CHAPTER VI
ON RELIABILITY ESTIMATION OF SYSTEM LIFE DISTRIBUTIONS

1. INTRODUCTION

So far we have discussed different probability models to describe system life and we have also obtained the reliability functions of those failure models. Consequently, the question arises how to evaluate the value of the reliability function from the practical data. As we have seen the reliability function of a system involves parameters of the corresponding failure model, so the evaluation of reliability needs the complete idea of these parameters. Very often, these parameters are unknown and they are to be estimated from the life data with the help of existing methods of statistical estimation theory.

Estimation of parameters as well as reliability depends on the nature of the experiment and invariably, on the nature of the data obtained from the experiment. Though it is desirable to conduct a complete experiment but sometimes it may not be possible rather feasible to conduct a complete experiment for various reasons such as total cost of operation may be very high, availability of number of components/systems for experiment may be limited, availability of sophisticated instruments needed for conducting the experiment may be insufficient, waiting time for experiment may be large, test may be of destructive nature, etc. In such cases a censored test (either a time censored or a failure censored) is carried out.

The most popular method of estimation used in the theory of reliability is the classical method of maximum likelihood. The maximum likelihood estimator (MLE) of the reliability function $R(t)$ has been obtained by several authors for different life models with different types of experiments.
In this chapter an attempt has been made to find estimators of the parameters as well as reliability for a two-component dependent series and parallel system life distributions arising from bivariate exponential and bivariate Burr.

These models have been discussed in detail in Chapter-V. It has been pointed out that estimators are very often biased. It is difficult to find out the exact expressions for the variance of estimators and hence their large sample approximations have been obtained.

2. **Estimation of Parameters Involved in System Life Distributions Arising Out of Bivariate Exponential Distribution (Morgenstern Model)**

In Chapter-V we have obtained life distributions of a two-component series system (equation (4.1)) and a two-component parallel system (equation (4.2)) from bivariate exponential (Morgenstern model) distribution, viz.,

\[ f^*(x) = \frac{2}{\theta} \left[ 1 + \varphi (1 - 3 \exp(-x/\theta) + 2 \exp(-2x/\theta)) \right] \exp(-2x/\theta) \]  
\[ (2.1) \]

and

\[ f^{**}(x) = \frac{2}{\theta} \left[ 1 - \left[ 1 + \varphi (1 - 3 \exp(-x/\theta) + 2 \exp(-2x/\theta)) \right] \exp(-x/\theta) \right] \exp(-x/\theta) \]  
\[ (2.2) \]

assuming components are identical in nature.

This shows that the above models involved only two parameters \( \theta \) and \( \varphi \) and our problem remains to estimate these parameters for which we have considered all three different types of test: complete, time censoring and failure censoring.

2.1. **Estimation of Parameters for a Two-Component Series System Life Model with Complete Sample**

Following equation (5.2) of Chapter-I, the log likelihood function can be written as
\[ L^* = \text{constant} - n \log \theta - (2/\theta) \sum_{i=1}^{n} x_i \]
\[ + \sum_{i=1}^{n} \log \left\{ 1 + \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right\}. \tag{2.3} \]

Differentiating \( L^* \) partially with respect to \( \theta \) and \( \beta \) respectively and equating the partial derivatives to zero, we obtain the likelihood equations as,

\[ \left( \frac{\partial L^*}{\partial \theta} \right) = 0 \Rightarrow \]
\[ 0 = -(n/\theta) + (2/\theta^2) \sum_{i=1}^{n} x_i \]
\[ \left( \frac{\partial L^*}{\partial \beta} \right) = 0 \Rightarrow \]
\[ 0 = \sum_{i=1}^{n} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \left/ \left( 1 + \beta \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right. \right) \tag{2.4} \]

Hence the MLE of \( \theta \) is

\[ \hat{\theta} = 2x + (\beta/n) \sum_{i=1}^{n} x_i \left( 4 \exp \left(-2x_i/\theta \right) - 3 \exp \left(-x_i/\theta \right) \right) \]
\[ \left/ \left( 1 + \beta \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right. \right) \tag{2.5} \]

The equation (2.5) is a polynomial in \( \beta \) of degree \((n-1)\). The exact solution for \( \beta \) is very difficult to find out. However, an approximate solution for \( \beta \) can be obtained. Hence the MLE of \( \beta \) is

\[ \hat{\beta} = \frac{\sum_{i=1}^{n} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right)}{\left( \sum_{i=1}^{n} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right)^2 \right)} \tag{2.6} \]

2.2. Estimation of parameters for a two-component series system life model with time censored sample

Following equation (5.3) of chapter-I, the log likelihood function can be written as
\( L^* = \text{Constant} - m \log \theta - (2/\theta)^m \sum_{i=1}^{m} x_i \)

\[
+ \sum_{i=1}^{m} \log \left[ 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right] \\
+ (n-m) \left[ \log \left[ 1 + \hat{\rho} \left( 1-\exp(-t_0/\hat{\theta}) \right)^2 \right] - 2(t_0/\hat{\theta}) \right]
\]

The likelihood equations are

\[
\frac{\partial L^*}{\partial \theta} = 0 \Rightarrow \\
0 = -(m/\theta) + (2/\theta^2) \sum_{i=1}^{m} x_i \\
+ (\hat{\rho}/\theta^2) \sum_{i=1}^{m} x_i \left( 4 \exp(-2x_i/\hat{\theta}) - 3 \exp(-x_i/\hat{\theta}) \right)/ \\
\left[ 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right] \\
+ 2(n-m) \left[ 1 + \hat{\rho} \left( 1-\exp(-t_0/\hat{\theta}) \right)^2 \right] \left( t_0/\theta^2 \right)/ \\
\left[ 1 + \hat{\rho} \left( 1-\exp(-t_0/\hat{\theta}) \right)^2 \right]
\]

and

\[
\frac{\partial L^*}{\partial \hat{\rho}} = 0 \Rightarrow \\
0 = \sum_{i=1}^{m} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right)/ \left( 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right) \\
+ (n-m) \left( 1 - \exp(-t_0/\hat{\theta}) \right)^2/ \left( 1 + \hat{\rho} \left( 1 - \exp(-t_0/\hat{\theta}) \right)^2 \right)
\]

Hence the MLE of \( \theta \) and \( \hat{\rho} \) can be expressed as

\[
\hat{\theta} = (1/m) \left[ 2 \sum_{i=1}^{m} x_i + \hat{\rho} \sum_{i=1}^{m} x_i \left( 4 \exp(-2x_i/\hat{\theta}) - 3 \exp(-x_i/\hat{\theta}) \right)/ \left( 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right) \right] \\
+ 2(n-m)t_0 \left[ 1 - \hat{\rho} \left( 1-\exp(-t_0/\hat{\theta}) \right) \exp(-t_0/\hat{\theta}) \right]/ \left( 1 + \hat{\rho} \left( 1-\exp(-t_0/\hat{\theta}) \right)^2 \right)
\]
\[ \hat{\theta} = \frac{(n-m) \left[ \left( 1 + \hat{\beta} \left( 1 - \exp \left(-t_i/\theta \right) \right)^2 \right]^{1/2}}{m \sum_{i=1}^{n} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right)} \]

respectively.

2.3. Estimation of parameters for a two-component parallel system life model with complete sample

Following equation (5.2.) of chapter-1, the log likelihood function can be written as

\[ \ln L^* = \text{Constant} - n \log \theta - \frac{1}{\theta^2} \sum_{i=1}^{n} x_i + \frac{1}{\theta^2} \sum_{i=1}^{n} \log \left[ 1 - \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right] \]

The likelihood equations are

\[ \frac{\partial \ln L^*}{\partial \theta} = 0 \Rightarrow \]

\[ 0 = \left( \frac{n}{\theta} + \frac{1}{\theta^2} \right) \sum_{i=1}^{n} x_i - \frac{1}{\theta} \sum_{i=1}^{n} \left( x_i \exp \left(-x_i/\theta \right) \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) \] 

\[ \left( 1 - \exp \left(-x_i/\theta \right) \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) \right] \]

and

\[ \frac{\partial \ln L^*}{\partial \hat{\beta}} = 0 \Rightarrow \]

\[ 0 = \sum_{i=1}^{n} \left( \exp \left(-x_i/\theta \right) \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) / \]

\[ \left( 1 - \exp \left(-x_i/\theta \right) \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) \right] \]

Hence the MLE of \( \theta \) is

\[ \hat{\theta} = \bar{x} - \frac{1}{n} \sum_{i=1}^{n} \left( x_i \exp \left(-x_i/\theta \right) \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) \] 

\[ \left( 1 - \exp \left(-x_i/\theta \right) \left( 1 + \hat{\beta} \left( 1 - 3 \exp \left(-x_i/\theta \right) + 2 \exp \left(-2x_i/\theta \right) \right) \right) \right] \]

The equation (2.15) is a polynomial in \( \hat{\beta} \) of degree \( n-1 \). The exact solution for \( \hat{\beta} \) is very hard to find out. However, an approximate
solution for $\rho$ can be obtained. Hence MLE of $\rho$ is

$$
\hat{\rho} = \prod_{1}^{n} \left[ \exp(-x_i/\hat{\theta}) \left( \exp(-x_i/\hat{\theta}) - 1 \right) \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right]
$$

$$
\times \left[ \prod_{1}^{n} \left[ \exp(-x_i/\hat{\theta}) \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right]^2 \right]^{-1} \tag{2.17}
$$

2.4. Estimation of Parameters for a Two-Component Parallel System Life Model with Time Censored Sample

Following equation (5.3) of Chapter I, the log likelihood function can be written as

$$
L^{**} = \text{Constant} - m \log \theta - \left( \frac{1}{\theta} \right) \sum_{1}^{m} x_i
$$

$$
+ \sum_{1}^{m} \log \left[ \exp(-x_i/\theta) \left( 1 + \rho \left( 1 - 3 \exp(-x_i/\theta) + 2 \exp(-2x_i/\theta) \right) \right) \right]
$$

$$
+ (n-m) \left[ \log \left\{ 2 - \exp(-t_0/\theta) \left( 1 + \rho \left( 1 - \exp(-t_0/\theta) \right)^2 \right) \right\} - (t_0/\theta) \right] \right]. \tag{2.18}
$$

