APPENDIX A

We first calculate $Z_{III}$ by applying the orthogonality condition [Eq. (2.6.5)] as in Sec 6 of Chapter 2. The orthogonality condition is given by

$$a^\mu \eta_\nu = 0$$

where

$$\xi_\mu = \left\{ \frac{\partial r_{II}}{\partial \zeta}, 0, 0, \frac{\partial r_{II}}{\partial \alpha} \right\}$$

and

$$\eta_\mu = \left\{ \frac{\partial t_{II}}{\partial \zeta}, 0, 0, \frac{\partial t_{II}}{\partial \alpha} \right\}$$

and contravariant components of the metric tensor $g^{\mu \nu}$ are obtained from Eq. (3.2.18). We find from (2.6.5) that

$$Z_{III}^2 = \cos \zeta \cos^2 \frac{\alpha}{2}$$

Now in order to have the same time coordinate $t_{II}$ at $r_I = a$ and $r_{III} = b$ we must have

$$Z_I^2 = Z_{III}^2$$

so using Eqs. (3.2.11) and (A1) we obtain

$$9t^2_*^{2/3} + 2(r_*^2 - a^2) = k \cos \zeta \cos^2 \frac{\alpha}{2} + \ell$$
where \( k \) and \( \ell \) are constants. Now using Eqs. (3.2.9) and (3.2.21) in Eq. (A3) we obtain, after putting \( r_I = a \) and \( \zeta = \zeta_0 \) in the resulting equation, the value of \( k \) and \( \ell \) as

\[
k = -4a^2 \csc^2 \zeta_0 \sec \zeta_0
\]

and

\[
\ell = 4a^2 \csc^2 \zeta_0
\]

Hence

\[
Z_{Ii}^2 = 4a^2 \csc^2 \zeta_0 \left(1 - \sec \zeta_0 \cos \zeta \cos^2 \frac{\alpha}{2}\right)
\]

**APPENDIX B**

The metrics of region I and region II are given by

\[
ds_I^2 = dt^2 - t_i^{4/3}(dr_i^2 + r_i^2 d\Omega^2)
\]

where we have taken \( k_1 = 0 \) and

\[
ds_{II}^2 = \left(1 - \frac{2m_1}{r_{II}}\right) dt_{II}^2 - \left(1 - \frac{2m_1}{r_{II}}\right)^{-1} dr_{II}^2 - r_{II}^2 d\Omega^2
\]

In Kruskal and Szekeres null coordinate system \((u_I, v_I)\) the SCH metric (3.2.2) is given by Eq. (3.3.6), viz.,

\[
ds_{II}^2 = -\frac{32m_1^3}{r_{II}} \exp \left(-\frac{r_{II}}{2m_1}\right) du_I dv_I - r_{II}^2 d\Omega^2
\]

where

\[
u_I = \left(\frac{r_{II}}{2m_1} - 1\right) \exp \left(\frac{r_{II}}{2m_1}\right)
\]

and
\[
\frac{u_I}{v_I} = \exp \left( \frac{t_{II}}{2m_1} \right). \tag{3.3.8}
\]

Now we consider
\[
X_I = \frac{1}{2a} (3t_I^{1/3} + r_I - a), \tag{B1}
\]
\[
Y_I = \frac{1}{2a} (3t_I^{1/3} - r_I + a). \tag{B2}
\]

and define the chart \([u_I(X_I), v_I(Y_I)]\) for the interior such that it joins continuously with the metric (3.3.6) at the boundary. Applying matching conditions for metrics (3.2.5) and (3.2.2) we obtain
\[
-g_{22} = r_{II}^2 = r_{II}^2 t_I^{4/3} = \frac{a^6}{81} (X_I + Y_I)^4 (X_I - Y_I + 1)^2 \tag{B3}
\]

