CHAPTER II
ON COMPOUND STACEY DISTRIBUTION

1. INTRODUCTION

The art of obtaining compound distribution is a quite old practice. The compound distributions are found to be very useful in various fields, particularly, in the field of biological sciences and recently have been used extensively by many scientists among others Takahasi (1965), Bhattacharya and Holla (1965), Bhattacharya (1966), Dubey (1968, 1970), Hutchinson (1981) in studying the life of a system whose parameters change randomly.

Here an attempt has been made to study the compound Stacey distributions which are found to be very useful in life testing. The choice of Stacey distribution lies in the fact that it includes Exponential, Gamma and Weibull distributions as special cases. Regarding the choice of the distribution function of the parameter \( \theta \) (a random variable) of the Stacey distribution we have considered two distributions, viz., Gamma and Inverted Gamma, so that a wide class of distributions can be covered as special cases and a member of this family of compound distributions can be chosen in any practical situation depending upon the nature of life test data. Moreover, relationships with various well known distributions have been shown. Properties like moments, failure rate, reliability have been calculated.
Stacey (1962) has obtained a distribution function having the probability density function (p.d.f.)

\[ f(x; \theta, a, \alpha) = \frac{e^{\theta a / \alpha} x^{a-1} \exp(-\theta x^\alpha)}{\Gamma(a/\alpha)}, \quad x > 0, \ (a, \theta, \alpha > 0). \quad (2.1) \]

The Gamma density function of \( \theta \) is given by

\[ g(\theta; \lambda, \beta) = \frac{\beta^{-1} \lambda^{\beta} \exp(-\lambda \theta)}{\Gamma(\beta)}, \quad \theta > 0, \ (\lambda, \beta > 0). \quad (2.2) \]

Now we can obtain the p.d.f. of a new distribution as

\[ h(x) = \int_0^\infty f(x; \theta, a, \alpha) g(\theta; \lambda, \beta) \, d\theta \]

\[ = \frac{\alpha \lambda^{\beta} x^{a-1}}{B(\beta, a/\alpha) (\lambda + x^\alpha)^{\beta + a/\alpha}}, \quad x > 0, \ (a, \alpha, \lambda, \beta > 0) \quad (2.3) \]

and hence the cumulative distribution function (c.d.f.) comes out as (vide appendix I)

\[ H(x) = \sum_{i=0}^{\infty} \frac{\alpha \lambda^{i} x^{\alpha}}{B(\beta, a/\alpha) (\lambda + x^\alpha)^{i + a/\alpha}} \quad (2.4) \]

The compound Stacey-Gamma distribution with p.d.f. given by (2.3) includes a wide class of distributions as special cases:

(i) Compound Gamma distribution (Dubey, 1970)

For \( \alpha = 1 \),

\[ h(x) = \frac{\lambda^{\beta} x^{a-1}}{B(a, \beta) (\lambda + x)^{\beta + a}}, \quad (2.5) \]

For \( a = a \),

\[
h(x) = \frac{\alpha \beta \lambda^\beta x^{\alpha-1}}{(\lambda + x^\alpha)^{\beta+1}}.
\]  

(2.6)

(iii) Compound Rayleigh-Gamma distribution

For \( a = a=2 \),

\[
h(x) = \frac{2\beta \lambda^\beta x}{(\lambda + x^2)^{\beta+1}}.
\]  

(2.7)

(iv) Burr type XII distribution (Burr, 1942)

For \( a = a, \lambda = 1 \)

\[
h(x) = \frac{\alpha \beta x^{\alpha-1}}{(1 + x^\alpha)^{\beta+1}}.
\]  

(2.8)

(v) Burr type III distribution (Burr, 1942)

For \( a = a, \lambda = 1 \) and putting \( x = 1/y \),

\[
h(y) = \frac{\alpha \beta}{y^{\alpha+1} (1+y^{-\alpha})^{\beta+1}}.
\]  

(2.9)

(vi) Burr type II distribution (Burr, 1942)

For \( a = a, \lambda = 1 \) and putting \( x^\alpha = \exp(-y) \),

\[
h(y) = \frac{\beta \exp(-y)}{(1 + \exp(-y))^{\beta+1}}.
\]  

(2.10)
(vii) Lomax distribution (Lomax, 1954)

For $a = \alpha = 1$,

$$h(x) = \frac{\beta \lambda^\beta}{(\lambda + x)^{\beta + 1}}.$$  \hfill (2.11)

(viii) Kappa distribution (Mielke, 1973)

