CHAPTER - V

RESOLVABILITY AND IRRESOLVABILITY IN BITOPOLITICAL SPACES

The triple \((X, \tau, \sigma)\) where \(X\) is a set and \(\tau\) and \(\sigma\) are topologies on \(X\) is called a bitopological space. Kelly \([38]\) initiated a systematic study of such spaces and thereafter many authors are interested in the problem of how far classical results in topological spaces can be generalized to bitopological spaces. Now it is to be noted that it is not always a routine matter to generalize the topological concepts in bitopological setting. For example, the definition of bitopological compactness given by authors like, say, Birsan \([8]\), Swart \([64]\), was criticized by Reilly in \([57]\) where he pointed out that such pairwise compact bitopological spaces \((X, \tau, \sigma)\) in presence of pairwise Hausdorff property convert in fact, to a compact Hausdorff topological space in the sense that \(\tau = \sigma\). Keeping an eye to such problems our main motivation in this chapter lies in finding whether the idea of resolvability and irresolvability can be strictly generalized to bitopological framework and if so, whether an analogue of Hewitt representation theorem for a topological space holds good for a bitopological space. The investigations are also there to generalize the consequences of Hewitt.
representation theorem and other properties of irresolvable topological spaces; see Theorem 5.2.1, Theorem 5.2.3, Theorem 5.2.8, and Theorem 5.3.1.

5.1. PAIRWISE RESOLVABLE AND PAIRWISE IRRESOLVABLE BITOPOLITICAL SPACES

Based on the classical notion of resolvability and irresolvability for topological spaces we define the corresponding notion for bitopological spaces as follows.

**Definition 5.1.1.** A bitopological space \((X, \tau, \sigma)\) is said to be pairwise resolvable if there exist disjoint subsets \(D\) and \(D^*\) such that \(X = D \cup D^*\) where \(D \in \mathcal{A}(\tau)\) and \(D^* \in \mathcal{A}(\sigma)\); otherwise \((X, \tau, \sigma)\) is said to be pairwise irresolvable.

That the pairwise resolvability of \((X, \tau, \sigma)\) is not governed by the resolvability of the topological spaces \((X, \tau)\) and \((X, \sigma)\) is shown by the following examples.

**Example 5.1.1.** Let \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X, \{a\}\}\) and \(\sigma = \{\emptyset, X, \{b\}, \{b, c\}\}\). Then \((X, \tau, \sigma)\) is pairwise resolvable but \((X, \tau)\) and \((X, \sigma)\) are both irresolvable.

**Example 5.1.2.** Let \(X = \{a, b\}\), \(\tau = \{\emptyset, X\}\) and \(\sigma = \{\emptyset, X, \{a\}, \{b\}\}\). Then \((X, \tau, \sigma)\) is pairwise irresolvable while \((X, \tau)\) is resolvable.
A subset $B$ of $(X, \tau, \sigma)$ is said to be pairwise resolvable if $(B, \tau/B, \sigma/B)$ is pairwise resolvable. $(X, \tau, \sigma)$ is said to be hereditarily pairwise irresolvable if every nonempty subset of $X$ is pairwise irresolvable.

The following theorem comes at once from the definition of a pairwise resolvable bitopological space.

**Theorem 5.1.1.** The following statements are equivalent for a bitopological space $(X, \tau, \sigma)$:

(i) $(X, \tau, \sigma)$ is pairwise resolvable.

(ii) there exists $D \in \mathcal{A}(\tau)$ such that $\text{int}_\sigma D = \emptyset$.

(iii) there exists $D^* \in \mathcal{A}(\sigma)$ such that $\text{int}_\tau D^* = \emptyset$.

(iv) there exists a pairwise resolvable subset $D \in \mathcal{A}(\tau)$.

(v) there exists a pairwise resolvable subset $D^* \in \mathcal{A}(\sigma)$.

**Corollary 5.1.1.** If $A$ is a pairwise resolvable subset of $(X, \tau, \sigma)$ then both $\text{cl}_\tau A$ and $\text{cl}_\sigma A$ are pairwise resolvable.

**Theorem 5.1.2.** Union of a collection of disjoint pairwise resolvable subsets in a bitopological space $(X, \tau, \sigma)$ is pairwise resolvable.

**Proof.** Let $\{A_\alpha\}$ be a collection of disjoint pairwise resolvable subsets in $(X, \tau, \sigma)$. Then for each $\alpha$, $A_\alpha = D_\alpha \cup D^*_\alpha$ where $D_\alpha$ and $D^*_\alpha$ are respectively $\tau$-dense and $\sigma$-dense in $(A_\alpha, \tau/A_\alpha, \sigma/A_\alpha)$ where $D_\alpha \cap D^*_\alpha = \emptyset$. Obviously, $\bigcup D_\alpha$ and
∪ₐ Dₐ* are respectively τ-dense and σ-dense disjoint subsets in (∪ₐ Aₐ, τ/∪ₐ Aₐ, σ/∪ₐ Aₐ).

We now prove the following theorem for a bitopological space which can be considered as a generalization of the Hewitt representation theorem.

Theorem 5.1.3. A pairwise irresolvable bitopological space (X, τ, σ) can be represented as X = G ∪ F = G* ∪ F*, where G, F, G*, F* are respectively τ-open, τ-closed, σ-open and σ-closed subsets of X and G, G* are hereditarily pairwise irresolvable and F, F* are pairwise resolvable with G ∩ F = ∅ and G* ∩ F* = ∅.

Proof. If (X, τ, σ) is hereditarily pairwise irresolvable, F and F* are chosen to be empty. Suppose that (X, τ, σ) is not hereditarily pairwise irresolvable. Then there exists a pairwise resolvable subset A₀, say. We shall first prove the existence of G and F. By Corollary 5.1.1, clₜA₀ is pairwise resolvable. If X − clₜA₀ is hereditarily pairwise irresolvable then we are done. Otherwise, let A₁ ⊆ X − clₜA₀ be pairwise resolvable. Then by Theorem 5.1.2, and Corollary 5.1.1, clₜ(A₀ ∪ A₁) is pairwise resolvable. If X − clₜ(A₀ ∪ A₁) is not hereditarily pairwise irresolvable then select a pairwise resolvable subset A₂ of X − clₜ(A₀ ∪ A₁). Suppose that sets Aₐ have been selected for every ordinal number β, β < α, α also being an ordinal. The set clₜ(∪ₐ Aₐ) is
pairwise resolvable and \( A = X - \operatorname{cl}_\tau (\bigcup_{\beta < \alpha} A_\beta) \) is nonempty, since \((X, \tau, \sigma)\) is pairwise irresolvable. If \( A \) is not hereditarily pairwise irresolvable then \( A_\alpha \) can be selected. Applying induction process, it follows that there exists some ordinal number \( \alpha_0 \) such that \( \operatorname{cl}_\tau (\bigcup_{\alpha < \alpha_0} A_\alpha) \neq X \) and \( \emptyset \neq G = X - \operatorname{cl}_\tau (\bigcup_{\alpha < \alpha_0} A_\alpha) \) is \( \tau \)-open and hereditarily pairwise irresolvable, \((X, \tau, \sigma)\) being pairwise irresolvable. Now \( F = \operatorname{cl}_\tau (\bigcup_{\alpha < \alpha_0} A_\alpha) \) is \( \tau \)-closed and pairwise resolvable and \( X = G \cup F \), as stated in the theorem.