From Eqs. (3.3.7) and (3.3.8) we have
\[
2r_{II} \frac{\partial r_{II}}{\partial u_I} = 8m_1^2 u_I \exp \left\{ - \left( \frac{r_{II} + t_{II}}{2m_1} \right) \right\} \tag{B4}
\]
and
\[
2r_{II} \frac{\partial r_{II}}{\partial v_I} = 8m_1^2 v_I \exp \left\{ - \left( \frac{r_{II} - t_{II}}{2m_1} \right) \right\}. \tag{B5}
\]

Now at the boundary \(r_I = a\),
\[
X_I = Y_I = \frac{1}{2a} 3t_I^{1/3} \tag{B6}
\]

Differentiating (B3) with respect to \(u_I\) and \(v_I\) we have at the boundary \(r_I = a\),
\[
- \frac{\partial g_{22}}{\partial u_I} \bigg|_{r_I=a} = 2r_{II} \frac{\partial r_{II}}{\partial u_I} \bigg|_{r_I=a} = \frac{dX_I}{du_I} \ 8m_1^2 X_I^3 (X_I + 1) \tag{B7}
\]
and
Where we have used Eq. (3.2.7) for the value of \( m_1 = \frac{2}{3} a^3 \). We can express Eqs. (B4) and (B5) in terms of \( X_t \) and \( Y_t \) respectively using Eqs. (3.2.7) and (3.2.8) at the boundary and then equate Eq. (B4) with (B7) and Eq. (B5) with (B8) which give on integration

\[
\begin{align*}
  u_t &= (X_t - 1) \exp \left( X_t + \frac{1}{2} X_t^2 + \frac{1}{3} X_t^3 \right), \\
  v_t &= (Y_t + 1) \exp \left( -Y_t + \frac{1}{2} Y_t^2 - \frac{1}{3} Y_t^3 \right),
\end{align*}
\]

In Eqs. (B9) and (B10) the integration constants turn out to be zero which are obtained by using Eqs (3.3.7) and (3.3.8) at the boundary \( r_t = a \) and for some time \( t_r \). Also we have taken the sign of \((u_t, v_t)\) to be \((+ , +)\) so that it match with the other boundary \( r_{III} = b \) of region III.

APPENDIX C

We consider the transformations

\[ \xi^\mu \rightarrow (r_{I}, \theta, \phi, t_{I}) \rightarrow (r_{IV}, \theta, \phi, t_{IV}) \]

and

\[ \eta^\mu \rightarrow (r_{III}, \theta, \phi, t_{III}) \rightarrow (r_{IV}, \theta, \phi, t_{IV}) \]

In comoving coordinates \((r_{I}, \theta, \phi, t_{I})\) we have

\[ U^\mu = \{0, 0, 0, 1\} \]

and in \((r_{III}, \theta, \phi, t_{III})\)
Thus we have

\[ U^1' = \frac{\partial r_{lv}}{\partial t_1}, \quad U^4' = \frac{\partial t_{lv}}{\partial t_1}, \quad U^2' = U^3' = 0 \] (C1)

and

\[ W^1' = \frac{\partial r_{lv}}{\partial t_{III}}, \quad W^4' = \frac{\partial t_{lv}}{\partial t_{III}}, \quad W^2' = W^3' = 0 \] (C2)

In calculating \( U^\mu' \), \( W^\mu' \) we shall use the fact that \( Z_I = Z_{III} \).

