CHAPTER - IV

MAXIMAL RESOLVABLE AND MINIMAL IRRESOLVABLE SPACES

A topological space \((X, \tau)\) with property \(R\) is minimal \(R\) (maximal \(R\)) if, whenever, \(\sigma\) is a topology on \(X\), strictly coarser (finer) than \(\tau\), \((X, \sigma)\) does not have the property \(R\). It clearly follows that for any coarser (finer) topology \(\sigma\) of \(\tau\) if \((X, \tau)\) is resolvable (irresolvable) then \((X, \sigma)\) is also resolvable (irresolvable) and this result may not hold for finer (coarser) topologies. Naturally the problem of characterizing maximal resolvable spaces and minimal irresolvable spaces arise. In this chapter we have characterized maximal resolvable spaces and specified the topology for a minimal irresolvable space. We have also shown that although the class of o.h.i. spaces is a subclass of the class of irresolvable spaces, the class of minimal o.h.i. spaces coincides with the class of minimal irresolvable spaces.

4.1 MAXIMAL RESOLVABLE SPACES

We, in this section deal with maximal resolvable spaces.
It is not difficult to check the existence of maximal resolvable spaces. One of the simple examples is the following.

**Example 4.1.1.** Let $X$ be an infinite set and $A = \{p, q\}$, $p, q \in X$, $p \neq q$. Let $\tau = \{B \subseteq X : A \subseteq B\} \cup \{\emptyset\}$. Then obviously, $(X, \tau)$ is resolvable. Also $(X, \tau)$ is maximal resolvable since every topology strictly finer than $\tau$ contains an isolated point.

We shall now prove Theorem 4.1.1 which characterizes maximal resolvable spaces. To prove it, we shall use the following Lemma.

**Lemma 4.1.1.** Let $(X, \tau)$ be resolvable and $A \subseteq X$ be a resolvable subspace. Then $(X, \tau|A)$ is resolvable; $\tau|A$ denoting the simple extension of $\tau$ by $A$.

**Proof.** There exists a subset $D$ of $A$ such that $D$ is dense in $(A, \tau|A)$ and $\text{int}_{\tau|A} D = \emptyset$. Two cases arise.

**Case I.** $X - \text{cl}_{\tau} A = \emptyset$.

Then $D$ is dense in $(X, \tau)$ and $\text{int}_\tau D = \emptyset$. Consider the topology $\tau(A)$. Clearly $D$ is dense in $(X, \tau(A))$ and $\text{int}_{\tau(A)} D = \emptyset$. So $(X, \tau(A))$ is resolvable in this case.

**Case II.** $X - \text{cl}_{\tau} A = B$ (say) $\neq \emptyset$. 
Since open subspace of a resolvable space is resolvable, 
\((B, \tau_B)\) is resolvable. Choose \(D^* \subseteq B\) such that \(D^*\) is dense in 
\((B, \tau_B)\) with \(\text{int}_{\tau_B}D^* = \emptyset\). Then \(D \cup D^*\) is dense in \((X, \tau)\) and 
\(\text{int}_\tau(D \cup D^*) = \emptyset\); for, if there exists a nonempty \(\tau\)-open subset \(0 \subseteq D \cup D^*\) then 
\(0 \cap D^* \neq \emptyset \implies 0 \cap B = 0'\) (say) \(\neq \emptyset \implies \text{int}_{\tau_B}D^* \neq \emptyset\) (since \(0' \cap A = \emptyset\)), a contradiction to the choice 
of \(D^*\). Now consider the topology \(\tau(A)\). Clearly \(D \cup D^*\) is 
dense in \((X, \tau(A))\) and \(\text{int}_{\tau(A)}(D \cup D^*) = \emptyset\); for, if \(\emptyset \neq 
\bigcup (V \cap A) \subseteq D \cup D^*\) for some \(U, V \in \tau\) then \(U = \emptyset\) and 
\(V \cap A \subseteq D \cup D^* \implies V \cap A \subseteq D\), a contradiction, since \(V \cap A\) is nonempty open in \((A, \tau/A)\) and 
\(\text{int}_{\tau/A}D = \emptyset\). Hence \((X, \tau(A))\) is resolvable in this case also. This completes the proof.