Consequently, the likelihood equations are

$$
0 = -(m/\theta) + (1/\theta^2) \sum_{1}^{m} x_i
$$

$$
- \left( \frac{1}{\theta^2} \right) \sum_{1}^{m} \left( x_i \exp(-x_i/\theta) \left( 1 + \rho \left( 1 - 3 \exp(-x_i/\theta) + 2 \exp(-2x_i/\theta) \right) \right) \right) / \left( 1 - \exp(-x_i/\theta) \left( 1 + \rho \left( 1 - 3 \exp(-x_i/\theta) + 2 \exp(-2x_i/\theta) \right) \right) \right)
$$

$$
+ (n-m) \left( t_0/\theta^2 \right) \left( 1 - \exp(-t_0/\theta) \left( 1 + \rho \left( 1 - 3 \exp(-t_0/\theta) + 2 \exp(-2t_0/\theta) \right) \right) \right) / \left( 2 - \exp(-t_0/\theta) \left( 1 + \rho \left( 1 - \exp(-t_0/\theta) \right)^2 \right) \right) \right] \tag{2.19}
$$

and

$$
0 = \sum_{1}^{m} \left( \exp(-x_i/\theta) \left( 1 - 3 \exp(-x_i/\theta) + 2 \exp(-2x_i/\theta) \right) \right) / \left( 1 - \exp(-x_i/\theta) \left( 1 + \rho \left( 1 - 3 \exp(-x_i/\theta) + 2 \exp(-2x_i/\theta) \right) \right) \right)
$$

$$
+ (n-m) \left( \exp(-t_0/\theta) \left( 1 - \exp(-t_0/\theta) \right)^2 \right) / \left( 2 - \exp(-t_0/\theta) \left( 1 + \rho \left( 1 - \exp(-t_0/\theta) \right)^2 \right) \right) \right]. \tag{2.20}
$$
Hence the MLE of $\theta$ and $\rho$ can be obtained as

$$\hat{\theta} = \frac{1}{m} \left[ \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} \left( x_i \exp(-x_i/\hat{\theta}) \left\{ 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right\} \right) \right] / \left( 1 - \exp(-x_i/\hat{\theta}) \left\{ 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right\} \right)$$

$$+ (n-m) t_0 \left[ 1 - \left( 1 + \hat{\rho} \left( 1 - 4 \exp(-t_0/\hat{\theta}) + 3 \exp(-2t_0/\hat{\theta}) \right) \right) \exp(-t_0/\hat{\theta}) \right] / \left( 2 - \exp(-t_0/\hat{\theta}) \left\{ 1 + \hat{\rho} \left( 1 - \exp(-t_0/\hat{\theta}) \right)^2 \right\} \right)$$

(2.21)

and

$$\hat{\rho} = \frac{(n-m) \left[ (2 - \exp(-t_0/\hat{\theta})) / (2 - \exp(-t_0/\hat{\theta}) \left\{ 1 + \hat{\rho} \left( 1 - \exp(-t_0/\hat{\theta}) \right)^2 \right\} - 1 \right] \left[ \sum_{i=1}^{m} \exp(-x_i/\hat{\theta}) \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right] / \left( 1 - \exp(-x_i/\hat{\theta}) \left\{ 1 + \hat{\rho} \left( 1 - 3 \exp(-x_i/\hat{\theta}) + 2 \exp(-2x_i/\hat{\theta}) \right) \right\} \right)^{-1}}$$

(2.22)

3. ESTIMATION OF PARAMETERS INVOLVED IN SYSTEM LIFE DISTRIBUTIONS ARISING OUT OF BIVARIATE BURR DISTRIBUTION (MORGENSTERN MODEL)

In chapter-5 we have obtained life distributions of a two-component series system (equation (4.21)) and two-component parallel system (equation (4.22)) from bivariate Burr (Morgenstern model) distribution, viz.,

$$f^*(x) = 2\alpha \lambda x^{\alpha-1}(1+x^\alpha)(-2\lambda + 1) \left\{ 1 + \rho \left\{ 1 - 3(1+x^\alpha)^{-\lambda} + 2(1+x^\alpha)^{-2\lambda} \right\} \right\}$$

(3.1)

and

$$f^{**}(x) = 2\alpha \lambda x^{\alpha-1}(1+x^\alpha)^{(-\lambda + 1)} \left[ 1 - (1+x^\alpha)^{-\lambda} \left\{ 1 + \rho \left\{ 1 - 3(1+x^\alpha)^{-\lambda} + 2(1+x^\alpha)^{-2\lambda} \right\} \right\} \right]$$

(3.2)

assuming components are identical in nature. Equations (3.1) and (3.2) reveal that these models contain three parameters $\alpha$, $\lambda$ and $\rho$ only, we have to estimate these parameters considering all three different types of tests i.e., complete, time censoring and failure censoring.
3.1. **Estimation of parameters for a two-component series system life model with complete sample**

Following equation (5.2) of chapter-I, the log likelihood function can be written as

\[ L^* = \text{Constant} + n \log \alpha + n \log \lambda + (\alpha - 1) \sum \limits_{1}^{n} \log x_i - (2 + 1) \sum \limits_{1}^{n} \log (1 + x_i^\alpha) \]

\[ + \sum \limits_{1}^{n} \log [1 + \rho (1 - 3 (1 + x_i^\alpha)^{-\lambda} + 2 (1 + x_i^\alpha)^{-2 \lambda})] \]  

(3.3)

Differentiating \( L^* \) partially with respect to \( \alpha \), \( \lambda \) and \( \rho \) respectively and equating the partial derivatives to zero, we get the likelihood equations as

\[ \left( \frac{\partial L^*}{\partial \alpha} \right) = 0 \Rightarrow \]

\[ 0 = \left( n / \alpha \right) + \sum \limits_{1}^{n} \log x_i (-2 + 1) \sum \limits_{1}^{n} \left( x_i^\alpha \log x_i / (1 + x_i^\alpha) \right) \]

\[ + \rho \lambda \sum \limits_{1}^{n} \left( \{ 3(1 + x_i^\alpha)^{-\lambda} - 4(1 + x_i^\alpha)^{-2 \lambda} \} x_i^\alpha \log x_i \right) \]

\[ (1 + \rho (1 - 3 (1 + x_i^\alpha)^{-\lambda} + 2 (1 + x_i^\alpha)^{-2 \lambda})) \]  

(3.4)

\[ \left( \frac{\partial L^*}{\partial \lambda} \right) = 0 \Rightarrow \]

\[ 0 = \left( n / \lambda \right) - 2 \sum \limits_{1}^{n} \log (1 + x_i^\alpha) + \rho \sum \limits_{1}^{n} \left( \{ 3(1 + x_i^\alpha)^{-\lambda} - 4(1 + x_i^\alpha)^{-2 \lambda} \} \log (1 + x_i^\alpha) \right) \]

\[ (1 + \rho (1 - 3 (1 + x_i^\alpha)^{-\lambda} + 2 (1 + x_i^\alpha)^{-2 \lambda})) \]  

(3.5)

and

\[ \left( \frac{\partial L^*}{\partial \rho} \right) = 0 \Rightarrow \]

\[ 0 = \sum \limits_{1}^{n} \left( 1 - 3 (1 + x_i^\alpha)^{-\lambda} + 2 (1 + x_i^\alpha)^{-2 \lambda} \right) \]

\[ (1 + \rho (1 - 3 (1 + x_i^\alpha)^{-\lambda} + 2 (1 + x_i^\alpha)^{-2 \lambda})) \]  

(3.6)

Hence MLE of \( \alpha \) and \( \lambda \) are
n = \frac{(2\lambda +1) \sum_{i=1}^{n} (x_i^* \log x_i)/(1+x_i^*)}{1 + \hat{\lambda} \sum_{i=1}^{n} [(3(1+x_i^*)^{-\lambda} + (2 \lambda + 1)(1+x_i^*) x_i^* \log x_i) / (1 + \hat{\lambda} \sum_{i=1}^{n} log x_i)]^{-1}}

\text{and}

\hat{\lambda} = \frac{n \sum_{i=1}^{n} \log (1+x_i^*) - \hat{\lambda} \sum_{i=1}^{n} [(3(1+x_i^*)^{-\lambda} - 2(1+x_i^*)^{-2\lambda}) \log (1+x_i^*)]}{(1+\hat{\lambda} \sum_{i=1}^{n} [(1-3(1+x_i^*)^{-\lambda} + 2(1+x_i^*)^{-2\lambda})]^{-1} \log (1+x_i^*)}^{-1} \cdot \quad (3.7)

Since the equation (3.5) is a polynomial in \rho of degree (n-1) the exact solution for \rho is very difficult to find out. However, an approximate solution of \rho can be obtained. Hence the MLE of \rho is

\hat{\rho} = \left[ \hat{\lambda} \sum_{i=1}^{n} [(1-3(1+x_i^*)^{-\lambda} + 2(1+x_i^*)^{-2\lambda})] \right]

\times \left[ \hat{\lambda} \sum_{i=1}^{n} [(1-3(1+x_i^*)^{-\lambda} + 2(1+x_i^*)^{-2\lambda})]^{-1} \log (1+x_i^*) \right]^{-1} \cdot \quad (3.8)

3.2. Estimation of parameters for a two-component series system life model with time censored sample

Following equation (5.3) of chapter-I, the log likelihood function can be written as

L^* = \text{Constant} + m \log \alpha + m \log \lambda + (a-1) \sum_{i=1}^{m} \log x_i - (2 \lambda + 1) \sum_{i=1}^{m} \log (1+x_i^*)

+ \sum_{i=1}^{m} \log \left[ 1 + \hat{\rho} \{ 1-3(1+t_i^*)^{-\lambda} + 2(1+t_i^*)^{-2\lambda} \} \right]

+ (n-m) \left[ \log \left\{ 1 + \hat{\rho} \{ 1-(1+t_o^*)^{-\lambda} \}^2 \right\} - 2 \lambda \log (1+t_o^*) \right] \cdot \quad (3.10)

The likelihood equations are
0 = (m/\alpha) + \frac{m}{1} \log x_1 - (2 \lambda + 1) \frac{m}{1} (x_1^\alpha \log x_1)/(1+x_1^\alpha)

+ \lambda \rho \frac{m}{1} \{[3(1+x_1^\alpha)^{-\lambda} - (\lambda + 1) - 4(1+x_1^\alpha)^{-2 \lambda}] (x_1^\alpha \log x_1) /

(1 + \rho \{1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2 \lambda}\})\}

+ 2 \lambda(n-m) \left[ \rho ((1+t_0^\alpha)^{-\lambda} - (1+n-t_0^\alpha)^{-2 \lambda})/(1 + \rho \{1-(1+t_0^\alpha)^{-\lambda}\}^2 \right] - 1 \right] \times \log (1+t_0^\alpha),

(3.11)

0 = (m/\lambda) - 2 \frac{m}{1} \log (1+x_1^\alpha) + \rho \frac{m}{1} \{[3(1+x_1^\alpha)^{-\lambda} - 4(1+x_1^\alpha)^{-2 \lambda}] \log (1+x_1^\alpha) / \n
(1 + \rho \{1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2 \lambda}\})\}

+ 2(n-m) \left[ \rho ((1+t_0^\alpha)^{-\lambda} - (1+n-t_0^\alpha)^{-2 \lambda})/(1 + \rho \{1-(1+t_0^\alpha)^{-\lambda}\}^2 \right] - 1 \right] \times \log (1+t_0^\alpha),

(3.12)

and

0 = \frac{m}{1} \{[3(1+x_1^\alpha)^{-\lambda} - 4(1+x_1^\alpha)^{-2 \lambda}] (x_1^\alpha \log x_1) /

(1 + \rho \{1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2 \lambda}\})\}

+ (n-m) \left[ 1-(1+t_0^\alpha)^{-\lambda} \right] \times \log (1+t_0^\alpha),

(3.13)

Hence the MLE of \alpha, \lambda and \rho can be expressed as

\hat{\alpha} = m \left[ (2 \lambda + 1) \frac{m}{1} (x_1^\hat{\alpha} \log x_1)/(1+x_1^\hat{\alpha}) \right.