To calculate \( \bar{\alpha} \) from Eq. (3.4.33) we express \( W^\mu U_\mu \) in \((r_{iv}, t_{iv})\) coordinates:

\[ W^\mu' U'_\mu = N \left[ W^4' U^4' - \frac{W^1' U^1'}{N^2} \right] \] (C3)

Now Eq. (3.4.30) in \((r_{iv}, t_{iv})\) coordinates becomes

\[ U^\mu = (\cos \bar{\alpha}) U^\mu' + \left( \frac{\rho_2}{\rho_1} \right)^{1/2} (\sin \bar{\alpha}) W^\mu' \] (C4)

Thus the velocity ratio is given by

\[ \frac{V^1}{V^4} = \frac{U^1}{U^4} \] (C5)

Now we shall derive the explicit expressions for \( U^1', U^4', W^1' \) and \( W^4' \) in different cases. With these expressions of \( U^\mu' \) and \( W^\mu' \) we can calculate the velocity ratio \( V^1/V^4 \) in \((r_{iv}, t_{iv})\) coordinates from Eq. (C5).
PART I: \( k_1 = 0; \quad k_3 = 0, +1, -1. \)

Case (i). \( k_3 = +1. \)

From Eqs. (3.2.9), (3.2.21), (3.2.23) and (3.2.24) we obtain

\[
U^1' = \frac{2}{3} r_I t_I^{-1/3},
\]

\[
U^4' = \frac{3L(1 - D)^2 \sin^2 \xi_b \cos \xi_b}{2a^2 t_I^{1/3} [D(1 - D)]^{1/2}(1 - D - \sin^2 \xi_c)},
\]

\[
W^1' = \sin \xi \cot \frac{\alpha}{2},
\]

\[
W^4' = \frac{(1 - D)^2 \cos \xi \cot \frac{\alpha}{2}}{[D(1 - D)]^{1/2}(1 - D - \sin^2 \xi_c)}.
\]

where

\[
D = \left(1 - \frac{Z_{III}^2}{4a^2} \sin^2 \xi_b \right) \cos \xi_b \sec \xi_c
\]

Substituting Eqs. (C6)–(C9), (C3) and (3.2.30) in Eq. (3.4.33) we obtain (3.4.44). Then Eq. (C5) gives the velocity ratio \( V^1/V^4 \), viz., Eq. (3.4.42).

PART II: \( k_1 = +1; \quad k_3 = 0, +1, -1. \)

Case (i). \( k_3 = 0. \)

From Eqs. (3.2.44), (3.2.52), (3.2.54) and (3.2.55) we obtain

\[
U^1' = \sin \xi \cot \frac{\alpha}{2},
\]

\[
U^4' = \frac{E^3}{2b(E^3 - 4c^2)} \tan \xi_a \cos \xi \cot \frac{\alpha}{2},
\]

\[
W^1' = \frac{2}{3} r_{III} t_{III}^{-1/3},
\]

104
\[ W^{4'} = \frac{E^{3}t_{III}^{-1/3}}{3(E^2 - 4c^2)}. \]  \hspace{1cm} (C13)

Case (ii). \( k_3 = +1 \).

From Eqs. (3.2.44), (3.2.61) and (3.2.63) we obtain
\[
U^1' = \sin \zeta \cot \frac{\alpha}{2}, \\
U^4' = \frac{(1 - Q)^{3/2} \tan \zeta \cot \xi \cos \zeta \cot \frac{\varphi}{2}}{Q^{1/2}(1 - Q - \sin^2 \xi)}, \\
W^1' = \sin \xi \cot \frac{\beta}{2}, \\
W^4' = \frac{(1 - Q)^{3/2} \cos \xi \cot \frac{\beta}{2}}{Q^{1/2}(1 - Q - \sin^2 \xi)}. \]  \hspace{1cm} (C14)-(C17)

Case (iii). \( k_3 = -1 \).