By using \( \sigma \)-closure we can similarly prove the existence of \( G^* \) and \( F^* \).

Now we come to the following definition:

**Definition 5.1.2.** A bitopological space \((X, \tau, \sigma)\) will be said to have a '\( \tau - \sigma \)' representation if \( X = G \cup F = G^* \cup F^* \) where \( G, F, G^*, F^* \) are respectively \( \tau \)-open, \( \tau \)-closed, \( \sigma \)-open and \( \sigma \)-closed subsets of \( X \) and \( G, G^* \) are hereditarily pairwise irresolvable and \( F, F^* \) are pairwise resolvable with \( G \cap F = \emptyset \) and \( G^* \cap F^* = \emptyset \).

From Theorem 5.1.3 it follows that a pairwise irresolvable bitopological space has a '\( \tau - \sigma \)' representation. Also, a pairwise resolvable bitopological space has always a '\( \tau - \sigma \)' representation if we consider \( G = G^* = \emptyset \) and \( F = F^* = X \).

**Remark 5.1.1.** '\( \tau - \sigma \)' representation of a bitopological space \((X, \tau, \sigma)\) may not be unique. Example 5.1.3 shows the
existence of a pairwise resolvable bitopological space
\((X, \mathcal{T}, \mathcal{G})\) where \(G \neq \emptyset\) and consequently \(F \neq X\).

**Example 5.1.3.** Let \(X = \{a, b, c, d\}\)
\[
\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \\
\{a, b, c\}\}
\]
\[
\mathcal{G} = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c\}, \\
\{a, c, d\}\}.
\]

Then \((X, \mathcal{T}, \mathcal{G})\) is pairwise resolvable, since \(X = D \cup D^*\)
where \(D = \{a, b\}, D^* = \{c, d\}\). But \(G = \{a\}\) is \(\mathcal{T}\)-open and
hereditarily pairwise irresolvable and \(F = \{b, c, d\}\) is
\(\mathcal{T}\)-closed and pairwise resolvable.

The following example shows that the '\(\mathcal{T} - \mathcal{G}\)' representa-
tion of a pairwise irresolvable bitopological space also
may not be unique.

**Example 5.1.4.** Let \(X = \{a, b, c, d\}\)
\[
\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}
\]
\[
\mathcal{G} = \{\emptyset, X, \{a\}, \{b\}, \{c, d\}\}.
\]

Then \((X, \mathcal{T}, \mathcal{G})\) is pairwise irresolvable.

Now take \(G = \{a\}, F = \{b, c, d\}\). Then \(G\) is \(\mathcal{T}\)-open and here-
ditarily pairwise irresolvable and \(F\) is \(\mathcal{T}\)-closed and
\[
\mathcal{T}/F = \{\emptyset, F, \{b\}, \{c, d\}\}, \quad \mathcal{G}/F = \{\emptyset, F\}.
\]
Obviously \(F = D \cup D^*\) where \(D = \{b, c\}, D^* = \{d\}\), showing that
\(F\) is pairwise resolvable.
Again if we take $G' = \{a, b\}$ and $F' = \{c, d\}$, then $G'$ is $\tau$-open and hereditarily pairwise irresolvable and $F'$ is $\tau$-closed and $\tau/F' = \{\emptyset, F'\}$, $\sigma/F' = \{\emptyset, F'\}$. Consequently $F'$ is pairwise resolvable.

Hence irrespective of the sets $G'$ and $F'$ we can conclude that $(X, \tau, \sigma)$ has two different $'\tau-\sigma'$ representations.

**Theorem 5.1.4.** If $(X, \tau, \sigma)$ is pairwise irresolvable and $X = G \cup F = G^* \cup F^*$ is a $'\tau-\sigma'$ representation of $X$ then

(i) $\text{int}_\sigma G \neq \emptyset$ and $\text{int}_\tau G^* \neq \emptyset$.
(ii) $\text{int}_\tau \text{int}_\sigma G \neq \emptyset$ and $\text{int}_\sigma \text{int}_\tau G^* \neq \emptyset$.

**Proof.** (i) If $\text{int}_\sigma G = \emptyset$ then $F \in \mathcal{D}(\sigma)$, which implies by Theorem 5.1.1 that $(X, \tau, \sigma)$ is pairwise resolvable, a contradiction. Hence $\text{int}_\sigma G \neq \emptyset$. Similarly it can be shown that $\text{int}_\tau G^* \neq \emptyset$.

(ii) If $\text{int}_\tau \text{int}_\sigma G = \emptyset$, then $X - \text{int}_\sigma G = \text{cl}_\sigma F \in \mathcal{D}(\tau)$, which by Corollary 5.1.1 and Theorem 5.1.1 implies $(X, \tau, \sigma)$ is pairwise resolvable, a contradiction. Hence $\text{int}_\tau \text{int}_\sigma G \neq \emptyset$. Similarly $\text{int}_\sigma \text{int}_\tau G^* \neq \emptyset$.

**Theorem 5.1.5.** Let $(X, \tau, \sigma)$ be pairwise irresolvable and let $S \in \tau \cap \sigma$. Then $S \subseteq G$ if and only if $S \subseteq G^*$, where $X = G \cup F = G^* \cup F^*$ is a $'\tau-\sigma'$ representation of $X$. 
Proof. Let \( S \subseteq G \). If \( S \cap F^* \neq \emptyset \) then \( S \cap F^* \) is both \( \tau / F^* \) open and \( \sigma / F^* \) open. Hence \( S \cap F^* \) would be pairwise resolvable, since \( F^* \) is so (this can be easily verified).

But \( S \cap F^* \subseteq G \), which implies that \( S \cap F^* \) is pairwise irresolvable, since \( G \) is hereditarily pairwise irresolvable. Thus a contradiction arises. Hence \( S \cap F^* = \emptyset \Rightarrow S \subseteq G^* \).