- \lambda \rho \frac{m}{1} \{[3(1+x_1^\hat{\alpha})^{-\lambda} - (\lambda + 1) - 4(1+x_1^\hat{\alpha})^{-2 \lambda}] (x_1^\hat{\alpha} \log x_1) /

(1 + \rho \{1-3(1+x_1^\hat{\alpha})^{-\lambda} + 2(1+x_1^\hat{\alpha})^{-2 \lambda}\})\}

\left. + 2 \lambda(n-m) \left[ \rho ((1+t_0^\hat{\alpha})^{-\lambda} - (1+n-t_0^\hat{\alpha})^{-2 \lambda})/(1 + \rho \{1-(1+t_0^\hat{\alpha})^{-\lambda}\}^2 \right] - 1 \right] \times \log (1+t_0^\hat{\alpha}) \right]^{-1}

(3.14)
\[ \hat{\lambda} = m \left[ 2 \sum_{i=1}^{m} \log (1+x_i^\hat{\lambda}) \right] \]

\[ - \hat{\rho} \sum_{i=1}^{m} \left( (3(1+x_i^\hat{\lambda})^{-\hat{\lambda}} - 4(1+x_i^\hat{\lambda})^{-2\hat{\lambda}}) \log (1+x_i^\hat{\lambda}) \right) / \]

\[ (1+\hat{\rho}(1-3(1+x_i^\hat{\lambda})^{-\hat{\lambda}} + 2(1+x_i^\hat{\lambda})^{-2\hat{\lambda}})) \]

\[ - 2(n-m) \left[ \hat{\rho} ((1+t_o^\hat{\lambda})^{-\hat{\lambda}} - (1+t_o^\hat{\lambda})^{-2\hat{\lambda}}) / (1+\hat{\rho}(1-(1+t_o^\hat{\lambda})^{-\hat{\lambda}})) \right] - 1 \]

\[ \cdot \log (1+t_o^\hat{\lambda}) \] \quad (3.15)

and

\[ \hat{\rho} = (n-m) \left[ (1+\hat{\rho}(1-(1+t_o^\hat{\lambda})^{-\hat{\lambda}}))^{-1} - 1 \right] \]

\[ \cdot \left[ \sum_{i=1}^{m} (1-3(1+x_i^\hat{\lambda})^{-\hat{\lambda}} + 2(1+x_i^\hat{\lambda})^{-2\hat{\lambda}}) / (1+\hat{\rho}(1-3(1+x_i^\hat{\lambda})^{-\hat{\lambda}} + 2(1+x_i^\hat{\lambda})^{-2\hat{\lambda}})) \right] \]

\[ \cdot \log (1+t_o^\hat{\lambda}) \] \quad (3.16)

respectively.

### 3.3. Estimation of parameters for a two-component parallel system life model with complete sample

Following equation (5.2) of chapter-I, the log likelihood function can be written as

\[ L^{**} = \text{Constant} + n \log \alpha + n \log \lambda + (a-1) \sum_{i=1}^{n} \log x_i - (\lambda +1) \sum_{i=1}^{n} \log (1+x_i^\alpha) + \sum_{i=1}^{n} \log [1-(1+x_i^\alpha)^{-\lambda} \{ 1+ \hat{\rho}(1-3(1+x_i^\alpha)^{-\lambda} + 2(1+x_i^\alpha)^{-2\lambda}) \}] \] \quad (3.17)

The likelihood equation are

\[ 0 = (n/\alpha) + \sum_{i=1}^{n} \log x_i - (\lambda +1) \sum_{i=1}^{n} (x_i^\alpha \log x_i) / (1+x_i^\alpha) \]

\[ + \lambda \sum_{i=1}^{n} ( (1+\hat{\rho}(1-6(1+x_i^\alpha)^{-\lambda} + 6(1+x_i^\alpha)^{-2\lambda})) \log (1+x_i^\alpha)^{-(\lambda +1)} x_i^\alpha \log x_i) / \]

\[ (1- \{ 1+ \hat{\rho}(1-3(1+x_i^\alpha)^{-\lambda} + 2(1+x_i^\alpha)^{-2\lambda}) \} (1+x_i^\alpha)^{-\lambda}) \] \quad (3.18)
\[ \begin{align*}
\hat{\theta} &= n \left[ \hat{\mu} + 1 \right] \sum_{i=1}^{n} \frac{\hat{x}_i \log x_i}{1+x_i} - \sum_{i=1}^{n} \log x_i \\
&= \frac{n}{\lambda} \left( \{ 1+ \hat{\rho} (1-6(1+x_1^a)^{-\lambda} + 6(1+x_1^a)^{-2\lambda} ) \} (1+x_1^a)^{-\lambda} \right) / \\
&= \hat{\beta} \left( 1+ \hat{\rho} (1-3(1+x_1^a)^{-\lambda} + 2(1+x_1^a)^{-2\lambda} ) \right) (1+x_1^a)^{-\lambda}. \\
\end{align*} \]

Hence the MLE of \( \theta \) and \( \lambda \) can be found out as

\[ \begin{align*}
\hat{\theta} &= n \left[ \hat{\mu} + 1 \right] \sum_{i=1}^{n} \left( \frac{\hat{x}_i \log x_i}{1+x_i} \right) - \sum_{i=1}^{n} \log x_i \\
&= \frac{n}{\lambda} \left( \{ 1+ \hat{\rho} (1-6(1+x_1^a)^{-\lambda} + 6(1+x_1^a)^{-2\lambda} ) \} (1+x_1^a)^{-\lambda} \right) / \\
&= \hat{\beta} \left( 1+ \hat{\rho} (1-3(1+x_1^a)^{-\lambda} + 2(1+x_1^a)^{-2\lambda} ) \right) (1+x_1^a)^{-\lambda}. \\
\end{align*} \]

The likelihood equation (3.20) is a polynomial in \( \rho \) of degree \( n-1 \). Exact solution of \( \rho \) can hardly be found out. Hence an approximate solution for \( \rho \) (which is MLE of \( \rho \)) can be obtained as

\[ \begin{align*}
\hat{\rho} &= \left[ \sum_{i=1}^{n} \left( 1-3(1+x_1^a)^{-\lambda} + 2(1+x_1^a)^{-2\lambda} \right) \right] 2 \left( 1+x_1^a \right)^{-2\lambda} \\
&= \left[ \sum_{i=1}^{n} \left( 1-3(1+x_1^a)^{-\lambda} + 2(1+x_1^a)^{-2\lambda} \right) \right] 2 \left( 1+x_1^a \right)^{-2\lambda}. \\
&= \hat{\beta} \left( 1+ \hat{\rho} (1-3(1+x_1^a)^{-\lambda} + 2(1+x_1^a)^{-2\lambda} ) \right) (1+x_1^a)^{-\lambda}. \\
\end{align*} \]
3.4. Estimation of parameters for a two-component parallel system life model with time censored sample

Following equation (5.3) of chapter-I, the log likelihood function can be written as

\[
L^* = \text{Constant} + m \log \alpha + m \log \lambda + (\alpha-1) \sum \log (1+x_i^\alpha) - (\lambda +1) \sum \log (1+x_i^\lambda) \\
+ \sum m \log[1-(1+x_i^\alpha)^{-\lambda} \{1+ \rho (1-3(1+x_i^\alpha)^{-\lambda} +2(1+x_i^\alpha)^{-2\lambda})\}] \\
+ (n-m)[\log[2-(1+t_o^\alpha)^{-\lambda} \{1+ \rho (1-(1+t_o^\alpha)^{-\lambda})^2\}] - \lambda \log(1+t_o^\alpha)] \\
\]

(3.24)

The likelihood equations are

\[
0 = \left(\frac{m}{\alpha}\right) + \sum \frac{m}{1} \log x_i - (\lambda +1) \sum \frac{m}{1} (x_i^\alpha \log x_i)/(1+x_i^\alpha) \\
+ \lambda \sum \left\{1+ \rho (1-6(1+x_i^\alpha)^{-\lambda} +6(1+x_i^\alpha)^{-2\lambda})\right\}(1+x_i^\alpha)^{-(\lambda +1)} x_i^\alpha \log x_i)/ \\
(1- \{1+ \rho (1-3(1+x_i^\alpha)^{-\lambda} +2(1+x_i^\alpha)^{-2\lambda})\}) (1+x_i^\alpha)^{-\lambda}) \\
+ \lambda(n-m) \left\{1+ \rho (1-4(1+t_o^\alpha)^{-\lambda} +3(1+t_o^\alpha)^{-2\lambda})\right\}(1+t_o^\alpha)^{-(\lambda +1)} \right\}/ \\
(2-(1+t_o^\alpha)^{-\lambda} \{1+ \rho (1-(1+t_o^\alpha)^{-\lambda})^2\}) - 1/(1+t_o^\alpha)] t_o^\alpha \log t_o, \\
\]

(3.25)

\[
0 = \left(\frac{m}{\lambda}\right) - \sum \frac{m}{1} \log (1+x_i^\alpha) \\
+ \frac{m}{1} \left\{1+ \rho (1-6(1+x_i^\alpha)^{-\lambda} +6(1+x_i^\alpha)^{-2\lambda})\right\} (1+x_i^\alpha)^{-\lambda} \log(1+x_i^\alpha)/ \\
(1- \{1+ \rho (1-3(1+x_i^\alpha)^{-\lambda} +2(1+x_i^\alpha)^{-2\lambda})\}) (1+x_i^\alpha)^{-\lambda}) \\
+ (n-m) \left\{1+ \rho (1-4(1+t_o^\alpha)^{-\lambda} +3(1+t_o^\alpha)^{-2\lambda})\right\}(1+t_o^\alpha)^{-\lambda} \log(1+t_o^\alpha)/ \\
(2-(1+t_o^\alpha)^{-\lambda} \{1+ \rho (1-(1+t_o^\alpha)^{-\lambda})^2\}) - \log(1+t_o^\alpha)\right\}. \\
\]

(3.26)
0 = \sum_{1}^{m} \left\{ (1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2\lambda}) \right\} (1+x_1^\alpha)^{-\lambda} / \\
(1- \left\{ 1 + \hat{\rho} \left[ (1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2\lambda}) \right] (1+x_1^\alpha)^{-\lambda} \right\} + (n-m) \left\{ (1+t_0^\alpha)^{-\lambda} \right\} (1-(1+t_0^\alpha)^{-\lambda})^2) / \\
(2-(1+t_0^\alpha)^{-\lambda} \left[ 1 + \hat{\rho} \left( 1-(1-t_0^\alpha)^{-\lambda} \right) ^2 \right]) . \\
(3.27)