From Eqs. (3.2.44), (3.2.78) and (3.2.80) we obtain
\[
U^1' = \sin \zeta \cot \frac{\alpha}{2}, \\
U^4' = \frac{(Q_1 - 1)^{3/2} \tan \zeta \coth \xi \cos \zeta \cot \frac{\varphi}{2}}{Q_1^{1/2}(Q_1 - 1 - \sinh^2 \xi)}, \\
W^1' = \sinh \xi \coth \frac{\beta}{2}, \\
W^4' = \frac{(Q_1 - 1)^{3/2} \cosh \xi \coth \frac{\beta}{2}}{Q_1^{1/2}(Q_1 - 1 - \sinh^2 \xi)}. \]  \hspace{1cm} (C18)-(C21)

Substituting Eqs. (C10)-(C13), (C3), (3.2.57), (3.2.58) in Eq. (3.4.33) we obtain (3.4.49) in case (i); Eq. (3.4.54) has been obtained by substituting Eqs. (C14)-(C17), (C3), (3.2.73), (3.2.74) in Eq. (3.4.33) in case (ii) and we obtain Eq. (3.4.59) by substituting Eqs. (C18)-(C21), (C3), (3.2.90), (3.2.91)
in Eq. (3.4.33) in case (iii). Then Eq. (C5) gives Eq. (3.4.47) in case (i), (3.4.52) in case (ii) and (3.4.57) in case (iii).

PART III: \( k_1 = -1; \quad k_3 = 0, +1, -1. \)

Case (i). \( k_3 = 0. \)

From Eqs. (3.2.95), (3.2.103), (3.2.105)–(3.2.107) we obtain

\[
U^1' = \sinh \zeta \coth \frac{\alpha}{2},
\]

\[
U^4' = \frac{E^3}{2b(E^2 - 4c^2)} \tanh \zeta \cosh \zeta \coth \frac{\alpha}{2},
\]

\[
W^1' = \frac{2}{3} r_{III} t_{III}^{-1/3},
\]

\[
W^4' = \frac{E^3 t_{III}^{-1/3}}{3(E^2 - 4c^2)}.
\]

Case (ii). \( k_3 = +1. \)

From Eqs. (3.2.95), (3.2.114) and (3.2.116) we obtain

\[
U^1' = \sinh \zeta \coth \frac{\alpha}{2},
\]

\[
U^4' = \frac{(1 - Q)^{3/2} \tanh \zeta \cot \xi_b \cosh \zeta \coth \frac{\alpha}{2}}{Q^{1/2}(1 - Q - \sin^2 \xi_b)},
\]

\[
W^1' = \sin \xi \cot \frac{\beta}{2},
\]

\[
W^4' = \frac{(1 - Q)^{3/2} \cos \xi \cot \frac{\beta}{2}}{Q^{1/2}(1 - Q - \sin^2 \xi_b)}.
\]

Case (iii). \( k_3 = -1. \)

From Eqs. (3.2.95), (3.2.131) and (3.2.133) we obtain
\[ U^{1'} = \sinh \xi \coth \frac{\alpha}{2}, \quad (C30) \]
\[ U^{4'} = \frac{(Q_1 - 1)^{3/2} \tanh \xi \coth \xi \cosh \xi \coth \frac{\alpha}{2}}{Q_1^{1/2}(Q_1 - 1 - \sinh^2 \xi_c)}, \quad (C31) \]
\[ W^{1'} = \sinh \xi \coth \frac{\beta}{2}, \quad (C32) \]
\[ W^{4'} = \frac{(Q_1 - 1)^{3/2} \cosh \xi \coth \frac{\beta}{2}}{Q_1^{1/2}(Q_1 - 1 - \sinh^2 \xi_c)}. \quad (C33) \]

Substituting Eqs. (C22)–(C25), (C3), (3.2.109), (3.2.110) in Eq. (3.4.33) we obtain (3.4.64) in case (i); Eq. (3.4.69) has been obtained by substituting Eqs. (C26)–(C29), (C3), (3.2.126), (3.2.127) in Eq. (3.4.33) in case (ii) and we obtain Eq. (3.4.74) by substituting Eqs. (C30)–(C33), (C3), (3.2.143), (3.2.144) in Eq. (3.4.33) in case (iii). Then Eq. (C5) gives Eq. (3.4.62) in case (i), (3.4.67) in case (ii) and (3.4.72) in case (iii).