**Theorem 4.1.1.** For a resolvable space \((X, \tau)\) the following 
are equivalent:

(i) \((X, \tau)\) is maximal resolvable,

(ii) the set of all open subsets of \(X\) = the set of all 
resolvable subsets of \(X\).

(iii) any continuous bijection \(f\) from a resolvable space 
\((Y, \sigma)\) onto \((X, \tau)\) is a homeomorphism.

**Proof :** (i) \(\implies\) (ii). Clearly every open subset is 
resolvable. Now suppose that \(A \subseteq X\) is resolvable but not 
open. Then by Lemma 4.1.1 \((X, \tau(A))\) is resolvable. Hence 
\((X, \tau)\) cannot be maximal resolvable since \(\tau(A)\) is strictly 
finer than \(\tau\).
(ii) $\implies$ (i). Suppose that $(X, \tau)$ is not maximal resolvable. Then there exists a topology $\tau'$, strictly finer than $\tau$, such that $(X, \tau')$ is resolvable. Let $U \in \tau'$ but $U \notin \tau$. Then $U$ is resolvable in $(X, \tau')$ and hence resolvable in $(X, \tau)$. This contradicts (ii).

(i) $\implies$ (iii). If $f : (Y, \delta) \to (X, \tau)$ is a continuous bijection then for $\tau' = \{ f(G) : G \in \delta \}$, $f : (Y, \delta) \to (X, \tau')$ is a homeomorphism and $(X, \tau')$ is resolvable (since the property of being resolvable is a topological property). Since $\tau' \supset \tau$ and $(X, \tau)$ is maximal resolvable it follows that $\tau' = \tau$.

(iii) $\implies$ (i). If $(X, \tau)$ is not maximal resolvable, then there exists a topology $\tau'$, strictly finer than $\tau$ such that $(X, \tau')$ is resolvable. The identity function $I : (X, \tau') \to (X, \tau)$ is a continuous bijection which is not a homeomorphism.

Note 4.1.1. By repeated application of simple extension, from any given resolvable space, one can arrive at a maximal resolvable space by using Lemma 4.1.1 and Theorem 4.1.1.

Corollary 4.1.1. Let $(X, \tau)$ be maximal resolvable. Then

(i) $(X, \tau)$ is extremally disconnected,

(ii) semi-open sets are open in $(X, \tau)$,

(iii) $(X, \tau)$ is $T_0$ if and only if $\Delta(\tau) > 2$ where $\Delta(\tau)$ denotes the dispersion character of $\tau$, the least cardinality of a nonempty open set.
Proof. (i). Let \( G \) be open and \( x \in \text{cl}_\tau G \setminus G \). Since \( G \) is resolvable, so is \( G \cup \{x\} \) and by Theorem 4.1.1, \( G \cup \{x\} \) is open. Thus \( \text{cl}_\tau G \cup \{ x \} \subseteq \bigcup_{x \in \text{cl}_\tau G} (G \cup \{x\}) \) is open. Hence \( (X, \tau) \) is extremally disconnected.

(ii). Proof of (ii) follows from proof of (i).

(iii). Suppose that \( \Delta(\tau) > 2 \). If \( (X, \tau) \) is not \( T_0 \) then there exists a pair of distinct points \( x, y \in X \) such that every open set containing \( x \) contains \( y \) and every open set containing \( y \) contains \( x \). Hence the topology of the subset \( A = \{x, y\} \) relative to \( \tau \) is indiscrete and consequently, \( A \) is resolvable. Since \( (X, \tau) \) is maximal resolvable, \( A \) should be open in \( (X, \tau) \). This contradicts our hypothesis that \( \Delta(\tau) > 2 \).