For $a = \beta = 1$, $\lambda = \alpha$ and putting $x = y/c$,

$$h(y) = \frac{(\alpha/c)}{(\alpha + (y/c)^{\alpha})^{1 + 1/\alpha}}.$$  \hfill (2.12)

(ix) Beta-k distribution (Mielke and Johnson, 1974)

For $a = \alpha$, $\lambda = 1$ and putting $x = d/y$,

$$h(y) = \frac{(\alpha \beta/d (y/d)^{\alpha \beta - 1}}{(1 + (y/d)^{\alpha})^{\beta + 1}}.$$  \hfill (2.13)

(x) Beta distribution of first kind (Dubey, 1970)

For $a = 1$ and putting $y = \lambda/(\lambda + x)$,

$$h(y) = \frac{y^{a-1} (1-y)^{a-1}}{B(a, \beta)}.$$  \hfill (2.14)

(xi) Beta distribution of second kind (Dubey, 1970)

For $a = 1$ and putting $y = x/\lambda$,

$$h(y) = \frac{y^{a-1}}{B(a, \beta)(1+y)^{a+\beta}}.$$  \hfill (2.15)
(xii) F-distribution with 2 $\alpha$ and 2 $\beta$ degrees of freedom (Dubey, 1970)

For $\alpha = 1$ and putting $y = \beta x / a \lambda$,

$$h(y) = \frac{(a/\beta)^a y^{a-1}}{B(a, \beta) (1 + ay/\beta)^{a+\beta}}$$

(2.16)

2.1 Moments

The moment generating function (m.g.f.), $m(t)$, of the compound Stacey-Gamma distribution comes out as

$$m(t) = \int_0^\infty \exp(tx) h(x) \, dx$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{k/a}}{k!} \frac{t^k}{\Gamma(\beta-k/a) \Gamma((a+k)/\alpha)} \Gamma(\beta) \Gamma(a/\alpha).$$

(2.17)

Hence the moment of order $k$ about the origin can be found as

$$\lambda_k = \lambda^{k/a} M_k,$$

(2.18)

For $k = 0, 1, 2, \ldots$, up through $k < \alpha \beta$, $M_k$ being given by

$$M_k = \frac{\Gamma(\beta-k/\alpha) \Gamma((a+k)/\alpha)}{\Gamma(\beta) \Gamma(a/\alpha)}.$$

(2.19)

Obviously, $M_0 = 1$. 

Consequently, the mean, the variance and the coefficient of variation (C.V.) of this distribution can be expressed as

\[ \mu = \frac{1}{\alpha} M_1, \]
\[ \sigma^2 = \frac{2}{\alpha} (M_2 - M_1^2), \]
\[ \text{C.V.} = \frac{(M_2 - M_1^2)^{1/2}}{M_1}. \]  

(2.20)

Clearly, the C.V. of this compound distribution is independent of \( \lambda \).

Now \( h'(x) > 0 \) if \( \lambda (a-1) - x^\alpha (1+\alpha \beta) > 0 \) provided \( a > 1 \)

Thus for \( a > 1 \), the distribution is unimodal and the mode being at

\[ x_M = \left( \frac{(a-1)}{1+\alpha \beta} \right)^{1/\alpha}, \]

with the highest modal ordinate

\[ h(x_M) = \frac{a-1}{\alpha} \frac{1+\alpha \beta}{1+\alpha \beta} \]
\[ \frac{\beta}{B(\beta, \alpha/\alpha)} \frac{1/\alpha}{(1+\alpha \beta)^{\beta+\alpha/\alpha}}. \]

(2.21)

It should be noted here that for \( \alpha = 1 \) and \( a > 1 \),

\[ \mu > x_M, \]

since

\[ \mu - x_M = \frac{(a+\beta)}{\beta(\beta+1)} > 0 \]

and hence (2.4) is a positively skewed distribution.
Here we consider the distribution of the parameter $\theta$ as Inverted Gamma with p.d.f. given by

$$g_1(\theta) = \frac{\lambda^\beta \exp(-\lambda/\theta)}{\Gamma(\beta) \theta^{\beta+1}}, \quad \theta > 0, \, (\lambda, \beta > 0). \quad (3.1)$$