Hence the MLE of \( \alpha, \lambda \) and \( \rho \) can be expressed as

\[ \hat{\alpha} = m \left[ (\lambda + 1) \sum_{1}^{m} (x_1^\alpha \log x_i)/(1+x_1^\alpha) \right] - \lambda \sum_{1}^{m} \left\{ (1 + \hat{\rho} \left[ (1-3(1+x_1^\alpha)^{-\lambda} + 6(1+x_1^\alpha)^{-2\lambda}) \right] (1+x_1^\alpha)^{-\lambda} \right\} - (n-m) \left( (1+t_0^\alpha)^{-\lambda} \right) \left( (1-(1+t_0^\alpha)^{-\lambda})^2 \right) / (1+(1-t_0^\alpha)) \right] ^{-1} \]

\[ (3.28) \]

\[ \lambda = m \left[ \sum_{1}^{m} \log (1+x_1^\alpha) \right] - \lambda \sum_{1}^{m} \left\{ (1 + \hat{\rho} \left[ (1-3(1+x_1^\alpha)^{-\lambda} + 6(1+x_1^\alpha)^{-2\lambda}) \right] (1+x_1^\alpha)^{-\lambda} \right\} \log (1+x_1^\alpha) / \\
\left(1- \left\{ 1 + \hat{\rho} \left[ (1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2\lambda}) \right] (1+x_1^\alpha)^{-\lambda} \right\} \right) - (n-m) \left\{ (1+t_0^\alpha)^{-\lambda} \right\} \left( (1-(1+t_0^\alpha)^{-\lambda})^2 \right) \log (1+t_0^\alpha) / \\
(2-(1+t_0^\alpha)^{-\lambda} \left[ 1 + \hat{\rho} \left( 1-(1-t_0^\alpha)^{-\lambda} \right)^2 \right] - \log (1+t_0^\alpha) \right] ^{-1} \]

\[ (3.29) \]

and

\[ \hat{\rho} = (n-m) \left[ (2-(1+t_0^\alpha)^{-\lambda}) \right] / \\
(2-(1+t_0^\alpha)^{-\lambda} \left[ 1 + \hat{\rho} \left( 1-(1+t_0^\alpha)^{-\lambda} \right)^2 \right] - 1 \right) \\
\times \left\{ (1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2\lambda}) (1+x_1^\alpha)^{-\lambda} \right\} / \\
(1- \left\{ 1 + \hat{\rho} \left[ (1-3(1+x_1^\alpha)^{-\lambda} + 2(1+x_1^\alpha)^{-2\lambda}) \right] (1+x_1^\alpha)^{-\lambda} \right\} \right] ^{-1} . \\
(3.30) \]
Remarks

1. In case of failure censored samples, the test terminates at $x_r$, the time-to-failure of the $r$th item. Hence $x_r$ is a random variable and $r$ is fixed. So, in all foregoing estimators obtained for time censored samples if we replace $t_0$ by $x_r$ and $m$ by $r$, we will get the estimators for failure censored samples.

2. It should be noted that estimator of one parameter needs the value of the other parameter. So when one parameter is known, the other can be obtained directly. But when both are unknown, the process of iteration is needed. The Newton-Raphson method for two unknowns may be used.

3. From equations (4.19) and (4.20) in chapter-V we see that the coefficient of variation is a function of $\rho$ only. Hence $\rho$ can be estimated as

$$\hat{\rho} = \frac{(7-6V^2) + (85-84V^2)^{1/2}}{(1+V^2)}$$  \hspace{1cm} \text{(for series)}$$

and

$$\hat{\rho} = \frac{(18V^{**2}-5) + (205-324V^{**2})^{1/2}}{(1+V^{**2})}$$  \hspace{1cm} \text{(for parallel)}$$

where $V^*$ and $V^{**}$ are the sample coefficient of variation.

Note that in each case the value of $\hat{\rho}$ will be chosen considering the limitations, $-1 \leq \rho \leq 1$.

4. PROPERTIES OF THE ESTIMATORS

In the foregoing sections we have considered maximum likelihood estimators of the parameters of failure distributions for series and parallel systems. In this section we will examine usefulness of these estimators in the light of classical properties of a good estimator, viz., unbiasedness, consistency and efficiency. The exact expressions
for various expectations, variances and covariances of the above estimators are very difficult to find out. However, for large samples, approximate values of them can be obtained.

4.1. Expectation of the estimators

The exact values of the expectation of the estimators are hard to find out. However, approximate values of them can be obtained using the well known result

$$E(f(X_1, \ldots, X_n)) \approx f(E(X_1), \ldots, E(X_n)).$$

All the estimators obtained before are biased and for large sample, the amount of bias is sufficiently small so that they are asymptotically unbiased.

4.2. Variance and covariance of the estimators

Exact expressions for variances and covariances of these above estimators are very difficult to obtain. However, approximate values for variances and covariances of the estimators can be obtained directly from the likelihood function as the sample size gets large, with the help of information matrix (I.M.).

The elements of the information matrix are obtained by taking the expected values of the negatives of the second order derivatives of the log likelihood function with respect to the parameters involved in the distribution function. The sample values would be replaced by expected values. Sometimes the exact expressions for various expectations of these second order derivatives are very difficult to find out. In such situations following Cohen (1965) approximations of these expected values can be obtained.
The usual method is to equate the variance–covariance matrix with the inverse of the information matrix and to find out the values of variances and covariances.

4.2.1. Variance and covariance of \((\hat{\theta}, \hat{\rho})\)

If \(L\) be the log likelihood function of the system life distribution arising out of bivariate exponential distribution, then the approximate variance–covariance matrix for \((\hat{\theta}, \hat{\rho})\) can be expressed as

\[
\begin{bmatrix}
V(\hat{\theta}) & \text{Cov}(\hat{\theta}, \hat{\rho}) \\
\text{Cov}(\hat{\rho}, \hat{\theta}) & V(\hat{\rho})
\end{bmatrix} = A^{-1},
\]

where

\[
A = \begin{bmatrix}
L_{\theta\theta} & L_{\theta\rho} \\
L_{\rho\theta} & L_{\rho\rho}
\end{bmatrix},
\]

the information matrix.

Case -1 Series system

For complete sample:

\[
L_{\theta\theta} = E(-\delta^2 L^*/\delta \theta^2)_{\theta=\hat{\theta}} = \frac{1}{n/3}[1+(\rho/3)+ (\rho/6U)(6+\rho)]
\]
\[
\times \{4 \exp(-'(6+\rho)/6) - 3 \exp(-(6+\rho)/12)\}
\]
\[
-(\rho(36 + 13\rho)/72 U)^2 \{8 \exp(-(6+\rho)/12)-3
\]
\[
- \rho(6 \exp(-(6+\rho)/6) - 8 \exp(-(6+\rho)/12)
\]
\[
+ 3) \} \exp(-(6+\rho)/12)]_{\theta=\hat{\theta}}, \quad (4.3)
\]

\[
L_{\rho\rho} = E(-\delta^2 L^*/\delta \rho^2)_{\theta=\hat{\theta}} = \frac{1}{n/2}(1 - (1/U))^2 \left. \right|_{\theta=\hat{\theta}} \quad (4.4)
\]
and

\[
L_{ep} = E(-\frac{2}{\theta^2} L^*/\theta \partial \theta)|_{\hat{\theta}, \hat{p}} = \frac{-n(6+p)/12 \theta^2}{\theta^2} \exp(-(6+p)/6) + 3 \exp(-(6 + p)/12))|_{\hat{\theta}, \hat{p}}
\]

\[(4.4')\]

where

\[
E(x) = \theta(6 + p)/12 \quad \text{and} \quad E(x^2) = \theta^2(36 + 13p)/72
\]

\[(4.8)\] in chapter v

and

\[
U = 1 + p\{1 - \exp(-(6 + p)/12)\} \{1 - 2 \exp(-(6 + p)/12)\}
\]

\[(4.5)\]

For type I censored sample

\[
L_{\theta \theta} = E(-\frac{2}{\theta^2} L^*/\theta \partial \theta)|_{\hat{\theta}, \hat{p}} = \frac{-n(6+p)/12 \theta^2}{\theta^2} \exp(-(6+p)/6) + 3 \exp(-(6 + p)/12))|_{\hat{\theta}, \hat{p}} \times \{4 \exp(-2 \alpha_1) - 3 \exp(- \alpha_1) + (p \alpha_1 \theta^2/ \theta^2)\}
\]

\[
\times \{8 \exp(-\alpha_1) - 3 - p(6 \exp(-2 \alpha_1) - 8 \exp(-\alpha_1) + 3)\}
\]

\[
\times \exp(-\alpha_1) - (4/3) (1-p)\theta t_0 (S - \rho(1-\exp(-t_0/\theta))
\]

\[
\times \exp(-t_0/\theta)^2 - (2p(1-p)\theta^2/ \theta^2)
\]

\[
\times \{S - 2 \exp(-t_0/\theta)\} \exp(-t_0/\theta)|_{\hat{\theta}, \hat{p}}
\]

\[(4.6)\]

\[
L_{ep} = E(-\frac{2}{\theta^2} L^*/\theta \partial \theta)|_{\hat{\theta}, \hat{p}} = \frac{-n(6+p)/12 \theta^2}{\theta^2} \exp(-(6+p)/6) + 3 \exp(-(6 + p)/12))|_{\hat{\theta}, \hat{p}} \times \{1 - \exp(-t_0/\theta)\}^3|_{\hat{\theta}, \hat{p}}
\]

\[(4.7)\]

and

\[
L_{pp} = E(-\frac{2}{\theta^2} L^*/\theta \partial \theta)|_{\hat{\theta}, \hat{p}} = \frac{n(6+p)/12 \theta^2}{\theta^2} \exp(-(6+p)/6) + 3 \exp(-(6 + p)/12))|_{\hat{\theta}, \hat{p}} \times \{1 - \exp(-t_0/\theta)\}^3|_{\hat{\theta}, \hat{p}}
\]

\[(4.8)\]

where

\[
E(m) = np_1
\]

\[(4.9)\]

\[
p_1 = F*(t_0)
\]

\[(4.10)\]
\[ a_1 = \mathbb{E}[x | x \leq t_0] \]
\[ = \frac{1}{2} \mathbb{F}(t_0)[(1/2)(1+2t_0/\theta) + (1+3t_0/\theta)exp(-3t_0/\theta) - (1+4t_0/\theta)exp(-4t_0/\theta)] , \quad (4.11) \]
\[ a_2 = \mathbb{E}[x^2 | x \leq t_0] \]
\[ = (\theta/2\mathbb{F}(t_0)) [(1/2)(1+(1+2t_0/\theta) + (1+3t_0/\theta)) exp(-2t_0/\theta) \]
\[ + \mathbb{F}(1/2)[(1+2(t_0/\theta) + (1+3t_0/\theta)) exp(-2t_0/\theta)] \]
\[ - (2/9)(2+(2+6(t_0/\theta) + 9(t_0/\theta)) exp(-3t_0/\theta)) \]
\[ + (1/3)(1+(1+4(t_0/\theta) + 8(t_0/\theta)) exp(-4t_0/\theta))] , \quad (4.12) \]
\[ M = 1 + \mathbb{F}[1 - \exp(-a_1)] \quad (1-2 \exp(-a_1)) \quad (4.13) \]
\[ S = 1 + \mathbb{F}[1 - \exp(-t_0/\theta)]^2 . \quad (4.14) \]