Conversely, let \( (X, \tau) \) be \( T_0 \). We show that \( \Delta(\tau) > 2 \). Since \( (X, \tau) \) is resolvable, it cannot contain an isolated point. If possible, let there be an open set \( A \) such that \( A \) consists of only two points \( x, y \), say, where \( x \neq y \). Since \( (X, \tau) \) is resolvable and \( A \) is open, \( A \) is resolvable. Hence the topology of \( A \) relative to \( \tau \) is indiscrete i.e., every open set in \( (X, \tau) \) containing \( x \) contains \( y \) and every open set containing \( y \) contains \( x \). This contradicts the fact that \( (X, \tau) \) is \( T_0 \). This completes the proof.

Before concluding the section we like to give some examples on product spaces. Hewitt [35] proved that a product space \( \prod X_\alpha \) with the product topology is resolvable if
one factor \( X_\alpha \) is resolvable. The following example shows that the converse is not true; in fact, a product space may be resolvable even if each factor is hereditarily irresolvable.

**Example 4.1.2.** Let \( X \) be an infinite set endowed with the discrete metric, i.e.,

\[
d(x, y) = \begin{cases} 
0 & \text{when } x = y \\
1 & \text{when } x \neq y.
\end{cases}
\]

Then the topology on \( X \) generated by \( d \) is the discrete topology. Now consider the countable infinite product \( \prod X_i \) where \( X_i = X \) for all \( i \). If \( x = (x_i) \) and \( y = (y_i) \) are two points in \( \prod X_i \), define, \( D(x, y) = \limsup \left\{ \frac{d(x_i, y_i)}{i} \right\} \).

It can be verified that \( D \) is a metric on \( \prod X_i \) that induces the product topology on \( \prod X_i \). Since \( \prod X_i \) is without isolated point and a metric space without isolated point is resolvable [35], it follows that \( \prod X_i \) is resolvable whereas each factor space is hereditarily irresolvable.

The following example shows that a product of maximal resolvable spaces may not be maximal resolvable.
Example 4.1.3. Let $X$ be an infinite set, $A = \{x, y\}$; $x, y \in X$, $x \neq y$. $\mathcal{T} = \{B \subseteq X : A \subseteq B\} \cup \{\emptyset\}$. Then $(X, \mathcal{T})$ is maximal resolvable. Now consider an infinite product $\prod_{\alpha} X_\alpha$ where $X_\alpha = X$ for each $\alpha$. Then the product space $\prod_{\alpha} X_\alpha$ is resolvable. Consider the points $p = (x_\alpha)$, $q = (y_\alpha)$ in $\prod_{\alpha} X_\alpha$ where $x_\alpha = x$ for each $\alpha$ and $y_\alpha = y$ for each $\alpha$. Then the subset $E = \{p, q\}$ of $\prod_{\alpha} X_\alpha$ is resolvable in the topology relative to the product topology, but not open. Hence by Theorem 4.1.1 it follows that $\prod_{\alpha} X_\alpha$ is not maximal resolvable.

4.2. MINIMAL IRRESOLVABLE SPACES

Now we shall deal with minimal irresolvable spaces. Let $|X|$ denote the cardinality of $X$. Obviously, if $(X, \mathcal{T})$ ($|X| > 1$) contains an isolated point, then it is minimal irresolvable if and only if $\mathcal{T} = \{\emptyset, \{p\}, X\}$ for some $p \in X$. But using Hewitt representation of a topological space, we shall prove that a minimal irresolvable space must contain an isolated point. To prove this result we require the following two lemmas.

Lemma 4.2.1. If a space $(X, \mathcal{T})$ is irresolvable with $|X| > 1$ then $\mathcal{T}$ is not the indiscrete topology.

Proof of Lemma 4.2.1 is easy.