Converse part can be similarly proved.

5.2. OPEN HEREDITARY IRRESOLVABILITY IN BITOPOLOGICAL SPACES

Knowing that o.h.i. topological spaces possess subtle interesting properties, we, in this section, have generalized this concept to bitopological spaces also. Using '\( \tau - \sigma \)' representation of a pairwise irresolvable bitopological space, open hereditary irresolvability in bitopological spaces has been characterized to some extent, some of which may be considered as generalizations of corresponding topological results.

Definition 5.2.1. A bitopological space \((X, \tau, \sigma)\) is said to be \( \tau \)-open hereditarily irresolvable (in brief, \( \tau \)-o.h.p.i.) (respectively, \( \sigma \)-open hereditarily irresolvable, in brief, \( \sigma \)-o.h.p.i.) if every nonempty \( \tau \)-open (respectively, \( \sigma \)-open) subset is pairwise irresolvable.
Definition 5.2.2. A subset $B$ of a bitopological space $(X, \tau, \sigma)$ is said to be $\tau$-preopen with respect to $\sigma$ if $B \subseteq \text{int}_\tau \text{cl}_\sigma B$ and $\sigma$-preopen with respect to $\tau$ if $B \subseteq \text{int}_\sigma \text{cl}_\tau B$.

Let us denote by $\text{PO}(\tau, \sigma)$, the collection of all $\tau$-preopen sets with respect to $\sigma$. The following result holds.

Theorem 5.2.1. $(X, \tau, \sigma)$ is $\tau$-o.h.p.i. ($\sigma$-o.h.p.i.) if and only if for every nonempty subset $A \in \text{PO}(\tau, \sigma)$, $\text{int}_\tau A \neq \emptyset$ ($A \in \text{PO}(\sigma, \tau)$, $\text{int}_\sigma A \neq \emptyset$).

Proof. Suppose that $(X, \tau, \sigma)$ is $\tau$-o.h.p.i. Let $A \in \text{PO}(\tau, \sigma)$ and $A \neq \emptyset$. Then

\[(5.2.1) \quad A \subseteq \text{int}_\tau \text{cl}_\sigma A = H, \text{ say,}\]

If possible, let $\text{int}_\tau A = \emptyset$. Then $(X - A) \cap H$ is dense in $(H, \tau/H)$. Since $(X, \tau, \sigma)$ is $\tau$-o.h.p.i., it follows from Theorem 5.1.1. that $\text{int}_\sigma (X - A) \neq \emptyset$. Let

\[(5.2.2) \quad H \cap K \subseteq X - A\]

where $K \in \sigma$ and $H \cap K \neq \emptyset$. Now from (5.2.1), $A \subseteq H \subseteq \text{cl}_\sigma A$.

So $\text{cl}_\sigma A \cap K \neq \emptyset$ i.e., $A \cap K \neq \emptyset$; a contradiction, since (5.2.1) and (5.2.2) imply $A \cap K \subseteq H \cap K \subseteq X - A$. Thus $\text{int}_\tau A \neq \emptyset$.

Conversely, suppose that for each $A \neq \emptyset \in \text{PO}(\tau, \sigma)$, $\text{int}_\tau A \neq \emptyset$. If possible, let there be a nonempty $\tau$-open
subset $O$ which is pairwise resolvable. Then $O = D \cup D^*$, $D \cap D^* = \emptyset$, $D$ is dense in $(0, \tau / 0)$ and $D^*$ is dense in $(0, \sigma / 0)$. Evidently, $\text{int}_\tau D^* = \emptyset$. But $D^* \subseteq 0 \subseteq \text{int}_\tau \text{cl}_\sigma D^*$, showing that $D^* \in \text{PO}(\tau, \sigma)$ i.e., $\text{int}_\tau D^* \neq \emptyset$ (by hypothesis); a contradiction. Hence $(X, \tau, \sigma)$ is $\tau\text{-o.h.p.i.}$ This completes the proof.

Equality of $\tau$ and $\sigma$ in the above theorem yields the equivalences (i) and (x) of Theorem 2.3.6.

We have noted in Theorem 2.3.6 that the subset $G$ in the Hewitt representation $X = F \cup G$ of a topological space $(X, \tau)$, has played an important role in characterizing o.h.i. spaces. We shall now investigate whether the subsets $G$ and $G^*$ in a $\tau\text{-}\sigma$ representation of a bitopological space can play such a role for obtaining similar bitopological results.

**Theorem 5.2.2.** Let $(X, \tau, \sigma)$ be pairwise irresolvable and let $X = G \cup F = G^* \cup F^*$ be a $\tau\text{-}\sigma$ representation of $X$. Then

(i) $\text{cl}_\sigma \text{int}_\sigma G = X$ if $\text{int}_\tau D \in \mathcal{A}(\sigma)$ for each $D \in \mathcal{A}(\sigma)$,

(Dually, $\text{cl}_\tau \text{int}_\tau G^* = X$ if $\text{int}_\sigma D \in \mathcal{A}(\tau)$ for each $D \in \mathcal{A}(\tau)$)

(ii) $(X, \tau, \sigma)$ is $\sigma\text{-o.h.p.i.}$ if $\text{cl}_\sigma \text{int}_\sigma G = X$

(Dually, $(X, \tau, \sigma)$ is $\tau\text{-o.h.p.i.}$ if $\text{cl}_\tau \text{int}_\tau G^* = X$).
Proof. (i). If possible, let $H$ be a nonempty $\sigma$-open set such that

\[(5.2.3) \quad H \cap \text{int}_\sigma G = \emptyset.\]

Then $H \subseteq X - \text{int}_\sigma G = \overline{G}$. Since $F$ is pairwise resolvable, by Corollary 5.1.1, $\overline{\sigma F}$ is pairwise resolvable. So there exist subsets $D', D''$ such that $\overline{\sigma F} = D' \cup D''$, where $D' \cap D'' = \emptyset$, $D'$ is dense in $(\overline{\sigma F}, \tau/\sigma F)$ and $D''$ is dense in $(\overline{\sigma F}, \sigma/\overline{\sigma F})$. We shall now construct a subset $D$ such that $D \in \mathfrak{A}(\sigma)$ but $\text{int}_\tau D \notin \mathfrak{A}(\sigma)$. Set $D = \text{int}_\sigma G \cup D''$. Clearly $D \in \mathfrak{A}(\sigma)$. We claim that $\text{int}_\tau D \subseteq \text{int}_\sigma G$. In fact, if $x \in \text{int}_\tau D - \text{int}_\sigma G$, then $x \in D'' \Rightarrow D'' \cap \text{int}_\tau D \notin \emptyset$ and hence $\overline{\sigma F} \cap \text{int}_\tau D \notin \emptyset \Rightarrow D' \cap \text{int}_\tau D \notin \emptyset$, which is not possible since $\text{int}_\tau D \subseteq \text{int}_\sigma G \cup D''$ and $D' \cap [\text{int}_\sigma G \cup D''] = \emptyset$. Thus $\text{int}_\tau D \subseteq \text{int}_\sigma G$. Consequently, by (5.2.3) $\text{int}_\tau D \notin \mathfrak{A}(\sigma)$. Hence (i) is proved.