**Case 2: Parallel System**

For complete sample,

\[ L_{\theta \theta} = \mathbb{E}[-\Delta^2 | \Delta^2 \theta^2] \bigg|_{\theta=\theta} \quad (4.15) \]
\[ = \left( \frac{n}{\theta^2} \right) \left[ (1/6)(12-\rho) - ((18-\rho)/6A) \right] \left[ 1 + \mathbb{F}(1-6\exp((\rho-18)/12) \]
\[ + 6 \exp((\rho-18)/6)) \exp((\rho-18)/12) + (252-13\rho)/72A^2 \]
\[ \times \left( 1 + \mathbb{F}(1 - 12 \exp((\rho-18)/12) + 21\exp((\rho-18)/6) - 8\exp((\rho-18)/4) \]
\[ + \mathbb{F}(3\exp((\rho-18)/6) - 3\exp((\rho-18)/4) + 6\exp((\rho-18)/3)) \right) \exp((\rho-18)/12) \]
\[ \right] \quad (4.15) \]
\[ L_{\rho \rho} = \mathbb{E}[-\Delta^2 | \Delta^2 \rho^2] \bigg|_{\rho=\rho} \quad (4.16) \]
\[ = \left( \frac{n}{\rho^2} \right) \left[ (1-\rho)/12A^2 \theta \right] \left[ 1 - 6\exp((\rho-18)/12) + 9\exp((\rho-18)/6) \]
\[ - 4 \exp((\rho-18)/4) \right] \exp((\rho-18)/12) \bigg|_{\rho=\rho} \quad (4.16) \]
and
\[ L_{\rho\rho} = E\left( -\frac{\partial^2 L^{**}}{\partial \rho^2} \right) \bigg|_{\theta = \hat{\theta}} \]
\[ = \left( n/\tilde{A}^2 \right) \left[ 1 - 3 \text{exp}((\rho - 18)/12) + 2 \text{exp}((\rho - 18)/6) \right] \text{exp}((\rho - 18)/6) \bigg|_{\theta = \hat{\theta}} \tag{4.17} \]
where \( E(x) = \theta(18 - \rho)/12 \) and \( E(x^2) = \theta^2 (252 - 13\rho)/72 \)
(Eq. (4.9) in chapter v)
and
\[ A = 1 - \left[ 1 - 3 \text{exp}((\rho-18)/12) + 2 \text{exp}((\rho - 18)/6) \right] \text{exp}((\rho - 18)/12). \tag{4.18} \]

For type I censored sample
\[ L_{\theta\theta} = E\left( -\frac{\partial^2 L^{**}}{\partial \theta^2} \right) \bigg|_{\rho = \hat{\rho}} \]
\[ = \left( n/\tilde{A}^2 \right) \left[ -p + 2p \left( \frac{\beta}{B} \right) - (2p \left( \frac{\beta}{B} \right)) \right] \left[ 1 + \rho \left( 1 - 6 \text{exp}(-\beta_1) + 6 \text{exp}(-2\beta_1) \right) \right] \text{exp}(-\beta_1) \]
\[ + (p_2 \beta_2/B^2) \left[ 1 + \rho (1 - 12 \text{exp}(-\beta_1) + 2 \text{exp}(-2\beta_1) + 6 \text{exp}(-3\beta_1) \right) \right] \text{exp}(-\beta_1) \]
\[ + \rho^2 (3 \text{exp}(-2\beta_1) - 3 \text{exp}(-3\beta_1) + 6 \text{exp}(-4\beta_1)) \text{exp}(-\beta_1) \]
\[ + 2(1-p_2) (t_0/D) \{ D \left[ 1 + \rho \left( 1 - 4 \text{exp}(-t_0/\theta) + 3 \text{exp}(-2t_0/\theta) \right) \right] \text{exp}(-t_0/\theta) \} \]
\[ + (l-p_2) (t_0/D^2) \{ [1 + \rho \left( 1 - 8 \text{exp}(-t_0/\theta) + 10 \text{exp}(-2t_0/\theta) + 2 \text{exp}(-3t_0/\theta) \right) \}
\[ + \rho^2 (1 - \text{exp}(-t_0/\theta))^2 \text{exp}(-2t_0/\theta) \} \text{exp}(-t_0/\theta) \bigg|_{\theta = \hat{\theta}} \tag{4.19} \]

\[ L_{\rho\theta} = E\left( -\frac{\partial^2 L^{**}}{\partial \rho \partial \theta} \right) \bigg|_{\rho = \hat{\rho}} \]
\[ = \left( n/\tilde{A}^2 \right) \left[ p_2 \beta_1 \left[ 1 - 6 \text{exp}(-\beta_1) + 9 \text{exp}(-2\beta_1) + 4 \text{exp}(-3\beta_1) \right] \right] \text{exp}(-\beta_1) \]
\[ + 2(1-p_2) (t_0/D^2) \left[ 1 - 4 \text{exp}(-t_0/\theta) + 4 \text{exp}(-2t_0/\theta) - 3 \text{exp}(-3t_0/\theta) \right] \text{exp}(-t_0/\theta) \]
\[ + (l-p_2) (t_0/D^2) \{ 1 - \text{exp}(-t_0/\theta)^4 \text{exp}(-2t_0/\theta) \} \bigg|_{\theta = \hat{\theta}} \tag{4.20} \]

\[ L_{\rho\rho} = E\left( -\frac{\partial^2 L^{**}}{\partial \rho^2} \right) \bigg|_{\rho = \hat{\rho}} \]
\[ = \left( n/\tilde{A}^2 \right) \left[ (p_2/B^2) \left[ 1 - 3 \text{exp}(-\beta_1) + 2 \text{exp}(-2\beta_1) \right] \right] \text{exp}(-2\beta_1) \]
\[ + ((l-p_2)/D^2) \{ 1 - \text{exp}(-t_0/\theta)^4 \text{exp}(-2t_0/\theta) \} \bigg|_{\theta = \hat{\theta}} \tag{4.21} \]
where
\[ E(m) = np_2, \quad (4.22) \]
\[ p_2 = F^{**}(t_0), \quad (4.23) \]
\[ \beta_1 = E(x|x < t_0) \]
\[ = (\theta/F^{**}(t_0))[(7/4) + ((1/4)\exp(-t_0/\theta) - 2)\exp(-t_0/\theta) \]
\[ + (t_0/\theta)(1/2)\exp(-t_0/\theta) - 2)\exp(-t_0/\theta) \]
\[ - \beta[(1/12) - (1+2(t_0/\theta))\exp(-2t_0/\theta) + (1+3t_0/\theta)\exp(-3t_0/\theta) \]
\[ - (1 + 4(t_0/\theta))\exp(-4t_0/\theta))] \quad (4.24) \]
and
\[ \beta_2 = E[x^2|x < t_0] \]
\[ = (\theta^2/F^{**}(t_0))[(7/2) + (1/2)\exp(-t_0/\theta) - 4)\exp(-t_0/\theta) + (t \]
\[ + (t_0/\theta)^2(\exp(-t_0/\theta) - 4)\exp(-t_0/\theta) \]
\[ + (t_0/\theta)^2(\exp(-t_0/\theta) - 2)\exp(-t_0/\theta) \]
\[ - \beta[(13/72) - (1/2) - (1+2(t_0/\theta) + 2(t_0/\theta)^2)\exp(-2t_0/\theta) \]
\[ + (2/9)(2 + 6(t_0/\theta) + 9(t_0/\theta)^2)\exp(-3t_0/\theta) \]
\[ - (1/8)(1+4(t_0/\theta) + 8(t_0/\theta)^2)\exp(-4t_0/\theta))] \quad (4.25) \]

Also \( B = 1 - [1 + \beta(1-\exp(-\beta_1/\theta)](1 - 2\exp(-\beta_1/\theta))\exp(-\beta_1/\theta) \quad (4.26) \)
and \( D = 2 - [1 + \beta(1 - \exp(-t_0/\theta)^2)]\exp(-t_0/\theta) \quad (4.27) \)

4.2.2. Variance and covariance of \((\hat{\alpha}, \lambda, \rho)\)

Let \( L \) denote log likelihood function of the life distribution arising out of bivariate Burr distribution. Then the approximate variance - covariance matrix for \((\hat{\alpha}, \lambda, \rho)\) can be expressed as

\[
\begin{bmatrix}
V(\hat{\alpha}) & \text{Cov}(\hat{\alpha}, \lambda) & \text{Cov}(\hat{\alpha}, \rho) \\
\text{Cov}(\hat{\alpha}, \lambda) & V(\lambda) & \text{Cov}(\lambda, \rho) \\
\text{Cov}(\hat{\alpha}, \rho) & \text{Cov}(\lambda, \rho) & V(\rho)
\end{bmatrix} = B^{-1}, \quad (4.29)
\]
where

\[
B = \begin{bmatrix}
L_{aa} & L_{a\lambda} & L_{a\rho} \\
L_{a\lambda} & L_{\lambda\lambda} & L_{\lambda\rho} \\
L_{a\rho} & L_{\lambda\rho} & L_{\rho\rho}
\end{bmatrix}
\]

the information matrix.

**Case-1 Series system**

For complete sample

\[
L_{aa} = \left| \frac{\partial^2 \ln L}{\partial \alpha^2} \right|_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}, \rho=\hat{\rho}} = \frac{2}{n} \left( \frac{n}{\alpha^2} + (2\lambda+1) \sum \frac{x^\alpha \log x}{1+x^\alpha} \right)
\]

\[
= \frac{n}{n^2} \left( \frac{n}{\alpha^2} + (2\lambda+1) \sum \frac{x^\alpha \log x}{1+x^\alpha} \right)
\]

\[
= \frac{1}{n} \sum \frac{x^\alpha \log x}{1+x^\alpha}
\]

\[
= \frac{1}{n} \sum (1+x_1^{\alpha})^{-1} (3-4(1+x_1^{\alpha})^{-1} (1+x_1^{\alpha})^{-\lambda}) x_1^{\alpha} \log x_1
\]

\[
(4.29)
\]

\[
L_{\lambda\lambda} = \left| \frac{\partial^2 \ln L}{\partial \lambda^2} \right|_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}, \rho=\hat{\rho}} = \frac{2}{n} \left( \frac{n}{\lambda^2} + (2\alpha+1) \sum \frac{x^{\alpha} \log x}{1+x^{\alpha}} \right)
\]

\[
= \frac{n}{n^2} \left( \frac{n}{\lambda^2} + (2\alpha+1) \sum \frac{x^{\alpha} \log x}{1+x^{\alpha}} \right)
\]

\[
= \frac{1}{n} \sum \frac{x^{\alpha} \log x}{1+x^{\alpha}}
\]

\[
= \frac{1}{n} \sum (1+x_1^{\alpha})^{-1} (3-4(1+x_1^{\alpha})^{-1} (1+x_1^{\alpha})^{-\lambda}) x_1^{\alpha} \log x_1
\]

\[
(4.30)
\]

\[
L_{\alpha\rho} = \left| \frac{\partial^2 \ln L}{\partial \alpha \partial \rho} \right|_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}, \rho=\hat{\rho}} = \frac{2}{n} \left( \frac{n}{\alpha \rho} \right)
\]

\[
= \frac{1}{n} \sum (1+x_1^{\alpha})^{-1} (3-4(1+x_1^{\alpha})^{-1} (1+x_1^{\alpha})^{-\lambda}) x_1^{\alpha} \log x_1
\]

\[
(4.31)
\]

