \( \alpha \)-topology if \( \mathcal{T} = \mathcal{T}^{\alpha} \).

Note that \( \mathcal{T} = \mathcal{T}^{\alpha} \) if and only if and only if every nowhere dense subset in \( (X, \mathcal{T}) \) is closed in \( (X, \mathcal{T}) \).
Theorem 6.1.3. Given any D-space \((X, \Theta)\), \(\tau^*(\Theta)\) is an \(\alpha\)-topology.

Proof. It suffices to prove that every nowhere dense subset in \(\tau^*(\Theta)\) is closed in \(\tau^*(\Theta)\). Let \(A\) be nowhere dense in \(\tau^*(\Theta)\). Then \(\text{int}_{\tau^*(\Theta)} \text{cl}_{\tau^*(\Theta)} A = \emptyset\), i.e.,

\[
(6.1.1) \quad \text{cl}_{\tau^*(\Theta)} \text{int}_{\tau^*(\Theta)} (X - A) = X.
\]

We show that \(X - A\) is open in \(\tau^*(\Theta)\). So let \(P \Theta E\). Then by Theorem 6.1.1(b), \(P \Theta X\) and hence by Theorem 6.1.2(ii) \(P\) is semi-open in \(\tau^*(\Theta)\). Hence it follows from equation (6.1.1) that

\[
\emptyset \neq P \cap (X - A) \subseteq \text{cl}_{\tau^*(\Theta)} \text{int}_{\tau^*(\Theta)} P = \text{cl}_{\tau^*(\Theta)} \left[ \text{int}_{\tau^*(\Theta)} P \cap \text{int}_{\tau^*(\Theta)} (X - A) \right].
\]

Hence \(P \cap (X - A) \subseteq \text{cl}_{\tau^*(\Theta)} \text{int}_{\tau^*(\Theta)} [P \cap (X - A)].\) Thus \(P \cap (X - A)\) is semi-open in \(\tau^*(\Theta)\), which shows again by Theorem 6.1.2(ii) that \(P \cap (X - A) \Theta X\). So \(P \Theta E\) and \((P \cap (X - A)) \Theta X\), which by axiom D3 imply \((P \cap (X - A)) \Theta E\). Thus \(P \Theta E \Rightarrow (P \cap (X - A)) \Theta E\). Hence \(X - A\) is open in \(\tau^*(\Theta)\) i.e., \(A\) is closed in \(\tau^*(\Theta)\).

Theorem 6.1.4. For any topological space \((X, \tau)\) there exists a D-space \((X, \Theta)\) such that \(\Theta\) is weakly compatible with \(\tau\).

Proof. Define \(\Theta\) by \(A \Theta B\) if and only if \(A\) is a nonempty semi-open set and \(B\) is dense in \((X, \tau)\). Then the axioms
D₁, D₂, D₃ hold readily. For D₄ we take C = \text{int}_\tau A. For D₅ we take 0 = X - \text{cI}_\tau \text{int}_\tau A, and for D₆ we take P = \text{int}_\tau (X - B).

Let us verify D₅. Suppose A ≠ ∅, A ⊆ X. Then by definition, A is not semi-open in \( \tau \). So 0 is open in \( \tau \) with A ∩ 0 ≠ ∅. Also if P \( \not\in \) E, and P ∩ 0 ≠ ∅ then P ∩ 0 is semi-open and hence (P ∩ 0) \( \not\in \) E. Again if ∅ ≠ B ⊆ A ∩ 0, then P = X = E will serve our purpose, because, otherwise if (P ∩ B) \( \not\in \) E i.e., B \( \not\in \) X then by definition of \( \theta \), \text{int}_\tau B ≠ ∅, but B ⊆ A ∩ 0 where \text{int}_\tau (A ∩ 0) = ∅. Thus D₅ is verified. Hence (X, \( \theta \)) becomes a D-space. Now we show that \( \tau^\ast(\theta) = \tau^\alpha \). Let A \( \in \) \( \tau^\alpha \). Let P \( \not\in \) E and P ∩ A ≠ ∅. Then by definition of \( \theta \), P is semi-open in \( \tau \) and since A \( \in \) \( \tau^\alpha \), from [49] it follows that P ∩ A is semi-open in \( \tau \). Hence (P ∩ A) \( \not\in \) E. Thus A \( \in \) \( \tau^\ast(\theta) \) i.e., \( \tau^\alpha \subseteq \tau^\ast(\theta) \). For the reverse inclusion, let A \( \in \) \( \tau^\ast(\theta) \). To show A \( \in \) \( \tau^\alpha \), it suffices to prove by [49] that whenever B is semi-open in \( \tau \), A ∩ B is semi-open in \( \tau \). Let B be semi-open in \( \tau \). If A ∩ B = ∅ then we are done. If A ∩ B ≠ ∅ then by definition of \( \theta \), B \( \not\in \) X and hence (A ∩ B) \( \not\in \) X (since A \( \in \) \( \tau^\ast(\theta) \)). Thus A ∩ B is semi-open in \( \tau \). Hence A \( \in \) \( \tau^\alpha \) i.e., \( \tau^\ast(\theta) \subseteq \tau^\alpha \). Consequently, \( \tau^\alpha = \tau^\ast(\theta) \). Hence by Definition 6.1.2(i), \( \theta \) is weakly compatible with \( \tau \).