Assuming the parameters $a$ and $\alpha$ of the Stacey distribution having p.d.f. given by (2.1) to remain constant, the p.d.f. of the new distribution has been obtained as

$$f(x) = \frac{(\beta+a/\alpha)/2 - 1+(a+\alpha\beta)/2}{\Gamma(\beta) \Gamma(a/\alpha)} \frac{2^\alpha \lambda x^\alpha K_{\beta-a/\alpha}(2\sqrt{\lambda x^\alpha})}{\Gamma(\beta+1/2)}, \, x > 0, (a, \alpha, \beta, \lambda > 0), \quad (3.2)$$

where $K_m(.)$ is the modified Bessel function of the second kind of order $m$ and therefore, the distribution function will come out as

(vide appendix II)

$$H_1(x) = \frac{2^{3-(\beta+a/\alpha)}}{\Gamma(\beta) \Gamma(a/\alpha)} \sqrt{\lambda x^\alpha} \left[2(\beta-1)K_{\beta-a/\alpha}(2\sqrt{\lambda x^\alpha}) - S_{\beta-2+a/\alpha, \beta-1-a/\alpha} \right]. \quad (3.3)$$

We may call this distribution as generalised Bessel distribution.
Two distributions (Bhattacharya, 1966) arise out of this distribution as special cases:

(i) For $\alpha = 1$,

\[ h_{1}(x) = \frac{2 \lambda^{(a+\beta)/2}}{\Gamma(a) \Gamma(\beta)} x^{-1+(a+\beta)/2} K_{\beta-a}(2\sqrt{\lambda x}). \]  

(ii) For $a = \alpha = \beta = 1$,

\[ h_{1}(x) = 2 \lambda K_{0}(2\sqrt{\lambda x}). \]

3.1 Moments

The m.g.f., $m(t)$, of compound Stacey-Inverted Gamma distribution has been found out as

\[ m(t) = \sum_{k=0}^{\infty} \lambda^{-k/\alpha} t^{k} \frac{\Gamma(\beta+k/\alpha) \Gamma((a+k)/\alpha)}{\Gamma(\beta) \Gamma(a/\alpha)}. \]  

Then the $k$-th order raw moment is expressed as

\[ \lambda_{k}' = \lambda^{-k/\alpha} w_{k}, \]  

for $k = 0, 1, 2 \ldots \ldots$ and where

\[ w_{k} = \frac{\Gamma(\beta+k/\alpha) \Gamma((a+k)/\alpha)}{\Gamma(\beta) \Gamma(a/\alpha)}. \]

Clearly $w_{0} = 1$. 
Hence the mean, the variance and the coefficient of variation (C.V.) are respectively as

\[ \mu = \frac{\lambda^{-1/\alpha}}{w_1}, \]

\[ \sigma^2 = \lambda^{-2/\alpha} (w_2 - w_1^2), \]  

and \[ CV = \frac{(w_2 - w_1^2)^{1/2}}{w_1} \] which is independent of \( \lambda \)

4. APPLICATION OF THE MODEL

The Stacey distribution (in effect the generalised Gamma distribution) covers Exponential, Weibull etc. as special cases and has been used as a failure model. Generally, a failure model is under the assumption of homogeneity in population but in many life test situations, due to some experimental constraints, the assumption of homogeneity is not tenable and the situations may be well described by a mixture of two or more distributions. For example, if the items come from two different sources (say, produced by two different machines), it is expected that some sort of heterogeneity will creep in and in such situations we may possibly assume a common distribution for both, but the parameters should be reasonably assumed to be different. Of course, when variation in the manufacturing process occurs continuously, the parameters labelling the distributions become continuous random variables. For instance, mean life of the items produced over a time period may vary from batch to batch and hence a distribution function may be assumed for mean life or for a parameter which
is a function of mean life and hence the failure model becomes a compound distribution. A similar situation arises in case of Bayesian estimation.

4.1 For Stacey-Gamma Distribution

The survival or reliability function is given by

\[ H(x) = 1 - H(x), \quad (4.1) \]

where \( H(x) \) is given by Eqn (2.4) and the conditional failure density function i.e., the hazard rate may be expressed by

\[ r(x) = \frac{a \lambda^\beta x^{\alpha-1}}{a B(\beta, \alpha) (\lambda + x)^{\beta+\alpha/\alpha} - \alpha \lambda^\beta x^{\alpha} \Gamma(\beta+\alpha, 1; 1+\alpha/\alpha; x/(\lambda + x))} \quad (4.2) \]