(ii). Let $0 (\neq \emptyset) \in \sigma$ and let $D \subseteq 0$ be such that $D$ is dense in $(0, \tau/0)$. Our proof will be complete if we can show that $\text{int}_\sigma (G \cap 0) \neq \emptyset$. Note that $\text{int}_\sigma G \cap 0$ is dense in $(0, \sigma/0)$, since $\overline{\sigma G} = X$ and $0 \in \sigma$. Consequently, $\text{int}_{\tau/0} (\text{int}_\sigma G \cap 0) \neq \emptyset$, since $G$ is hereditarily pairwise irresolvable. So $G \cap D \neq \emptyset$. Let us now verify that $G \cap D$ is dense in $(0 \cap G, \tau/0 \cap G)$. Let $\emptyset \neq H \in \tau/0 \cap G$. Then $H = 0 \cap G \cap H'$ for some $H' \in \tau$ and $0 \cap (G \cap H') \in \tau/0$ and since $D$ is dense in $(0, \sigma/0)$, $0 \cap (G \cap H') \cap D = (0 \cap G \cap H') \cap (G \cap D) = H \cap (G \cap D) \neq \emptyset$. Hence
$G \cap D$ is dense in $(0 \cap G, \tau/0 \cap G)$. Now since $G$ is hereditarily pairwise irresolvable, $0 \cap G$ is pairwise irresolvable. Consequently, $\text{int}_{\delta/0 \cap \sigma} (G \cap D) \neq \emptyset$. Let $H'' \in \sigma$ such that

$$(5.2.4) \quad \emptyset \neq 0 \cap G \cap H'' \subset G \cap D.$$  

Since $\text{cl}_{\sigma} \text{int}_{\sigma} G = X$, it is clear that

$$(5.2.5) \quad (0 \cap H'') \cap \text{int}_{\sigma} G \neq \emptyset.$$  

Hence by (5.2.4) and (5.2.5), it follows that $0 \cap H'' \cap \text{int}_{\sigma} G \subset D$ i.e., $\text{int}_{\delta/0} D \neq \emptyset$.

We shall now show that the converse statements of (i) and (ii) of Theorem 5.2.2 also hold in some restricted bitopological spaces. Before that we like to mention the following observation.

Note 5.2.1. If $A$ is a pairwise irresolvable subset of $(X, \tau, \sigma)$ and $\tau/A$ is indiscrete then $\sigma/A$ must be discrete, because, otherwise $(A, \sigma/A)$ contains a proper dense subset whose complement in $A$ is dense in $(A, \tau/A)$, resulting $A$ pairwise resolvable.

Definition 5.2.3. $(X, \tau, \sigma)$ will be said to satisfy the \textquoteleft $\tau - \sigma$\textquoteright condition (respectively, \textquoteleft $\sigma - \tau$\textquoteright condition) if for each proper $\tau$-open subset $A$ and each proper $\sigma$-open subset $B$, 

A \cap B \neq \emptyset \implies \text{int}_\tau(A \cap B) \neq \emptyset \) (respectively, \( \text{int}_\sigma(A \cap B) \neq \emptyset \)).

**Theorem 5.2.3.** Let \((X, \mathcal{G})\) be dense-in-itself. Suppose that \((X, \tau, \mathcal{G})\) is pairwise irresolvable and satisfies the '\(\tau - \mathcal{G}\)' condition. If \(X = G \cup F = G^* \cup F^*\) is a '\(\tau - \mathcal{G}\)' representation of \(X\) then the following statements are equivalent:

(i) \((X, \tau, \mathcal{G})\) is \(\mathcal{G}\)-o.h.p.i.,
(ii) \(\text{int}_\tau D \in \mathcal{A}(\mathcal{G})\) for each \(D \in \mathcal{A}(\mathcal{G})\),
(iii) \(\text{cl}_\mathcal{G} \text{int}_\mathcal{G} G = X\).

**Proof.** (i) \(\implies\) (ii). If possible suppose that there exists a nonempty \(\mathcal{G}\)-open subset \(H\) and a \(\mathcal{G}\)-dense set \(D\) such that

\[(5.2.6)\quad H \cap \text{int}_\tau D = \emptyset.\]

Since \(H\) is pairwise irresolvable and \(H \cap D\) is dense in \((H, \mathcal{G}/H)\)

\[(5.2.7)\quad \text{int}_{\tau/H}(H \cap D) \neq \emptyset.\]

Since \((X, \mathcal{G})\) is dense-in-itself and \(H \in \mathcal{G}\), \((H, \mathcal{G}/H)\) is not discrete and hence, \(H\) being pairwise irresolvable, from Note 5.2.1 it follows that \((H, \tau/H)\) is not indiscrete. Hence by \((5.2.7)\) we can choose a proper \(\tau\)-open subset \(K\) such that \(K \cap H \subseteq H \cap D\). Now using the '\(\tau - \mathcal{G}\)' condition, it follows that \(\emptyset \neq \text{int}_\tau(K \cap H) \subseteq H \cap D\), contradicting \((5.2.6)\). Hence (ii) follows.
(ii) $\Rightarrow$ (iii): See Theorem 5.2.2 (i).

(iii) $\Rightarrow$ (i). See Theorem 5.2.2 (ii).

The dual of the above result is stated below.

**Theorem 5.2.4.** Let \((X, \tau)\) be dense-in-itself. Suppose that \((X, \tau, 6)\) is pairwise irresolvable and satisfies the '6 - \tau' condition. If \(X = G \cup F = G^\tau \cup F^\tau\) is a '\tau - 6' representation of \(X\) then the following statements are equivalent:

1. \((X, \tau, 6)\) is T-o.h.p.i.,
2. \(\text{int}_6 D \in A(\tau)\) for each \(D \in A(\tau)\),
3. \(\text{cl}_\tau \text{int}_\tau G^\tau = X\).