Theorem 6.1.5. The following statements are equivalent for a topological space \((X, \tau)\):

(i) \( \tau \) is an \( \alpha \)-topology,

(ii) there exists a D-space \((X, \theta)\) for which \( \theta \) is compatible with \( \tau \).

Proof follows from Theorem 6.1.3 and Theorem 6.1.4.
6.2. CHARACTERIZATION OF SOME KNOWN SPACES

Definition 6.2.1. In a $D$-space $(X, \theta)$

(i) $\theta$ will be said to satisfy the conditions (A) if for any two subsets $A, B$

$$A \not\in B \implies A \in X - B ,$$

(ii) $\theta$ will be said to satisfy the condition (B) if for any two subsets $A, B$

$$X \not\in A \text{ and } X \not\in B \implies X \not\in (A \cap B) ,$$

(iii) $\theta$ will be said to satisfy the condition (C) if for any two subsets $A, B$

$$A \not\in X \text{ and } B \not\in X \implies (A \cap B) \not\in X ,$$

(iv) $\theta$ will be said to satisfy the condition (D) if for any two subsets $A, B$

$$A \not\in B \implies B \not\in A ,$$

(v) $\theta$ will be said to satisfy the condition (E) if for any two subsets $A, B$

$$A \not\in X, \ B \not\in X, \ A \cap B \not\in \emptyset \implies (A \cap B) \not\in X .$$

Theorem 6.2.1. Given a $D$-space $(X, \theta)$, $\tau^*(\theta)$ is irresolvable if and only if $\theta$ satisfies the condition (A).
Proof. If $\tau^*(\theta)$ is irresolvable then any two dense subsets in $\tau^*(\theta)$ intersect. Hence if $A \not\subset B$ then by Theorem 6.1.2(i) $A \not\subset X - B$. Conversely, let the condition (A) hold. Then for any dense subset $D$ in $\tau^*(\theta)$, $X \not\subset D$ and hence $X \not\subset X - D$. Therefore by axioms $D_6$ and $D_4$ we have $\text{int}_{\tau^*(\theta)} D \neq \emptyset$ i.e., $X - D$ is not dense in $\tau^*(\theta)$. Thus $\tau^*(\theta)$ is irresolvable.

Theorem 6.2.2. The following statements are equivalent for a topological space $(X, \tau)$:

(i) $(X, \tau)$ is irresolvable,

(ii) there exists a D-space $(X, \theta)$ where $\theta$ is weakly compatible with $\tau$ and $\theta$ satisfies the condition (A).

Proof: (i) $\Rightarrow$ (ii). Define $\theta$ by $A \theta B$ if and only if $A$ is a nonempty semi-open set and $B$ is a dense set in $(X, \tau)$. Then we have already seen in Theorem 6.1.4 that with this definition of $\theta$, $(X, \theta)$ becomes a D-space such that $\theta$ is weakly compatible with $\tau$. Hence $\tau$ and $\tau^*(\theta)$ have the same collection of dense subsets. Since $\tau$ is irresolvable, $\tau^*(\theta)$ is so and hence by Theorem 6.2.1, $\theta$ satisfies the condition (A).

(ii) $\Rightarrow$ (i). By Theorem 6.2.1, $\tau^*(\theta)$ is irresolvable and since $\theta$ is weakly compatible with $\tau$, $\tau$ is irresolvable.