\[
L_{\lambda\rho} = \left| \frac{\partial^2 \ln L}{\partial \lambda \partial \rho} \right|_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}, \rho=\hat{\rho}} = \frac{2}{n} \left( \frac{n}{\lambda \rho} \right)
\]

\[
= \frac{1}{n} \sum (1+x_1^{\alpha})^{-1} (3-4(1+x_1^{\alpha})^{-1} (1+x_1^{\alpha})^{-\lambda}) x_1^{\alpha} \log x_1
\]

\[
(4.32)
\]

\[
L_{\alpha\lambda} = \left| \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \right|_{\alpha=\hat{\alpha}, \lambda=\hat{\lambda}, \rho=\hat{\rho}} = \frac{2}{n} \left( \frac{n}{\alpha \lambda} \right)
\]

\[
= \frac{1}{n} \sum (1+x_1^{\alpha})^{-1} (3-4(1+x_1^{\alpha})^{-1} (1+x_1^{\alpha})^{-\lambda}) x_1^{\alpha} \log x_1
\]

\[
(4.33)
\]
\[ L_{\lambda \rho} = \left| \mathbb{E}(\vartheta^2 \mathbf{L}^*)/(\vartheta \lambda \varphi) \right| \bigg|_{\alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho}} = - \frac{1}{\lambda} \sum_{i=1}^{n} (1/A) \{3-4(1+x_i^\alpha)^{-\lambda} \} (1+x_i^\alpha)^{-\lambda/2} \log (1+x_i^\alpha) |_{\alpha = \alpha, \lambda = \lambda, \rho = \rho} \tag{4.34} \]

\[ L_{\rho \rho} = \left| \mathbb{E}(\vartheta^2 \mathbf{L}^*)/(\vartheta \rho^2) \right| \bigg|_{\alpha = \alpha, \lambda = \lambda, \rho = \rho} = - \frac{1}{\lambda} \sum_{i=1}^{n} (1/A) \{1-(1+x_i^\alpha)^{-\lambda} \} \{1-2(1+x_i^\alpha)^{-\lambda} \} |_{\alpha = \alpha, \lambda = \lambda, \rho = \rho} \tag{4.35} \]

where

\[ A = 1+ \rho \{1-(1+x_i^\alpha)^{-\lambda} \} \{1-2(1+x_i^\alpha)^{-\lambda} \} \tag{4.36} \]

For type I Censored sample

\[ L_{\alpha \alpha} = \mathbb{E}(\vartheta^2 \mathbf{L}^*)/(\vartheta \alpha^2) = -\frac{1}{\lambda} \sum_{i=1}^{n} (1/A) \{3-4(1+x_i^\alpha)^{-\lambda} \} (1+x_i^\alpha)^{-\lambda/2} \log (1+x_i^\alpha) \]

\[ = \left| \frac{(m/\alpha)^2}{2(\lambda+1)} \sum_{i=1}^{m} (x_i^\alpha \log x_i)/(1+x_i^\alpha)^2 \right| + \frac{\rho}{\lambda} \sum_{i=1}^{m} (1/A) \{3-4(1+x_i^\alpha)^{-\lambda} \} (1+x_i^\alpha)^{-\lambda/2} \log (1+x_i^\alpha)^2 \]

\[ = \frac{\rho}{\lambda} \sum_{i=1}^{m} (1/A) \left\{ A \{ \lambda (8(1+x_i^\alpha)^{-\lambda} -3) + 4(1+x_i^\alpha)^{-\lambda} -3 \} \right\} \]

\[ = \frac{\rho}{\lambda} (1+x_i^\alpha)^{-\lambda} \{3-4(1+x_i^\alpha)^{-\lambda} \} \{1+(1+t_0^\alpha)^{-\lambda} \} (1+t_0^\alpha)^{-\lambda+1} \}

\[ - 2 \lambda (n-m) \left\{ \right\} \frac{\rho}{\mathbb{E}} (1- (1+t_0^\alpha)^{-\lambda}) \right\} (1+t_0^\alpha)^{-\lambda+1} \}

\[ + 2 \rho (n-m) \left\{ \right\} = \left( (1+x_i^\alpha)^{-\lambda} \right)^2 - (1/(1+t_0^\alpha)^2) \}

\[ \lambda (t_0^\alpha \log t_0^\alpha)^2 \bigg|_{\alpha = \alpha, \lambda = \lambda, \rho = \rho} \tag{4.37} \]
\[ L_{\alpha \lambda} = E\left( (\mathfrak{D}^2\mathfrak{D}^*)/(\mathfrak{D} \alpha \mathfrak{D} \lambda) \right) = - (\mathfrak{D}^2\mathfrak{D}^*)/(\mathfrak{D} \alpha \mathfrak{D} \lambda) \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ = 2 \sum_{1}^{m} \frac{\alpha x_i \log x_i}{(1+x_i^\alpha)} \]

\[- \rho \sum_{1}^{m} \frac{(l/A) \{3-4(l+x_1^\alpha)^{-\lambda}\} (1+x_1^\alpha)^{-\lambda + 1} x_1^\alpha \log x_i}{1+x_1^\alpha} \]

\[- \rho \lambda \sum_{1}^{m} \frac{(1/A^2)[A\{3-8(l+x_1^\alpha)^{-\lambda}\} + \rho (l+x_1^\alpha)^{-\lambda} \{3-4(l+x_1^\alpha)^{-\lambda}\}]^2}{l+x_1^\alpha} \]

\[ x_{1+x_1^\alpha}^{\alpha - (\lambda + 1)} x_1^\alpha \log x_i (1+x_1^\alpha) - 2(n-m) \{ (p/B)(1-(1+t_0^\alpha)^{-\lambda}) \}

\[ x_{1+t_0^\alpha}^{\alpha - (\lambda + 1)} - (1/(1+t_0^\alpha)) \} t_0^\alpha \log t_0 \]

\[ + (2/B^2)(n-m) \rho \lambda \left[ \frac{1}{B} \{1-2(1+t_0^\alpha)^{-\lambda}\} \right] (1/(1+t_0^\alpha)) + 2 \rho \{1-(1+t_0^\alpha)^{-\lambda}\}^2 (1+t_0^\alpha)^{-\lambda} \]

\[ x_{1+t_0^\alpha}^{\alpha - \lambda} \log t_0 (1+t_0^\alpha) \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ (4.38) \]

\[ L_{\alpha \rho} = E\left( \mathfrak{D}^2\mathfrak{D}^*/(\mathfrak{D} \alpha \mathfrak{D} \rho) \right) = - (\mathfrak{D}^2\mathfrak{D}^*)/(\mathfrak{D} \alpha \mathfrak{D} \rho) \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ = - \lambda \sum_{1}^{m} \frac{(1/A^2) \{3-4(l+x_1^\alpha)^{-\lambda}\} (1+x_1^\alpha)^{-\lambda + 1} x_1^\alpha \log x_i}{1+x_1^\alpha} \]

\[- (2 \lambda /B^2)(n-m) \{1-(1+t_0^\alpha)^{-\lambda}\} \right] (1+t_0^\alpha)^{-\lambda + 1} t_0^\alpha \log t_0 \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ (4.39) \]

\[ L_{\alpha \lambda} = E\left( (\mathfrak{D}^2\mathfrak{D}^*/(\mathfrak{D} \lambda \mathfrak{D} \lambda)) \right) = - (\mathfrak{D}^2\mathfrak{D}^*)/(\mathfrak{D} \lambda \mathfrak{D} \lambda) \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ = (m/\lambda^2) + \rho \sum_{1}^{m} \frac{(1/A^2)[A \{3-4(l+x_1^\alpha)^{-\lambda}\} + \rho (l+x_1^\alpha)^{-\lambda} \{3-4(l+x_1^\alpha)^{-\lambda}\}]^2}{l+x_1^\alpha} \]

\[ x_{1+x_1^\alpha}^{\alpha} \log^2(1+x_1^\alpha)+(2/B^2)(n-m) \left[ \frac{1}{B} \{1-2(1+t_0^\alpha)^{-\lambda}\} \right] \]

\[ + 2 \rho \{1-(1+t_0^\alpha)^{-\lambda}\}^2 (1+t_0^\alpha)^{-\lambda} \log^2(1+t_0^\alpha) \bigg| \alpha = \tilde{\alpha}, \lambda = \tilde{\lambda}, \rho = \tilde{\rho} \]

\[ (4.40) \]
\[ L_{\lambda \rho} = |E\left(-\left(\partial^2 I^*\right)/\partial \lambda \partial \rho\right)\rangle = \left(-\left(\partial^2 I^*\right)/\partial \lambda \partial \rho\right)|_{\lambda = \hat{\lambda}, \rho = \hat{\rho}} = \sum_{l=1}^{m} \left(1/A^2\right) \left[3-4(1+x_{i}^\alpha)^{-\lambda}\right] (1+x_{i}^\alpha)^{-\lambda} \log (1+x_{i}^\alpha) \]

\[ - \left(2/B^2\right)(n-m) \left[1(1+\alpha_{0})^{-\lambda}\right] (1+\alpha_{0})^{-\lambda} \log (1+\alpha_{0}) |_{\alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho}} \]

\[ L_{\rho \rho} = |E\left(-\left(\partial^2 I^*\right)/\partial \rho \partial \rho\right)\rangle = \left(-\left(\partial^2 I^*\right)/\partial \rho \partial \rho\right)|_{\alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho}} = \sum_{l=1}^{m} \left(1/A^2\right) \left[1-(1+x_{i}^\alpha)^{-\lambda}\right]^2 \left[1-2(1+x_{i}^\alpha)^{-\lambda}\right]^2 \]

\[ + \left(1/B^2\right)(n-m) \left[1-(1+\alpha_{0})^{-\lambda}\right]^4 \left[1-(1+\alpha_{0})^{-\lambda}\right]^2 \]

where \(A\) is given by Eq. (4.36) and

\[ B = 1 + \rho \left[1-(1+\alpha_{0})^{-\lambda}\right]^2 \]

Case 2. Parallel system

For complete sample

\[ L_{\alpha \alpha} = |E\left(-\left(\partial^2 I^{**}\right)/\partial \alpha \partial \alpha\right)\rangle = \left(-\left(\partial^2 I^{**}\right)/\partial \alpha \partial \alpha\right)|_{\alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho}} = |(n/\alpha^2) + (\lambda + 1) \sum_{l=1}^{n} \left(x_{i}^\alpha \log^2 x_{i}\right)/(1+x_{i}^\alpha)^2 \]

\[ - \lambda \sum_{l=1}^{n} \left(1/U\right) \left[1+\rho \left[1-6(1+x_{i}^\alpha)^{-\lambda} + 6(1+x_{i}^\alpha)^{-2\lambda}\right] (1+x_{i}^\alpha)^{-\lambda} x_{i}^\alpha \log^2 x_{i} \right] \]

\[ + \sum_{l=1}^{n} \left(1/U^2\right) \left[\lambda + 1 + \rho \left[\lambda + 1-6(2\lambda + 1)(1+x_{i}^\alpha)^{-\lambda} + 6(3\lambda + 1)(1+x_{i}^\alpha)^{-2\lambda}\right]\right] \]

\[ + \left[1+\rho \left[1-6(1+x_{i}^\alpha)^{-\lambda} + 6(1+x_{i}^\alpha)^{-2\lambda}\right]\right] \lambda \left(1+x_{i}^\alpha\right)^{-\lambda} \]

\[ \left(1+x_{i}^\alpha\right)^{-\lambda} x_{i}^\alpha \left(\lambda + 2\right) \left(x_{i}^\alpha \log x_{i}\right)^2 \]