**Remark 5.2.1.** A 6-o.h.p.i. bitopological space \((X, \tau, 6)\) for which \((X, 6)\) is dense-in-itself and which satisfies the '\tau - 6' condition may not collapse into a topological space, that is to say, \(\tau\) may be different from 6. For example, consider

\[ X = \{a, b, c, d\}, \ \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \]

\[ 6 = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}. \]

The following two examples guarantee that the conditions imposed in Theorem 5.2.3 are not redundant.
Example 5.2.1. Let \( X = \{a, b, c\} \)
\[ \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \delta = \{\emptyset, X, \{a, b\}, \{c\}\}. \]

Then \((X, \tau, \delta)\) satisfies the '\(\tau - \delta\)' condition and statement (i) of Theorem 5.2.3, but fails to satisfy the statement (ii).

Note that \((X, \delta)\) is not dense-in-itself.

Example 5.2.2. Let \( X = \{a, b, c, d, e, f\} \),
\[ \tau \text{ = topology on } X \text{ generated by the base } \mathcal{B} \text{ where} \]
\[ \mathcal{B} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}, \{d, f\}\} , \]
and \[ \delta = \{\emptyset, X, \{a, b\}, \{c, d\}, \{e, f\}, \{a, b, c, d\}, \{a, b, e, f\}, \{c, d, e, f\}\}. \]

Then \((X, \tau, \delta)\) satisfies the statement (i) of Theorem 5.2.3 but does not satisfy the statement (ii), because, \(\{a, c, f\} \notin \mathcal{A}(\delta)\) but \(\text{int}_\tau \{a, c, f\} = \{a, c\} \notin \mathcal{A}(\delta)\). Also \((X, \delta)\) is dense-in-itself whereas \((X, \tau, \delta)\) fails to satisfy the '\(\tau - \delta\)' condition, because \(\{e, f\} \notin \delta\) and \(\{d, f\} \notin \tau\) but \(\{e, f\} \cap \{d, f\} = \{f\}\) has empty interior in \((X, \tau)\).

Definition 5.2.4. A bitopological space \((X, \tau, \delta)\) is said to be pairwise hyperconnected if \(A \cap B \neq \emptyset\) for every nonempty \(\tau\)-open subset \(A\) and nonempty \(\delta\)-open subset \(B\).
The following characterizations of pairwise hyperconnected bitopological spaces can be easily obtained.

**Theorem 5.2.5.** The following are equivalent for a bitopological space 
\((X, \tau, \sigma)\).

\begin{enumerate}[(i)]
  \item \((X, \tau, \sigma)\) is pairwise hyperconnected,
  \item \(\mathcal{A}(\tau) = \mathcal{P}(\tau, \tau)\),
  \item \(\mathcal{A}(\sigma) = \mathcal{P}(\tau, \sigma)\),
  \item for every subset \(A\) of \(X\), either \(A \in \mathcal{A}(\tau)\) or \(X - A \in \mathcal{A}(\sigma)\).
\end{enumerate}

**Theorem 5.2.6.** If \((X, \tau, \sigma)\) is pairwise hyperconnected and pairwise irresolvable then \((X, \tau, \sigma)\) is \(\tau\)-o.h.p.i. and \(\sigma\)-o.h.p.i.

Proof follows from Theorem 5.2.1 and Theorem 5.2.5.

**Theorem 5.2.7.** \((X, \tau, \sigma)\) is pairwise hyperconnected and pairwise irresolvable if and only if for every subset \(A\) of \(X\), either \(\text{int}_\sigma A \in \mathcal{A}(\tau)\) or \(\text{int}_\tau (X - A) \in \mathcal{A}(\sigma)\).

Proof follows from Theorem 5.2.5.

We shall now prove Theorem 5.2.8 which may be considered as a generalization of Theorem 1.2.2.

**Theorem 5.2.8.** Let \((X, \tau, \sigma)\) be pairwise irresolvable and \(X = G \cup F = G^* \cup F^*\) be a '\(\tau - \sigma\)' representation of \(X\). If

\begin{enumerate}[(i)]
  \item every \(\sigma\)-open subset of \(G\) is semi-open in \((X, \tau)\) and
  \item every nonempty \(\tau\)-open subset of \(\text{int}_\sigma G\) is not nowhere dense in \((X, \sigma)\) then
\end{enumerate}

\[\mathcal{A} = \{\text{int}_\sigma D : D \in \mathcal{A}(\tau)\}\]

is a filterbase on \(X\).
Proof. By Theorem 5.1.1, \( \text{int}_\sigma D \neq \emptyset \) for every \( D \in \mathfrak{A}(\tau) \). We now show that \( \text{int}_\sigma G \subseteq \text{cl}_\sigma \text{int}_\sigma D \) for all \( D \in \mathfrak{A}(\tau) \). Let \( B \) be any \( \sigma \)-open subset of \( G \). Let \( H \in \tau/B \) and \( H \neq \emptyset \). Then \( H = B \cap K \), \( K \in \tau \). By condition (i), \( \text{int}_\tau B \cap K \neq \emptyset \). This implies for every \( D \in \mathfrak{A}(\tau) \), \( D \cap H \neq \emptyset \) i.e., \( \text{cl}_\tau/B (B \cap D) = B \). Since \( B \) is \( \sigma \)-o.h.p.i. and \( B \cap D \in \text{PO}(\sigma/B, \tau/B) \), from Theorem 5.2.1, \( \text{int}_\sigma/B (B \cap D) \neq \emptyset \), i.e., \( \text{int}_\sigma D \cap B \neq \emptyset \). Thus \( \text{int}_\sigma G \subseteq \text{cl}_\sigma \text{int}_\sigma D \) for every \( D \in \mathfrak{A}(\tau) \). Hence it follows that \( \text{int}_\sigma (D_1 \cap D_2) \neq \emptyset \) and

\[(5.2.8) \quad \text{int}_\sigma G \subseteq \text{cl}_\sigma \text{int}_\sigma (D_1 \cap D_2),\]

for every pair \( D_1, D_2 \in \mathfrak{A}(\tau) \).

We now show that for every pair \( D_1, D_2 \in \mathfrak{A}(\tau) \), there exists \( D_3 \in \mathfrak{A}(\tau) \) such that \( \text{int}_\sigma (D_1 \cap D_2) \subset \text{int}_\sigma D_3 \).