The following theorem is obvious from Theorem 6.2.2.
Theorem 6.2.3. The following statements are equivalent for a topological space \((X, \tau)\):

i) \((X, \tau)\) is resolvable,

ii) there exists a D-space \((X, \theta)\) where \(\theta\) is weakly compatible with \(\tau\) and \(\theta\) does not satisfy the condition (A).

Note 6.2.1. We know that a topological space \((X, \tau)\) is resolvable if and only if it possesses a pair of disjoint dense subsets \(D\) and \(X - D\). Now in a D-space \((X, \theta)\), if \(\theta\) does not satisfy the condition (A) then there exist subsets \(P\) and \(E\) such that \(P \cap E = \emptyset\) and \(E \subset X - E\). This exactly implies that \(\tau^*(\theta)\) possesses disjoint dense subsets \(E\) and \(X - E\), which implies by Theorem 6.2.3, that \(\tau^*\) possesses disjoint dense subsets \(E\) and \(X - E\) since \(\tau^* = (\tau^*(\theta))^\alpha = \tau^*(\theta)\), i.e., \(\tau\) possesses disjoint dense sets \(E\) and \(X - E\). Thus resolvability as well as irresolvability (Theorem 6.2.2.) have been described in terms of a relation on \(\mathcal{P}(X)\).

Theorem 6.2.4. Given a D-space \((X, \theta)\), \(\tau^*(\theta)\) is o.h.i. if and only if \(\theta\) satisfies the condition (B).

Proof. We know from Theorem 1.2.5. and Theorem 2.3.6. that a topological space \((X, \tau)\) is o.h.i. if and only if the collection of dense subsets forms a filter on \(X\). Now if \(\tau^*(\theta)\) is o.h.i. then using Theorem 6.1.2(i) we see that \(\theta\) satisfies the condition (B).
Conversely, if \( \Theta \) satisfies the condition (B) then by Theorem 6.1.1.(b) and Theorem 6.1.2(i) it follows that the collection of dense subsets in \( \tau^*(\Theta) \) forms a filter and hence \( \tau^*(\Theta) \) is o.h.i.

**Corollary 6.2.1.** Given any D-space \((X, \Theta)\), \( \tau^*(\Theta) \) is sub-maximal if and only if \( \Theta \) satisfies the condition (B).

Proof follows directly from Theorem 2.3.6.

**Theorem 6.2.5.** The following statements are equivalent for a topological space \((X, \mathcal{T})\):

(i) \((X, \mathcal{T})\) is o.h.i.,

(ii) there exists a D-space \((X, \Theta)\) for which \( \Theta \) is weakly compatible with \( \mathcal{T} \) and \( \Theta \) satisfies the condition (B).

Proof follows from Theorem 6.2.4 and an argument similar to that in the proof of Theorem 6.2.2.

**Theorem 6.2.6.** Given a D-space \((X, \Theta)\), \( \tau^*(\Theta) \) is hyperconnected if and only if \( \Theta \) satisfies the condition (C).

Proof. Suppose \( \tau^*(\Theta) \) is hyperconnected and let \( A \Theta X, B \Theta X \). Obviously, \( A, B \) are nonempty semi-open in \( \tau^*(\Theta) \). Hence \( A \cap B \) is semi-open, being a superset of a nonempty open set in the hyperconnected space \((X, \tau^*(\Theta))\). So \((A \cap B) \Theta X \).

Conversely, let the condition (C) hold for the D-space \((X, \Theta)\). Take any two nonempty open sets \( O_1 \) and \( O_2 \) in \( \tau^*(\Theta) \).
Then $0_1 \emptyset X$ and $0_2 \emptyset X$ and hence $(0_1 \cap 0_2) \emptyset X$ and so by axiom $D_1$, $0_1 \cap 0_2 \neq \emptyset$. Thus $\tau^*(\emptyset)$ is hyperconnected.

**Theorem 6.2.7.** The following statements are equivalent for a topological space $(X, \tau)$:

(i) $(X, \tau)$ is hyperconnected,

(ii) there exists a D-space $(X, \emptyset)$ for which $\emptyset$ is weakly compatible with $\tau$ and $\emptyset$ satisfies the condition (C).

Proof follows from Theorem 6.2.6. and an argument similar to that in the proof of Theorem 6.2.2.

