CHAPTER - I

INTRODUCTION

Our understanding of physical processes in the microscopic realm is largely based on the quantum theory of scattering and the amount of relevant experimental data available. A scattering experiment is performed by sending a beam of particles towards a target (usually stationary in the laboratory) in the form of a wave packet. In the remote past when the packet was prepared the particles were free because they did not interact with the scatterer. In the course of time, the probing particles or the missiles interact with the target and are scattered in all directions. One observes the energy, angular distribution, intensity, polarization and other compatible characteristics of the scattered beam. The mathematical theory of scattering enables us to compute these observable quantities from a given law of interaction.

In elementary quantum mechanics the two-particle scattering problem is treated as follows.

We consider a particle with positive energy $E$ moving in a
spherically symmetric potential $V(r)$ with a wave function whose radial part is $u(r, E)$. The Schrödinger equation is then

$$\left[ \frac{d^2}{dr^2} - V(r) + E \right] u(r, E) = 0 \quad (1.1)$$

with the boundary condition

$$u(0, E) = 0. \quad (1.2)$$

In writing (1.1) we consider only the s-wave scattering and omit the subscript $\ell = 0$. If the potential satisfies the condition

$$\int_0^{\infty} |V(r)| \, dr < \infty, \quad (1.3)$$

then the asymptotic behaviour of the wave function is given by

$$u(r, E) \sim \sin (kr + \delta(E)) \quad (1.4)$$

with $k = E^{1/2}$ as $r \to \infty$. The function $\delta(E)$ in (1.4) is the phase-shift for s-wave scattering at energy $E$, and is a measurable quantity. Historically, for the first twenty years (1926-1946) after the discovery of wave mechanics, physicists were concerned with the problem of calculating phase shift $\delta(E)$ produced by a given potential $V(r)$. This is referred to as a "direct scattering problem". Potentials were either deduced
from atomic or nuclear models or simply guessed, and the resulting phase-shift compared with experiment. About 1946 it became clear that many different potentials were consistent with the experimental data, so that the choice of a potential, for example, in the case of neutron-proton scattering was determined more by personal taste than by experiments. Confronted with this situation, people began to ask the question, "Is there a way to solve the problem backward, to begin with a phase-shift $\delta(E)$, supposedly measured for all energies $E$, and to calculate by strict mathematical procedure the potential $V(r)$ that would produce the phase shift?" This is the purpose of the well known "inverse scattering problem". The 'direct' and 'inverse' problems relating to two-particle scattering have been extensively discussed in the literature. In fact, with the availability of high speed digital computers we can solve the Schrödinger equation for potential scattering and calculate the scattering matrix easily. The inverse problem can also be solved with almost equal ease and reliability. As a result many people have the feeling that we do not need to have the theory of potential scattering. In fact, when we are dealing with simple potential scattering we need only to calculate on-shell or elastic scattering matrices. In such a case the above feeling in a practical sense is justified. Moreover, if we go a little further beyond simple potential scattering, we immediately encounter the problem
of treating off-shell scattering or wave matrices.

In scattering reactions involving more than two particles energy and momentum are not necessarily conserved in the internal two-body system since momentum can be absorbed by the other particles. In other words, the different pairs of particles do not scatter elastically from each other and therefore in addition to the on-shell two-particle amplitude we require the knowledge of off-shell amplitudes to determine the properties of a many-particle system. The dynamical origin of the off-shell effects may be visualized better in terms of quantum field theoretic ideas. Wick \[1\] showed that the finite range of nuclear forces \[2\] arises naturally from the coupling of the uncertainty principle with special theory of relativity. This was stimulated by experiment e.g., by the discovery of pions. Ever since we have known that nuclei must contain pions in addition to neutrons and protons. Despite that most of nuclear physics has been developed by using phenomenological nuclear potentials. Naturally, these potentials do not take explicit account of these mesonic degrees of freedom. However, there have been attempts to construct the so called meson theoretic potentials using in many cases either a combination of field theory and dispersion theory or one boson exchange models \[3\] roughly correlated with empirical boson mass spectrum. Although such models can after
considerable effort and a generous use of empirical parameters, provide a reasonably quantitative description of two-nucleon elastic scattering, this is no guarantee that these models provide a correct description even of the nucleonic degrees of freedom at short distances, due to the fact that there are always an infinite number of ways to describe this short range behaviour which lead to identical elastic scattering amplitudes.

Thus the stringent test of the nuclear force models comes from confronting them with three-nucleon data. Unfortunately the test fails. The "realistic potentials" underbind the triton by 1 to 1.5 MeV, and predict electrostatic form factors with a first minimum at higher energy and a second maximum of such smaller magnitude [4] than given by the experimental results of e - He$^3$ elastic scattering. So we see that the mesonic degrees of freedom do really matter in a quantitative sense, whether due to the fact that they give short range behaviour quite different from (though phase equivalent to) the "realistic potentials". The hidden mesonic degrees of freedom associated with the phenomenological potentials may be attributed to the so called off-shell effects [5].

A class of nuclear reactions are believed to probe the off-shell two-nucleon force. These include (p,2p) reactions [6], bremsstrahlung [7], and pion production near threshold [8]. To describe them one usually envisages reaction
mechanisms which involve two processes:

(1) a pair of nucleons interact with each other, and
(2) a break up or production process occurs.

The second step limits the extent of time taken for step one so that the scattering of nucleons do not conserve energy.

There are three energies involved in a general two-nucleon scattering theory: the initial relative energy, \( k_i^2/2m \), and the center of mass energy of propagation \( E \). In the usual notation the partial wave \( T \) matrix for a scattering is written as \( T_\ell(k_i, k_f, E) \). If the three energies are all unequal, the \( T \) matrix is said to be fully off-shell. The fully off-shell \( T \) matrix plays a role in the theories of three-particle system [9].