(4.44)
\[ L_{x_1} = |E(\mathcal{R}(2L^{**}))/\mathcal{R} \alpha \lambda) = -(\mathcal{R}(2L^{**}))/\mathcal{R} \alpha \lambda) \mid_{\alpha=\lambda, \lambda, p=\hat{p}} \]

\[ = \frac{n}{1} \sum x_1^{x_1} \log x_1/(1+x_1^{x_1}) \cdot \frac{n}{1} \sum (1/U)[1+\rho(1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda})] \]

\[ * (1+x_1^{x_1}) - (\lambda+1) x_1 \log x_1 \]

\[ + \frac{n}{1} \sum (1/U^2) (1+\rho(1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda})] \]

\[ + (1+x_1^{x_1})^{-\lambda} \cdot \{(1+\rho(1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda})] \}^2 \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \] \hspace{1cm} (4.45)

\[ L_{\alpha \rho} = |E((2L^{**}))/\mathcal{R} \alpha \rho) = -(\mathcal{R}(2L^{**}))/\mathcal{R} \alpha \rho \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \]

\[ = -\lambda \frac{n}{1} \sum (1/U^2) [1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda}] \]

\[ * (1+x_1^{x_1}) - (\lambda+1) x_1 \log x_1 \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \] \hspace{1cm} (4.46)

\[ L_{\lambda \lambda} = |E(-((2L^{**}))/\mathcal{R} \lambda \lambda) = -(\mathcal{R}(2L^{**}))/\mathcal{R} \lambda \lambda \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \]

\[ = \frac{n}{1} \sum (1/U^2) [1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda}] \]

\[ + (1+x_1^{x_1})^{-\lambda} \cdot \{(1+\rho(1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda})] \}^2 \]

\[ \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \] \hspace{1cm} (4.47)

\[ L_{\lambda \rho} = |E(-((2L^{**}))/\mathcal{R} \lambda \rho) = -(\mathcal{R}(2L^{**}))/\mathcal{R} \lambda \rho \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \]

\[ = \frac{n}{1} \sum (1/U^2) [1-6(1+x_1^{x_1})^{-\lambda} + 6(1+x_1^{x_1})^{-2\lambda}] \]

\[ \log (1+x_1^{x_1}) \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \] \hspace{1cm} (4.48)

\[ L_{\rho \rho} = |E(-(2L^{**}))/\mathcal{R} \rho \rho) = -(\mathcal{R}(2L^{**}))/\mathcal{R} \rho \rho \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \]

\[ = \frac{n}{1} \sum (1/U^2) (1+x_1^{x_1})^{-2\lambda} \cdot \{1-6(1+x_1^{x_1})^{-\lambda}\} \cdot \{1-2(1+x_1^{x_1})^{-\lambda}\} \mid_{\alpha=\lambda, \lambda, \rho=\hat{p}} \] \hspace{1cm} (4.49)

where

\[ U = 1-(1+x_1^{x_1})^{-\lambda} \cdot \{1+\rho(1-6(1+x_1^{x_1})^{-\lambda})\} \cdot \{1-2(1+x_1^{x_1})^{-\lambda}\} \] \hspace{1cm} (4.50)
For type I Censored sample

\[ I_{\alpha} = |E(-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2})| = (-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2}) \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \]

\[ = (m/\alpha^{2}) + (\lambda + 1) \sum_{1}^{m} (x_{i}^{2} \log^{2} x_{i})/(1+x_{i}^{2}) \]

\[ - \lambda \sum_{1}^{m} (1/\mu^{2})[l + \rho [1 - 6(1+x_{i}^{2})^{-\lambda} + 6(1+x_{i}^{2})^{-2\lambda}] (1+x_{i}^{2})^{-\lambda} x_{i}^{2} \log^{2} x_{i}] \]

\[ + \sum_{1}^{m} (1/\mu^{2})[l + \rho [1 - 6(1+x_{i}^{2})^{-\lambda} + 6(1+x_{i}^{2})^{-2\lambda}] (1+x_{i}^{2})^{-\lambda} x_{i}^{2} \log^{2} x_{i}] \]

\[ + \lambda (1+x_{i}^{2})^{-\lambda} \{ [1 + \rho [1 - 6(1+x_{i}^{2})^{-\lambda} + 6(1+x_{i}^{2})^{-2\lambda}] ]^{2} (1+x_{i}^{2})^{-\lambda} \} \]

\[ + \lambda (n-m) [1 + \rho [1 - 4(1+t_{o}^{2})^{-\lambda} + 3(1+t_{o}^{2})^{-2\lambda}] (1+t_{o}^{2})^{-\lambda}] \times t_{o}^{2} \log^{2} t_{o} \]

\[ + (\lambda /\nu^{2}) (n-m) \{ [1 - 1 + \rho [1 - 4(1+t_{o}^{2})^{-\lambda} + 3(1+t_{o}^{2})^{-2\lambda}] ]^{2} (1+t_{o}^{2})^{-\lambda} \} \]

\[ + \lambda (n-m) (1+t_{o}^{2})^{-2} (t_{o}^{2} \log^{2} t_{o})^{2} \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \]

\[ \{ E(-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2}) \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \} \]

\[ \{ E(-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2}) \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \} \]

\[ (4.51) \]

\[ I_{\lambda \lambda} = |E(-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2})| = (-\sigma^{2}\mu^{*})/(\sigma^{2}\lambda^{2}) \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \]

\[ = (m/\lambda^{2}) + \sum_{1}^{m} (1/\nu^{2})[1 + \rho [1 - 12(1+x_{i}^{2})^{-\lambda} + 24(1+x_{i}^{2})^{-2\lambda} + 8(1+x_{i}^{2})^{-3\lambda}] \]

\[ + \rho^{2} [3 - 8(1+x_{i}^{2})^{-\lambda} + 6(1+x_{i}^{2})^{-2\lambda}] (1+x_{i}^{2})^{-2\lambda} \log (1+x_{i}^{2}) \]

\[ + (n-m) (2/\nu^{2}) [1 + \rho [1 - 8(1+t_{o}^{2})^{-\lambda} + 10(1+t_{o}^{2})^{-2\lambda} + 2(1+t_{o}^{2})^{-3\lambda}] \]

\[ + \rho^{2} (1+t_{o}^{2})^{-2\lambda} [1 - (1+t_{o}^{2})^{-\lambda}]^{2} (1+t_{o}^{2})^{-\lambda} (\log (1+t_{o}^{2}))^{2} \mid \alpha = \hat{\alpha}, \lambda = \hat{\lambda}, \rho = \hat{\rho} \]

\[ (4.52) \]
\[ I_{\alpha \rho} = E \left( -\left( \partial^2 I^{**}/(\partial \alpha \partial \rho) \right) \right) = E \left( -\left( \partial^2 I^{**}/(\partial \alpha \partial \rho) \right) \right)_{\alpha = \hat{\alpha}; \lambda = \hat{\lambda}; \rho = \hat{\rho}} \]

\[ = \sum_{l=1}^{m} \frac{1}{(1+2\rho)} \left[ \frac{1}{(1+2\rho)} \left[ 1+2\rho \left( 1+2\rho \right) \right] \right] \left( 1+2\rho \right) \log (1+2\rho) \]

\[ + \left( 2/2 \right) (n-m) \left[ 1+2\rho \left( 1+2\rho \right) \right] \left( 1+2\rho \right) \log (1+2\rho) \]

\[ \text{and} \]

\[ I_{\rho \rho} = E \left( -\left( \partial^2 I^{**}/(\partial \rho \partial \rho) \right) \right) = E \left( -\left( \partial^2 I^{**}/(\partial \rho \partial \rho) \right) \right)_{\alpha = \hat{\alpha}; \lambda = \hat{\lambda}; \rho = \hat{\rho}} \]

\[ = \sum_{l=1}^{m} \left( 1+2\rho \right) \left[ 1+2\rho \left( 1+2\rho \right) \right] \left( 1+2\rho \right) \log (1+2\rho) \]

\[ + \left( 2/2 \right) (n-m) \left[ 1+2\rho \left( 1+2\rho \right) \right] \left( 1+2\rho \right) \log (1+2\rho) \]
where \( U \) is given by Eq. (4.50) and

\[
V = 2 - \left[ 1 + \rho \{ 1 - (1 + t_o^a)^{-\lambda} \} \right] (1 + t_o^a)^{-\lambda}.
\]

(4.57)

**Remark**

Similarly we can construct variance-covariance matrix of the estimators for type II censored samples. Note that in this case \( x_r \), the failure-time of the \( r \)th failure, is a random variable. So \( t_o \) will be replaced by \( E(x_r) \), which is the mean of the \( r \)th order statistic. Also \( m \) will be replaced by \( r \).

5. **ESTIMATION OF RELIABILITY**

Reliability functions of series and parallel systems (with two identical components) arising out of bivariate exponential as well as Burr distributions have been estimated in this section. Exact expressions of these estimators are difficult to find out. However, approximate values of them can be obtained. Expectations and variances of these estimators have been found out.

5.i. **Estimation of reliability function for systems arising out of bivariate exponential distribution (Morgenstern model) and its properties.**

The reliability function involves two parameters, viz., \( \theta \) and \( \rho \). So estimated reliability becomes a function of estimators of the parameters. Mathematically,

\[
\hat{R}(\theta, \rho) = R(\hat{\theta}, \hat{\rho}).
\]

(5.1)

Also

\[
E[\hat{R}(\theta, \rho)] = R(E(\hat{\theta}), E(\hat{\rho})).
\]

(5.2)
and
\[ V[\hat{R}(\theta, \rho)] = \left[ \left( \frac{\partial R}{\partial \theta} \right)^2 V(\theta) + \left( \frac{\partial R}{\partial \rho} \right)^2 V(\rho) \right. \]
\[ + 2 \left( \frac{\partial R}{\partial \theta} \right) \left( \frac{\partial R}{\partial \rho} \right) \text{Cov}(\hat{\theta}, \hat{\rho}) \right]_{\theta = \hat{\theta}, \rho = \hat{\rho}}. \]  
(5.3)

The expressions \( V(\theta), V(\rho) \) and \( \text{Cov}(\hat{\theta}, \hat{\rho}) \) for series and hot parallel systems can be obtained from section 4.

For series system
\[ \hat{R}^*(\theta, \rho) = \{ 1 + \hat{\rho} (1-\exp(-t/\theta))^2 \} \exp(-2t/\theta). \]  
(5.4)
\[ \frac{\partial}{\partial \hat{\theta}} \hat{R}^* = (t/\theta) \hat{\theta}^* (t; \hat{\theta}, \hat{\rho}). \]  
(5.5)
\[ \frac{\partial}{\partial \hat{\rho}} \hat{R}^* = \{ 1-\exp(-t/\theta) \}^2 \exp(-2t/\theta). \]  
(5.6)

Hence \( V(\hat{R}^*) \) is known.