**Theorem 6.2.8.** Given a D-space $(X, \emptyset)$, $\tau^*(\emptyset)$ is hyperconnected and irresolvable if and only if $\emptyset$ satisfies the condition (D).

Proof. Suppose that $\tau^*(\emptyset)$ is hyperconnected and irresolvable. Also let $A \emptyset B$. Then by Theorem 6.1.2. and Theorem 6.1.1., $A$ is semi-open and $B$ is dense in $\tau^*(\emptyset)$. Since $\tau^*(\emptyset)$ is hyperconnected and irresolvable, $A$ is dense and $B$ is semi-open and hence $B \emptyset X$ and $E \emptyset A$ for some $E \subseteq X$. Thus by axiom $D_3$, $B \emptyset A$. Hence $\emptyset$ satisfies the condition (D).

Conversely, suppose that the condition (D) holds for $\emptyset$. Let $A(\neq \emptyset) \in \tau^*(\emptyset)$. Then $A \emptyset X$ and hence $X \emptyset A$ implies $A$ is dense in $\tau^*(\emptyset)$. Thus $\tau^*(\emptyset)$ is hyperconnected. Again if $D$ is dense in $\tau^*(\emptyset)$, then $X \emptyset D \implies D \emptyset X \implies \text{int}^* \tau^*(\emptyset) D \neq \emptyset$, by axiom $D_4$. Thus $\tau^*(\emptyset)$ is irresolvable also.
Theorem 6.2.9. The following statements are equivalent for a topological space \((X, \tau)\):

(i) \((X, \tau)\) is hyperconnected and irresolvable,

(ii) there exists a D-space \((X, \theta)\) for which \(\theta\) is weakly compatible with \(\tau\) and \(\theta\) satisfies the condition \((D)\).

Proof. \((i) \implies (ii)\). Define \(\theta\) as in the proof of \((i) \implies (ii)\) in Theorem 6.2.2. Then \(\theta\) also satisfies condition \((D)\).

\((ii) \implies (i)\). Let \(A (\neq \emptyset) \in \tau\). We claim \(X \not\in A\). For, if \(X \not\in A\) then by axiom \(D_6\), there exists \(P \subseteq X - A\) such that \(P \not\in X\). Then by condition \((D)\), \(X \not\in P\) i.e., \(P\) is dense in \(\tau^*(\theta)\) and hence dense in \(\tau\). But \(P \cap A = \emptyset\), a contradiction. Therefore \(X \not\in A\) and thus \(A\) is dense in \(\tau^*(\theta)\) i.e., dense in \(\tau\). Hence \((X, \tau)\) is hyperconnected. Also \((X, \tau)\) is irresolvable because if \(D\) is dense in \(\tau\), then \(D\) is dense in \(\tau^*(\theta)\) and consequently \(X \not\in D\) and by condition \((D)\) and axioms \(D_1\) and \(D_3\), \(X \not\in X - D\) i.e., \(X - D\) is not dense in \(\tau^*(\theta)\) and hence not dense in \(\tau\).

As we see that a hyperconnected irresolvable topological space is o.h.i. then it is natural to ask whether in a D-space \((X, \theta)\), if \(\theta\) satisfies the condition \((D)\) then it must satisfy the condition \((B)\). An affirmative answer to this question is given by the following argument.
Suppose \( \emptyset \) satisfies the condition (D). Let \( X \emptyset A \) and \( X \emptyset B \). Then \( A \emptyset X \) and \( B \emptyset X \). If possible, let \( X \not\subseteq (A \cap B) \). Then by axiom \( D_6 \), there exists \( C \subseteq X - (A \cap B) \) such that \( C \emptyset X \). Therefore \( X \emptyset C \) and by axioms \( D_3 \) and \( D_1 \) we get \( A \emptyset C \), \( B \emptyset C \), \( A \cap C \not\emptyset \), \( B \cap C \not\emptyset \). Now by axiom \( D_4 \), there exists a nonempty set \( D \subseteq C \) such that if \( P \emptyset E \) and \( P \cap D \not\emptyset \) then \( (P \cap D) \emptyset E \). We now contend that \( C \) satisfies the condition that if \( P \emptyset E \) and \( P \cap C \not\emptyset \) then \( (P \cap C) \emptyset E \). So let \( P \emptyset E \) and \( P \cap C \not\emptyset \). We claim that \( P \cap D \not\emptyset \). For, if \( P \cap D = \emptyset \), then \( P \subseteq X - D \). Now \( P \emptyset E \Rightarrow E \emptyset P \Rightarrow E \emptyset X - D \). But \( X \emptyset X \Rightarrow D \emptyset X \). Hence \( E \emptyset X - D \) and \( D \emptyset X \) imply \( D \emptyset X - D \), contradicting the axiom \( D_1 \). Thus \( P \cap D \not\emptyset \). Now it follows that \( (P \cap D) \emptyset E \). Therefore \( E \emptyset (P \cap D) \Rightarrow E \emptyset (P \cap C) \Rightarrow (P \cap C) \emptyset E \). Hence our contention is true. Now