Let \( D_1, D_2 \in \mathfrak{A}(\tau) \) and let

\[ A = [\text{int}_\sigma (D_1 \cap D_2) \cap \text{int}_\sigma G] \cup [(G - \text{int}_\sigma G) \cup E] \]

where

\[ F = E \cup E^* \text{, } E \cap E^* = \emptyset \text{, } E \text{ is dense in } (F, \tau/F) \text{ and } E^* \text{ is dense in } (F, \sigma/F). \]

We verify that \( A \in \mathfrak{A}(\tau) \) and \( \text{int}_\sigma A \subseteq \text{int}_\sigma (D_1 \cap D_2) \). Let \( H \in \tau \), \( H \neq \emptyset \). We must show that \( H \cap A \neq \emptyset \). If \( H \cap E \neq \emptyset \) then clearly \( H \cap A \neq \emptyset \). Suppose that \( H \cap E = \emptyset \). Then \( H \cap F = \emptyset \) implies that \( H \subset G \). If \( H \cap (G - \text{int}_\sigma G) \neq \emptyset \), then clearly \( H \cap A \neq \emptyset \). If possible, let \( H \cap (G - \text{int}_\sigma G) = \emptyset \). Then \( H \subset \text{int}_\sigma G \). By (ii) in the hypothesis, \( \text{int}_\sigma \text{cl}_\sigma H \cap \text{int}_\sigma G \neq \emptyset \). Hence by (5.2.8), \( A \cap H \neq \emptyset \) i.e., \( A \in \mathfrak{A}(\tau) \). Now
let \( x \in \text{int}_6 A \). Then \( x \notin E \), otherwise, \( \text{int}_6 A \cap E^* \neq \emptyset \); a contradiction. If \( x \in G - \text{int}_6 G \) then \( \text{int}_6 A \notin G \), a contradiction. Hence \( x \in \text{int}_6 (D_1 \cap D_2) \cap \text{int}_6 G \). Thus \( \text{int}_6 A \subseteq \text{int}_6 (D_1 \cap D_2) \). Hence we have shown that \( \mathcal{A} = \{ \text{int}_6 D : D \in \mathcal{A}(\tau) \} \) is a filterbase on \( X \).

Replacing conditions (i) and (ii) in Theorem 5.2.8 by their duals, we can show that \( \mathcal{B} = \{ \text{int}_\tau D : D \in \mathcal{D}(\sigma) \} \) is a filterbase on \( X \).

The condition (i) of Theorem 5.2.8 can be replaced by condition (i) of the following Theorem.

**Theorem 5.2.9.** Let \((X, \tau, \sigma)\) be pairwise irresolvable and \( X = G \cup F = G^* \cup F^* \) be a \('\tau-\sigma' representation of \( X \). If

(i) \( (\text{int}_6 G, \sigma / \text{int}_6 G) \) is hyperconnected and

(ii) every \( \tau \)-open subset of \( \text{int}_6 G \) is not nowhere dense in \((X, \sigma)\) then \( \mathcal{A} = \{ \text{int}_6 D : D \in \mathcal{D}(\tau) \} \) is a filterbase on \( X \).

**Proof.** We have \( \text{int}_6 D \neq \emptyset \) for every \( D \in \mathcal{D}(\tau) \). We first show that \( \text{int}_6 D \cap \text{int}_6 G \neq \emptyset \) for every \( D \in \mathcal{D}(\tau) \). The set \((D \cap G) \cup E \in \mathcal{A}(\tau)\) where \( E \) is chosen as in Theorem 5.2.8. Thus \( \text{int}_6 [(D \cap G) \cup E] \neq \emptyset \). Also one can easily verify that \( \emptyset \neq \text{int}_6 [(D \cap G) \cup E] = \text{int}_6 (D \cap G) = \text{int}_6 D \cap \text{int}_6 G \).

Now it follows by using (i) that \( \text{int}_6 (D_1 \cap D_2) \neq \emptyset \) for every pair \( D_1, D_2 \in \mathcal{D}(\tau) \). Using (ii) and following the proof of Theorem 5.2.8, we can verify that the set
belongs to \( \mathcal{A}(\tau) \) and \( \operatorname{int}_\sigma A \subseteq \operatorname{int}_\sigma (D_1 \cap D_2) \). Hence \( \mathcal{A} \) is a filterbase on \( X \).

### 5.3. Minimal Pairwise Irresolvable and Maximal Pairwise Resolvable Bitopological Spaces

According to Raghavan and Reilly [54], a bitopological space \((X, \tau, \sigma)\) is minimal pairwise Hausdorff if \((X, \tau, \sigma)\) is pairwise Hausdorff and if \((X, \tau', \sigma')\) is pairwise Hausdorff with \( \tau' \subseteq \tau \) and \( \sigma' \subseteq \sigma \) then \( \tau' = \tau \) and \( \sigma' = \sigma \).

Following this, we now define

**Definition 5.3.1.** A bitopological space \((X, \tau, \sigma)\) is minimal pairwise irresolvable if \((X, \tau, \sigma)\) is pairwise irresolvable and if \((X, \tau', \sigma')\) is pairwise irresolvable with \( \tau' \subseteq \tau \) and \( \sigma' \subseteq \sigma \) then \( \tau' = \tau \) and \( \sigma' = \sigma \).

In Note 5.2.1, we have noticed that for a pairwise irresolvable bitopological space \((X, \tau, \sigma)\), if \( \tau \) (respectively \( \sigma \)) is indiscrete then \( \sigma \) (respectively \( \tau \)) must be discrete. We now point out the fact that if \( \tau \) (respectively \( \sigma \)) is indiscrete and \( \sigma \) (respectively \( \tau \)) is discrete then at the same time \((X, \tau, \sigma)\) is minimal pairwise irresolvable. On the other hand, if \((X, \tau, \sigma)\) is minimal pairwise irresolvable, then it is not necessarily true that \( \tau \) (respectively \( \sigma \)) is indiscrete and \( \sigma \) (respectively \( \tau \)) is discrete. See Example 5.3.1.
Example 5.3.1. Let $X$ be an infinite set and $p, q \in X$, $p \neq q$, and let $\tau = \{\emptyset, X, \{p\}, \{q\}, \{p, q\}\}$ and $\sigma = \{\emptyset, X, \{p, q\}\}$. Then $(X, \tau, \sigma)$ is minimal pairwise irresolvable.

Thus, unlike the case of a minimal irresolvable topological space, the topologies of a minimal pairwise irresolvable bitopological space cannot be specified. However, we now prove

Theorem 5.3.1. Let $(X, \tau, \sigma)$ be pairwise irresolvable, where $|X| > 1$ and let $\{\text{int}_\sigma D : D \in \mathcal{D}(\tau)\}$ and $\{\text{int}_\tau D : D \in \mathcal{D}(\sigma)\}$ be filterbases on $X$. Then $(X, \tau, \sigma)$ is minimal pairwise irresolvable if and only if $\tau = \sigma = \{\emptyset, \{p\}, X\}$ for some $p \in X$.

Proof. Necessity: We shall prove this part in three steps.