Lemma 4.2.2. Let $(X, \mathcal{T})$ be irresolvable and let $X = F \cup G$ be the Hewitt representation of $(X, \mathcal{T})$. If $W$ is a nonempty
open subset of \( G \) then \( \sigma(W) = \{ x \} \cup \{ \in \mathcal{T} : U \subset W \} \) is a coarser irresolvable topology on \( X \).

**Proof.** It is clear that \( \sigma(W) \) is a topology on \( X \) with \( \sigma(W) \subset \mathcal{T} \). Now suppose that \( (X, \sigma(W)) \) is resolvable, i.e., there exist disjoint subsets \( D \) and \( E \) of \( X \) which are dense in \( (X, \sigma(W)) \). If \( D^* = D \cap W \) and \( E^* = E \cap W \) then \( D^* \) and \( E^* \) are nonempty. Since \( G \) is hereditarily irresolvable, \( W \) is an irresolvable subspace of \( (X, \mathcal{T}) \). Hence either \( D^* \) or \( E^* \) fails to be dense in \( (W, \mathcal{T}/W) \), say \( D^* \). So \( W \) is not a subset of \( \text{cl}_{\mathcal{T}}(W \cap D) \). Hence there exists a nonempty \( \mathcal{T} \)-open subset \( U \) with \( U \subset W \) and \( U \cap W \cap D = U \cap D = \emptyset \). But \( U \in \sigma(W) \). So we have a contradiction. Consequently, \( (X, \sigma(W)) \) is irresolvable.

**Theorem 4.2.1.** Let \( (X, \mathcal{T}) \) be irresolvable with \( |X| > 1 \). Then \( (X, \mathcal{T}) \) is minimal irresolvable if and only if there exists \( p \in X \) such that \( \mathcal{T} = \{ \emptyset, \{ p \}, X \} \).

**Proof.** Clearly, if \( \mathcal{T} = \{ \emptyset, \{ p \}, X \} \) for some \( p \in X \), then \( (X, \mathcal{T}) \) is minimal irresolvable. Now suppose that \( (X, \mathcal{T}) \) is minimal irresolvable. If \( X = F \cup G \) is the Hewitt representation of \( (X, \mathcal{T}) \) then \( G \neq \emptyset \). If \( G = \{ p \} \) for some \( p \in X \), then by Lemma 4.2.2, \( \sigma(G) = \{ \emptyset, \{ p \}, X \} = \mathcal{T} \) and we are done. Otherwise, \( |G| > 1 \) and by Lemma 4.2.1 there exists a proper nonempty \( \mathcal{T} \)-open subset \( W \) of \( G \) (\( G \) being irresolvable and \( \mathcal{T} \)-open). If \( |W| > 1 \), by the same argument, there exists a proper nonempty \( \mathcal{T} \)-open subset \( V \) of \( W \). By Lemma 4.2.2, \( \sigma(V) = \mathcal{T} \) and thus \( W \in \sigma(V) \). According to the definition of \( \sigma(V) \),
it follows that $W = X$. But this is a contradiction to the fact that $W$ is a proper subset of $G$. Consequently we have $W = \{p\}$ for some $p \in X$ and by Lemma 4.2.2, we have $\tau = \sigma(W) = \{\emptyset, \{p\}, X\}$. This completes the proof.

As a straightforward consequence of Lemma 4.2.2 and Theorem 4.2.1 we get

**Theorem 4.2.2.** Let $(X, \tau)$ be a space with $|X| > 1$. Then $(X, \tau)$ is minimal o.h.i. if and only if $(X, \tau)$ is minimal irresolvable.

**Remark 4.2.1.** D.R. Anderson [2] has demonstrated the existence of a large class of connected irresolvable Hausdorff spaces which have no isolated point. Theorem 4.2.1 now indicates the existence of irresolvable spaces whose topologies have no minimal irresolvable subtopology.

For the works of this chapter we like to refer to our published paper [16].