4.2 For Stacey-Inverted Gamma Distribution

Here the reliability function can be obtained from

\[ H_1(x) = 1 - H_1(x), \quad (4.3) \]

where \( H_1(x) \) has been obtained in Eqn. (3.3) and the hazard rate will come out as

\[ r_1(x) = \frac{\alpha \lambda (2\sqrt{\lambda x})^{\beta+\alpha/\alpha} - x^{\alpha} \phi(x)}{\Gamma(\beta) \Gamma(s/\alpha) \cdot 2^{\beta-2a/\alpha} - x^{\alpha} \phi(x)} \quad (4.4) \]

where \( \phi(x) \) is given by Eqn. (2) in appendix II.
Appendix - I

Derivation of Compound Stacey-Gamma Distribution Function

\[ H(x) = \int \mathcal{P}(x; \theta) g(\theta) d\theta \]

\[ = \frac{\lambda^\beta}{\Gamma(\beta) \Gamma(a/\alpha)} \int_0^\infty \left( \frac{a}{a} \right)^{\frac{1}{2}(\frac{a}{\alpha} - 1)} \exp\left(-\frac{1}{2} \theta x^\alpha\right) \exp\left(-\frac{1}{2} \theta x^\alpha\right) M_{\frac{1}{2}\left(\frac{a}{\alpha} - 1\right)}, \frac{1}{2} \frac{a}{\alpha} \cdot \theta^\beta - 1 \exp\left(-\lambda \theta\right) d\theta \]

\[ = \frac{\alpha \lambda^\beta x^\alpha}{\alpha \Gamma(\beta) \Gamma(a/\alpha)} \int_0^\infty \frac{1}{2} \left(\frac{a}{\alpha} - 1\right)^{\beta - 1} \exp\left\{-(\lambda + x^\alpha) \theta\right\} \]

\[ \cdot M_{\frac{1}{2}\left(\frac{a}{\alpha} - 1\right)}, \frac{1}{2} \frac{a}{\alpha} \cdot \theta^\beta - 1 \exp\left(-\lambda \theta\right) d\theta , \quad (1) \]

where \( M \) represents a Whittaker's function defined by Eqn. (5.8) in chapter I.

Now making use of equation (11) on page 215 of Erdélyi (1954), we obtain

\[ H(x) = \frac{\alpha \lambda^\beta x^\alpha}{a \cdot B(\beta, a/\alpha) \left(\lambda + x^\alpha\right)^{\beta + a/\alpha}} \cdot _2F_1(\beta + a/\alpha, 1; 1 + a/\alpha; x^\alpha/(\lambda + x^\alpha)) , \quad (2) \]

\(_2F_1\) is Gauss's hypergeometric function, defined by Eqn. (5.1) in chapter I.
Appendix - II

Derivation of Compound Stacey-Inverted Gamma Distribution Function

\[ H_1(x) = \int_0^x h_1(x) \, dx \]

\[ = \frac{2\alpha \lambda (\beta + a/\alpha) / 2}{\Gamma(\beta) \Gamma(a/\alpha)} \int_0^x x^{\lambda} (a + \alpha \beta) - 1 \cdot K_{\beta - a/\alpha} x^\alpha \, dx \]

Now using the transformation

\[ z = 2\sqrt{\lambda} x^\alpha \]

we can write

\[ H_1(x) = \frac{2^{2-(\beta + a/\alpha)}}{\Gamma(\beta) \Gamma(a/\alpha)} \int_0^z z^{-1 + \beta + a/\alpha} K_{\beta - a/\alpha}(z) \, dz. \]

To evaluate the integral on the right hand side we take help of the equation (1) on page 85 of Luke (1962).

We then have

\[ H_1(x) = \frac{2^{3-(\beta + a/\alpha)}}{(\beta) (a/\alpha)} \sqrt{\lambda} x^\alpha \phi(x), \quad (1) \]

where

\[ \phi(x) = 2(\beta - 1) K_{\beta - a/\alpha}(2\sqrt{\lambda} x^\alpha) \cdot S_{\beta - 2 + a/\alpha, \beta - 1 - a/\alpha} \cdot \frac{(2\sqrt{\lambda} x^\alpha)}{(2\sqrt{\lambda} x^\alpha)} - K_{\beta - a/\alpha - 1} \cdot S_{\beta - 1 + a/\alpha, \beta - a/\alpha} \cdot \frac{(2\sqrt{\lambda} x^\alpha)}{(2\sqrt{\lambda} x^\alpha)} \quad (2) \]

provided

\[ 2(a/\alpha) - 1, 2\beta - 1 \] are not necessarily integers and \( S_{a, b}(z) \) is Lommel's function defined by Eqn. (5.5) in chapter I.