In step I, we show that $\text{int}_\tau D \in \mathcal{D}(\tau)$ for every $D \in \mathcal{D}(\sigma)$. In step II, we show that there can have only one nonempty proper $\tau$-open subset and one nonempty proper $\sigma$-open subset of $X$.

In step III, we show that $\tau = \sigma = \{\emptyset, \{p\}, X\}$ for some $p \in X$.

Step I. Let $D \in \mathcal{D}(\sigma)$. Clearly $\text{int}_\tau D \neq \emptyset$. Let

$$\tau' = \{U \in \tau : U \subseteq \text{int}_\tau D\} \cup \{X\}.$$ 

Then $\tau'$ is a topology on $X$ and $\tau' \subseteq \tau$. We claim that $(X, \tau', \sigma)$ is pairwise irresolvable. If possible, let $(X, \tau', \sigma)$ be pairwise resolvable. Then there are subsets $D_1, D_2$ such that $D_1 \in \mathcal{D}(\tau')$, $D_2 \in \mathcal{D}(\sigma)$, $X = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Clearly $\text{int}_\tau D_2 \neq \emptyset$. Also $\text{int}_{\tau'} D_2 = \emptyset$, which implies $\text{int}_\tau D \cap \text{int}_\tau D_2 = \emptyset$. This contradicts the fact that $\{\text{int}_\tau D : D \in \mathcal{D}(\sigma)\}$ is a filterbase on
X. Hence \((X, \tau', \delta)\) is pairwise irresolvable and since \((X, \tau, \delta)\) is minimal pairwise irresolvable, it follows that \(\tau = \tau'\). Thus \(\text{int}_\tau D \in \mathcal{A}(\tau)\).

Similarly it can be shown that for \(D \in \mathcal{A}(\tau)\), \(\text{int}_\delta D \in \mathcal{A}(\delta)\).

**Step II.** Let \(W \neq \emptyset\) be a proper \(\tau\)-open subset of \(X\). Consider \(\tau'_W = \{U \in \tau : U \subseteq W\} \cup \{X\}\). We claim that \(\tau = \tau'_W\). For any \(D \in \mathcal{A}(\delta)\),

\[
\text{int}_\tau D \in \mathcal{A}(\tau) \implies W \cap \text{int}_\tau D = H \text{ (say)} \neq \emptyset \implies H \in \tau'_W \quad \text{and} \\
\text{int}_\tau D \neq \emptyset. \text{ Thus } (X, \tau'_W, \delta) \text{ is pairwise irresolvable and hence } \tau = \tau'_W. \text{ Similarly it can be shown that } \delta = \delta'_W \text{ where } \delta'_W = \{V \in \delta : V \subseteq W\} \cup \{X\}, W' \text{ being a nonempty } \delta\text{-open subset of } X.

Now we can conclude that there can have only one nonempty proper \(\tau\)-open subset of \(X\) and similarly there can have only one nonempty proper \(\delta\)-open subset of \(X\).

**Step III.** Since \((X, \tau, \delta)\) is pairwise irresolvable and \(|X| > 1\), \(\tau\) and \(\delta\) cannot be simultaneously indiscrete. Also if one is indiscrete then the other will be discrete. But by step II, it then follows that \(|X| = 1\), which contradicts our hypothesis. Hence neither \(\tau\) nor \(\delta\) can be indiscrete. Let \(W \in \tau\) with \(X \neq W \neq \emptyset\) and \(W^* \in \delta\) with \(X \neq W^* \neq \emptyset\). Now we claim that \(W = W^*\). If possible, let \(p \in W - W^*\). Then \(\{p\} \in \mathcal{A}(\tau) \implies X - \{p\} \notin \mathcal{A}(\delta) \implies \{p\} \in \delta\) \(\implies \{p\} = W^*\), a contradiction. Hence \(W \subseteq W^*\). Similarly, it can be shown that \(W^* \subseteq W\) and hence \(W = W^*\). Consequently, \(W = W^* = \{p\}\).
for some \( p \in X \), since \((X, \tau, \sigma)\) is pairwise irresolvable. Hence 
\[ \tau = \sigma = \{\emptyset, \{p\}, X\} .\]

Sufficiency: Follows easily.

The following example shows that a pairwise irresolvable bitopological space \((X, \tau, \sigma)\) may not collapse into a topological space even if \(\{\text{int}_\tau D : D \in \mathcal{A}(\sigma)\}\) and \(\{\text{int}_\sigma D : D \in \mathcal{A}(\tau)\}\) are filterbases.

**Example 5.3.2.** Let \(X\) be an infinite set and \(p, q \in X, p \neq q\).

Set \(\tau = \{\emptyset, X, \{p\}, \{q\}, \{p, q\}\}\) and \(\sigma = \{\emptyset, X, \{q\}, \{p, q\}\}\).

**Definition 5.3.2.** A bitopological space \((X, \tau, \sigma)\) is said to be maximal pairwise resolvable if \((X, \tau, \sigma)\) is pairwise resolvable and if \((X, \tau', \sigma')\) is pairwise resolvable with \(\tau \subset \tau'\) and \(\sigma \subset \sigma'\) then \(\tau = \tau'\) and \(\sigma = \sigma'\).

It is easy to verify that if \((X, \tau, \sigma)\) is maximal pairwise resolvable then both \((X, \tau)\) and \((X, \sigma)\) are extremally disconnected. For, if \(\emptyset \neq A \in \tau\) then \((X, \tau(\text{cl}_\tau A), \sigma)\) is pairwise resolvable and hence \(\text{cl}_\tau A \in \tau\). Similarly, for \(\emptyset \neq B \in \sigma\), \(\text{cl}_\sigma B \in \sigma\).

The notion of pairwise extremally disconnected bitopological spaces is already in literature. We recall the following definition owing to Balasubramanian [6].
Definition 5.3.3. \((X, \tau, \sigma)\) is said to be pairwise extremally disconnected if for each \(U \in \tau\) and each \(V \in \sigma\), \(\text{cl}_{\sigma} U \in \tau\) and \(\text{cl}_{\tau} V \in \sigma\).

We now investigate when a maximal pairwise resolvable bitopological space is pairwise extremally disconnected.

Theorem 5.3.2. Let \((X, \tau, \sigma)\) be maximal pairwise resolvable. If

(i) for each pair of \(\tau\)-open sets \(U, V\)

\[
U \cap \text{cl}_{\sigma} V \neq \emptyset \Rightarrow \text{int}_{\tau}(U \cap \text{cl}_{\sigma} V) \neq \emptyset
\]

and

(ii) for each pair of \(\sigma\)-open sets \(U, V\)

\[
U \cap \text{cl}_{\tau} V \neq \emptyset \Rightarrow \text{int}_{\sigma}(U \cap \text{cl}_{\tau} V) \neq \emptyset,
\]

then \((X, \tau, \sigma)\) is pairwise extremally disconnected.