\[
A \emptyset X \text{ and } A \cap C \not\emptyset \Rightarrow (A \cap C) \emptyset X,
\]

\[
B \emptyset X \text{ and } B \cap C \not\emptyset \Rightarrow (B \cap C) \emptyset X.
\]

Therefore, \( X \emptyset (A \cap C) \) and \( (B \cap C) \emptyset X \Rightarrow (B \cap C) \emptyset (A \cap C) \) (by axiom \( D_3 \)) \( B \cap C \cap A \not\emptyset \) (by axiom \( D_4 \)), which contradicts the fact that \( C \subseteq X - (A \cap B) \). Hence \( X \emptyset (A \cap B) \). Thus \( \emptyset \) satisfies the condition (B).

**Theorem 6.2.10.** Given a \( D \)-space \( (X, \emptyset) \), \( \tau^*(\emptyset) \) is extremally disconnected if and only if \( \emptyset \) satisfies the condition (E).

**Proof.** Suppose that \( \tau^*(\emptyset) \) is extremally disconnected. Then by Proposition 2 in [28], every semi-open set is \( \alpha \)-open and
hence open in \( \mathcal{T}^*(\theta) \). Consequently, \( \theta \) satisfies the condition (E).

Conversely, let the condition (E) hold. Let \( A \) be a nonempty semi-open set in \( \mathcal{T}^*(\theta) \). Then \( A \not\in X \) and by definition of \( \mathcal{T}^*(\theta) \), it follows that \( A \) is open in \( \mathcal{T}^*(\theta) \). Thus again by Proposition 2 in [28] it follows that \( \mathcal{T}^*(\theta) \) is extremally disconnected.

**Theorem 6.2.11.** The following statements are equivalent for a topological space \((X, \mathcal{T})\):

1. \((X, \mathcal{T})\) is extremally disconnected,
2. there exists a D-space \((X, \theta)\) for which \( \theta \) is weakly compatible with \( \mathcal{T} \) and \( \theta \) satisfies the condition (E).

**Proof.** (i) \(\Rightarrow\) (ii). Define \( \theta \) as in the proof of (i) \(\Rightarrow\) (ii) in Theorem 6.2.2. Then \((X, \theta)\) is a D-space and \( \theta \) is weakly compatible with \( \mathcal{T} \) i.e., \( \mathcal{T}^*(\theta) = (\mathcal{T}^*(\theta))^\alpha = \mathcal{T}^\alpha \). Now suppose \( A \in X, B \not\in X, A \cap B \neq \emptyset \). Then \( A, B \) are semi-open in \( \mathcal{T} \). Since \((X, \mathcal{T})\) is extremally disconnected, \( A, B \in \mathcal{T}^\alpha \) i.e., \( A, B \in \mathcal{T}^*(\theta) \). Hence by definition of \( \mathcal{T}^*(\theta) \), \((A \cap B) \not\in X \).

(ii) \(\Rightarrow\) (i). Let \( A \) be a nonempty semi-open set in \((X, \mathcal{T})\). We show that \( A \in \mathcal{T}^\alpha \). Now since \( \mathcal{T}^*(\theta) = \mathcal{T}^\alpha \), \( A \) is semi-open in \( \mathcal{T}^*(\theta) \). Hence \( A \not\in X \) and since \( \theta \) satisfies the condition (E), by definition of \( \mathcal{T}^*(\theta) \), \( A \) becomes open in \( \mathcal{T}^*(\theta) \). Hence \( A \in \mathcal{T}^\alpha \). Thus \((X, \mathcal{T})\) is extremally disconnected.