For parallel system
\[ \hat{R}^{**}(\theta, \rho) = \{ 2- \{ 1 + \hat{\rho} (1-\exp(-t/\theta))^2 \} \exp(-t/\theta) \} \exp(-t/\theta). \]  
(5.7)
\[ \frac{\partial}{\partial \hat{\theta}} \hat{R}^{**} = (t/\theta) \hat{\theta}^{**} (t; \hat{\theta}, \hat{\rho}). \]  
(5.8)
\[ \frac{\partial}{\partial \hat{\rho}} \hat{R}^{**} = - \{ 1-\exp(-t/\theta) \}^2 \exp(-2t/\theta). \]  
(5.9)

Thus \( V(\hat{R}^{**}) \) is known.

5.2. Estimation of reliability function for systems arising out of

bivariate Burr distribution (Morgenstern model) and its properties

Here the reliability function involves three parameters, viz., \( \alpha, \lambda, \rho \). So estimated reliability is given by
\[ \hat{R}(\alpha, \lambda, \rho) = R(\hat{\alpha}, \hat{\lambda}, \hat{\rho}). \]  
(5.10)

Also
\[ E[\hat{R}(\alpha, \lambda, \rho)] = R[E(\hat{\alpha}), E(\hat{\lambda}), E(\hat{\rho})]. \]  
(5.11)
and
\[ V[\hat{R}(\alpha, \lambda, \rho)] = \left[ \left( \frac{\partial^2 R}{\partial \hat{\alpha} \partial \hat{\lambda}} \right)^2 V(\hat{\alpha}) + \left( \frac{\partial^2 R}{\partial \hat{\lambda} \partial \hat{\rho}} \right)^2 V(\hat{\lambda}) + \left( \frac{\partial^2 R}{\partial \hat{\alpha} \partial \hat{\rho}} \right)^2 V(\hat{\rho}) + 2 \left( \frac{\partial^2 R}{\partial \hat{\alpha} \partial \hat{\lambda}} \right) \left( \frac{\partial^2 R}{\partial \hat{\alpha} \partial \hat{\rho}} \right) \text{Cov}(\hat{\alpha}, \hat{\lambda}) \text{Cov}(\hat{\alpha}, \hat{\rho}) + 2 \left( \frac{\partial^2 R}{\partial \hat{\lambda} \partial \hat{\rho}} \right) \left( \frac{\partial^2 R}{\partial \hat{\lambda} \partial \hat{\rho}} \right) \text{Cov}(\hat{\lambda}, \hat{\rho}) + 2 \left( \frac{\partial^2 R}{\partial \hat{\alpha} \partial \hat{\rho}} \right) \left( \frac{\partial^2 R}{\partial \hat{\lambda} \partial \hat{\rho}} \right) \text{Cov}(\hat{\alpha}, \hat{\lambda}) \text{Cov}(\hat{\alpha}, \hat{\rho}) \right]_{\hat{\alpha}=\alpha, \hat{\lambda}=\lambda, \hat{\rho}=\rho} \] (5.12)

The expressions for \( V(\hat{\alpha}) \), \( V(\hat{\lambda}) \), \( V(\hat{\rho}) \), \( \text{Cov}(\hat{\alpha}, \hat{\lambda}) \), \( \text{Cov}(\hat{\lambda}, \hat{\rho}) \) and \( \text{Cov}(\hat{\alpha}, \hat{\rho}) \) for the series and parallel systems can be obtained from section 4.

For series system

\[ \hat{R}^*(\alpha, \lambda, \rho) = (1+\hat{\alpha})^{-2\lambda} \left[ 1 + \hat{\rho} \left\{ 1 - \left( (1+\hat{\alpha})^{-\lambda} \right)^2 \right\} \right]. \] (5.13)

\[ \left( \frac{\partial \hat{R}^*}{\partial \hat{\alpha}} \right) = -(t/\hat{\alpha}) \log t f^*(t; \hat{\alpha}, \hat{\lambda}, \hat{\rho}) \] (5.14)

\[ \left( \frac{\partial \hat{R}^*}{\partial \hat{\lambda}} \right) = -(t^{-\hat{\alpha}/\hat{\lambda}})(1+\hat{\alpha}) \log(1+\hat{\alpha}) f^*(t; \hat{\alpha}, \hat{\lambda}, \hat{\rho}) \] (5.15)

\[ \left( \frac{\partial \hat{R}^*}{\partial \hat{\rho}} \right) = (1+\hat{\alpha})^{-2\lambda} \left\{ 1 - \left( (1+\hat{\alpha})^{-\lambda} \right)^2 \right\}. \] (5.16)

Hence \( V(\hat{R}^*) \) is known.

For parallel system

\[ \hat{R}^{**}(\alpha, \lambda, \rho) = \left[ 2 \left\{ 1 + \hat{\rho} \left( 1 - (1+\hat{\alpha})^{-\lambda} \right)^2 \right\} \left( 1+\hat{\alpha} \right)^{-\lambda} \right]. \] (5.17)

\[ \left( \frac{\partial \hat{R}^{**}}{\partial \hat{\alpha}} \right) = -(t/\hat{\alpha}) \log t f^{**}(t; \hat{\alpha}, \hat{\lambda}, \hat{\rho}) \] (5.18)

\[ \left( \frac{\partial \hat{R}^{**}}{\partial \hat{\lambda}} \right) = -(t^{-\hat{\alpha}/\hat{\lambda}})(1+\hat{\alpha}) \log(1+\hat{\alpha}) f^{**}(t; \hat{\alpha}, \hat{\lambda}, \hat{\rho}) \] (5.19)

\[ \left( \frac{\partial \hat{R}^{**}}{\partial \hat{\rho}} \right) = -(1+\hat{\alpha})^{-2\lambda} \left\{ 1 - \left( (1+\hat{\alpha})^{-\lambda} \right)^2 \right\}. \] (5.20)

So we can obtain \( V(\hat{R}^{**}) \).
PART - II

SOME CONTRIBUTIONS

TO

RELIABILITY SAMPLING PLANS
The reliability of an item (either a component or a system) as we have defined (in chapter I) as the probability that it performs its intended purpose adequately up to a specified time period under the given operating conditions or environment. The term adequate performance indicates that criteria must be established which clearly specify or define what is considered to be satisfactory operation. Hence reliability measures the performance quality of the product in terms of performance parameters like mean - time - to - failure, mean-time between failures, failure rate, etc. One method of measuring this performance quality of a product is to test a batch of items for a period of time and to note the failure times or the number of failures. Such a process is known as life testing. On the basis of life test, decision regarding the performance of the items is taken.

A reliability sampling plan is comprising a stopping rule for life testing together with a well defined decision rule for necessary action. Such a plan has to specify:

i. how many prototypes are to be tested,
ii. when should the units be checked, i.e., continuously or at instant of time previously decided upon,
iii. whether or not to replace unit that failed,
iv. whether or not additional units if needed, should be supplied to continue the test,
v. when should the test be stopped,
vi. what mathematical model will satisfactorily describe the distribution of failure times,
vii) What constitutes a failure? Many products which do not fail suddenly, deteriorate gradually and in such cases it is essential to decide beforehand the discrepancy in performance which constitutes failures.

viii. What are the criteria of acceptance/rejection,

ix. If there is a change in the manufacturing process, how the distribution of failure times alter.

To evaluate the quality of a product the techniques of attributes as well as variables acceptance sampling are well utilised by quality control (QC) engineers as an effective and economical method. Similarly, time sampling (along with a suitable life test) is used by reliability engineers to verify the reliability of items (either components or systems).

In general, reliability sampling plans bear a striking similarity to those of acceptance sampling plans used in product control. This is what we expect, because, the mathematical basis for both types of sampling is the same, though we might be sampling for different parameters. The basic differences between these two types of plans are:

i. In case of attribute sampling, the QC sampling plans are based on number of defectives whereas reliability sampling plans are based on number of failures with respect to time,

ii. In case of variable sampling, a measurable quality characteristic for product is used for QC plans whereas reliability plans are based on failure times only.

In product control, an assumption that the product must come from the controlled manufacturing process is made, likewise, in reliability sampling we assume a certain degree of maturity of design that must be
demonstrated before a sample is taken. In other words, there must be an indication of a good degree of belief that items are homogeneous in the sense that each was built to conform to the same engineering requirements (i.e., specifications) and was manufactured essentially under the same operating conditions using similar methods and processes.

Various sampling plans - acceptance or demonstration plans have been developed among others by Liberman and Kulin (1957), Goode and Kao (1961), Epstein (1960), Bagchi (1972, 1973, 1976), Mukherjee and Bagchi (1976, 1979). While a lot of works has been done for items (either a single component or a complex of components considered as a single entity) but a little has been reported so far for the system considering its functional diagram. But when the structure or the configuration of a complex system is known completely, it is always useful to consider reliability of each component separately and hence to determine the reliability of the complex system.

A reliability sampling plan can be classified into various categories depending on the nature of life tests and decision rules. A sampling plan based on number of failures is known as attribute sampling plan whereas a sampling plan based on times of failure is known as variable sampling plan. A plan which is based on a life test of fixed sample size is known as fixed sample size plan whereas in a sequential sampling plan the test is carried out sequentially. A life test can be both replacement and non-replacement types. In a non-replacement type test, items are tested simultaneously under identical conditions but an item is not replaced when it fails. On the other hand, in a replacement type test, failed item is always repaired or replaced during the operation time.
A sampling plan for two-component standby redundant systems have been derived by Bagchi (1976) considering components are statistically independent. In this dissertation a break through has been made by considering a sampling plan for two-component coherent dependent system. Also a two-stage sampling plan originated by Bagchi (1972) has been considered for two-component coherent systems.

Determination of reliability involves costs in practice. A design can be improved to the level as we desire by extending our technical knowledge as well as providing best type of materials. But such an improvement will involve a large amount of expenditure which may not be possible rather feasible from the economic standpoint of the organisation. So in determining the performance quality of an item the cost factor should always be accounted for. In fact, the choice of a sampling plan is largely dependent on the total cost that have to be incurred in using the sampling plan. An optimum economic sampling plan must have a good discriminating power of eliminating bad lots as well as ensuring minimum inspection costs.

In practice, the price of a lot is determined according to its performance quality. A producer expects a good return for superior quality of items; whereas, a consumer does not intend to pay a higher price for items of marginal quality. The idea of price adjusted sampling plan (PASP) for product inspection introduced by Foster (1970). He ruled out the possibility of rectifying inspection and the price of a lot is adjusted by a function of the observed number of defectives in the sample and by an appropriate choice of this function the consumer can take care of the lot quality.
Subsequently, the plan has been modified by Foster and Perry (1972), Mohapatra and Banerjee (1972), and Dey (1977). But nothing has been reported so far about a price adjusted reliability sampling plan. Here an attempt has been made to derive an economic reliability sampling inspection plan usable for components, for mixture of components, and for mixture of mortal and immortal items.

As we have already mentioned that in replacement type life test, items failed are renewed immediately (i.e., time for replacement is not allowed) during the operational period. As attempt to develop attribute sampling plan for items as well as for two component dependent systems (series and parallel) has been made considering total number of renewals during the operational period following an age replacement policy.