Proof. Let \(0 \in \tau\) and if possible, suppose that \(\text{cl}_{\sigma} 0 \notin \tau\). Now since \((X, \tau, \sigma)\) is pairwise resolvable, \(X = D \cup D^*\), \(D \cap D^* = \emptyset\), \(D \in A(\tau)\), \(D^* \in A(\sigma)\). Consider \((X, \tau(\text{cl}_{\sigma} 0), \sigma)\). We claim that \(\text{int}_{\tau(\text{cl}_{\sigma} 0)} D^* = \emptyset\), for, if there exists a nonempty \(\tau(\text{cl}_{\sigma} 0)\)-open set \(A = U \cup (V \cap \text{cl}_{\sigma} 0) \subseteq D^*\), for some \(U, V \in \tau\), then \(U = \emptyset\) (since \(\text{int}_{\tau} D^* = \emptyset\)) and \(V \cap \text{cl}_{\sigma} 0 \neq \emptyset\), implying by condition (i) that \(\text{int}_{\tau} D^* \neq \emptyset\), a contradiction. Thus \(\text{int}_{\tau(\text{cl}_{\sigma} 0)} D^* = \emptyset\), and hence \((X, \tau(\text{cl}_{\sigma} 0), \sigma)\) is pairwise resolvable. Since \((X, \tau, \sigma)\) is maximal pairwise resolvable, \(\tau = \tau(\text{cl}_{\sigma} 0)\) i.e., \(\text{cl}_{\sigma} 0 \in \tau\).

Similarly we can show by using condition (ii) that for each \(V \in \sigma\), \(\text{cl}_{\tau} V \in \sigma\). Hence \((X, \tau, \sigma)\) is pairwise extremally disconnected.
The following Lemma will be used in proving Theorem 5.3.3.

Lemma 5.3.1. Let \((X, \tau, \sigma)\) be maximal pairwise resolvable and \(A\) be a pairwise resolvable subset of \(X\). Then

(i) if \(X - \text{cl}_\tau A\) is a nonempty pairwise resolvable subset then \(A \in \tau\) and

(ii) if \(X - \text{cl}_\sigma A\) is a nonempty pairwise resolvable subset then \(A \in \sigma\).

Proof. (i). Let \(A = D \cup D'\), where \(D \cap D' = \emptyset\), \(D\) is dense in \((A, \tau/A)\) and \(D'\) is dense in \((A, \sigma/A)\) and let \(X - \text{cl}_\tau A = D_1 \cup D'_1\) where \(D_1 \cap D'_1 = \emptyset\), \(D_1\) is dense in \((X - \text{cl}_\tau A, \tau/X - \text{cl}_\tau A)\) and \(D'_1\) is dense in \((X - \text{cl}_\tau A, \sigma/X - \text{cl}_\tau A)\). Consider \((X, \tau(A), \sigma)\).

Then \(D \cup D'_1 \in \mathcal{A}(\tau(A))\). For, let \(\mathcal{A} \neq \emptyset \cup (V \cap A) \in \tau(A)\). If \(U = \emptyset\) then \(V \cap A \cap D \neq \emptyset\). If \(U \neq \emptyset\) and \(U \cap A = \emptyset\) then \(U \subset X - \text{cl}_\tau A \Rightarrow U \cap D_1 \neq \emptyset\). Now we show that \((\text{cl}_\tau A - D) \cup D'_1 \in \mathcal{A}(\sigma)\). Let \(\mathcal{A} \neq \emptyset \subset H \in \sigma\). Suppose that \(H \cap D'_1 = \emptyset\). Then \(H \cap (X - \text{cl}_\tau A) = \emptyset \Rightarrow H \subset (\text{cl}_\tau A - D) \subset (\text{cl}_\tau A - D) \cup D'_1 \in \mathcal{A}(\sigma)\).

Now \(X = (D \cup D'_1) \cup ((\text{cl}_\tau A - D) \cup D'_1)\) where \(D \cup D'_1 \in \mathcal{A}(\tau(A))\) and \((\text{cl}_\tau A - D) \cup D'_1 \in \mathcal{A}(\sigma)\). Since \((D \cup D'_1) \cap ((\text{cl}_\tau A - D) \cup D'_1) = \emptyset\), it follows that \((X, \tau(A), \sigma)\) is pairwise resolvable.
Since \((X, \tau, \mathcal{D})\) is maximal pairwise resolvable, it follows that 
\(\tau = \tau(A)\) i.e., \(A \in \mathcal{T}\). Similarly (ii) can be proved.

**Theorem 5.3.3.** Let \((X, \tau, \mathcal{D})\) be maximal pairwise resolvable. Then for any pairwise resolvable subset \(A\), \(X - \text{cl}_{\tau} A \in \mathcal{D} \implies A \in \mathcal{T}\).

*Proof.* Two cases arise.

**Case I.** \(\text{cl}_{\tau} A = X\).

Let \(A = D \cup D^*\), where \(D \cap D^* = \emptyset\), \(D\) is dense in \((A, \tau/A)\) and \(D^*\) is dense in \((A, \mathcal{D}/A)\). Since \(\text{cl}_{\tau} A = X\), then \(D \in \mathcal{D}(\tau), D^* \cup (X - A) \in \mathcal{D}(\mathcal{D})\). Also \(D \in \mathcal{D}(\tau(A))\). Hence \((X, \tau(A), \mathcal{D})\) is pairwise resolvable. Consequently, \(\tau = \tau(A)\) and hence \(A \in \mathcal{T}\).

**Case II.** \(X - \text{cl}_{\tau} A \neq \emptyset\).

Since \((X, \tau, \mathcal{D})\) is pairwise resolvable and \(X - \text{cl}_{\tau} A\) is both \(\tau\)-open and \(\mathcal{D}\)-open, it is clear that \(X - \text{cl}_{\tau} A\) is pairwise resolvable. Hence by Lemma 5.3.1, it follows that \(A \in \mathcal{T}\). This completes the proof.

In case of topological spaces, we have seen that in a maximal resolvable space, any resolvable subset is open (see Theorem 4.1.1). So Theorem 5.3.3 above can be thought of as some sort of generalization of the corresponding result in topological spaces.
A number of results of this chapter on resolvability and irresolvability in bitopological spaces have been published in a paper in Soochow J. of Mathematics [17]. However we regret to say that there was an error in Example 2.1 of the paper which has been duly corrected in the thesis.