In the reactions described above, the time for the two-nucleon scattering process is restricted on one side only. After their interaction, the two nucleons proceed to infinity unscattered. The center of mass energy of propagation must therefore be equal to the final relative kinetic energy. The \( T \) matrix applicable in this case \( T_\ell(k_i, k, k^2/2m) \) is called a half-off-shell \( T \) matrix.

In two-nucleon scattering, there are no other particles to provide additional scatterings and thus constrain the time
available for two-nucleon interaction. As a result the three energies must be equal. The \( T \) matrix for the elastic process \( T_\ell (k, k, k^2/2m) \) is called on-shell.

The \( T \) matrix introduced above relate to the outgoing wave boundary condition. The scattered wave involves an outgoing spherical wave in addition to the incident plane wave. We can consider a corresponding situation with a plane wave in the incident channel and standing waves in all other channels. The scattering operator appropriate to this situation is the \( K \) matrix. Like the \( T \) matrix, the matrix element of the \( K \) operator can also be on-shell, half-off-shell and fully off-shell. Both \( T \) and \( K \) matrices have important applications in the study of nuclear reactions. In principle, a \( K \) matrix formulation is sometimes preferable to a formulation based on the \( T \) matrix, because the former permits approximate calculations incorporating the conditions of unitarity. This point has been discussed in some detail by Rodberg and Thaler [10]. But unfortunately, the calculation of the off-shell two-body \( K \) matrix poses certain problems. In contrast to the \( T \) matrix, the integral equation for the \( K \) operator does not have a Lippmann-Schwinger iteration [11]. Thus it is of some interest to look for a suitable mathematical framework to derive expressions for \( K \) matrices for the so called realistic nucleon-nucleon potential. Work in this direction has been
initiated by Kouri and Levin [12], who related the matrix elements of the K operator to the matrix elements of an altered K. They could thus avoid the problem in solving the associated Lippmann-Schwinger equation.

The purpose of the present work is two-fold.

(i) To develop a differential equation approach to the K matrix theory.

(ii) To seek a fully off-energy-shell generalization of the variable-phase-approach (VPA) to potential scattering [13] and study the spatial behavior of T and K matrices for some selected potentials.

The differential equation approach (DEA) to off-shell scattering has recently received considerable attention. Particularly, introduction of the Jost type functions by Fuda and Whiting [14] has initiated a number of works in this area. Calculations of T matrices for Morse and Woods-Saxon potentials have followed immediately [15].

In the DEA, the K matrix is obtained from an inhomogeneous form of the Schrödinger equation in which the inhomogeneous term is proportional to a free wave. The solution of this equation has been referred to as the off-shell wave function. For a K matrix calculation one is concerned with the off-shell wave function with standing wave boundary condition.
An important element of our treatment is that apart from its conceptual simplicity, we can also derive certain calculational advantages thereby. For example, exact expressions for $K$ matrices for local potentials can easily be constructed by using the present approach.

Pioneered by Morse and Allis [16], the VPA to potential scattering has been beautifully expounded by Calogero [13] and others. The mathematical foundation of this approach is based on the well known connection between the second order differential equation and first order equations of the Riccati type [17]. The Schrödinger equation can thus be reduced to a non-linear equation of the first order. The physical content of this approach is that a function which satisfies the Riccati equation (a phase function $\delta(r)$) has at each point the meaning of the phase shift of the wave function for scattering by the potential at that point. The equation for $\delta(r)$ is solved with the initial condition $\delta(0) = 0$. This implies that a completely amputed potential can produce no phase shift. Here phase shift is defined by $\delta = \lim_{r \to \infty} \delta(r)$. For further details the reader is referred to the excellent monograph of Calogero [13] and review article of Babikov [18]. The VPA has proven fruitful in theoretical and numerical applications of scattering theory. Some advantages of the method are ease of
interpretation, convenience for generation of approximations and determination of bounds on the error \( [13, 18] \). Equations for on-shell interpolating \( T \) and \( K \) matrices are readily obtained within the framework of Calogero's approach. Sobel \( [19] \) could treat the half-off-shell case by using a formalism which closely parallels the treatment of Calogero \( [13] \). But the fully off-shell case cannot be treated in a similar manner. We shall show in this work that the DEA indicated above can be combined with the powerful mathematical technique of invariant imbedding \( [20] \) to develop a fully off-shell generalization of the VPA. The need for such an investigation may be visualised as follows.

It has long been known that local nuclear potentials contain a hard core, a phenomenological intermediate region and a one pion exchange tail. However, it is found that different potentials have the same value for \( r \) greater than certain \( r' \), but differ for \( r \) less than \( r' \), although they fit the scattering data \( [21] \). Thus one may say that scattering experiments have probed the potential down to \( r' \) (the one pion exchange tail) but not closer. It is, therefore, of some interest to express off-shell matrix elements in a form which exhibits dependence on the different regions of \( r \)-space. The off-shell equations thus obtained will delineate, on the one hand, the spatial behaviour of \( T \) and \( K \) operators and on the other hand meet the basic requirements of the
few-body dynamics.

In the next chapter we review the basic results of the elastic or on-energy-shell scattering with particular emphasis on the role of Jost functions in the nonrelativistic scattering theory. The main idea here is to sketch the background for the motivation of our work.

In chapter 3 we try to bring out the concept of off-shell scattering by using the integral form of the Schrödinger equation and thus introduce $T$ and $K$ matrices formally. We also discuss certain properties of the $K$ matrix and indicate that the integral equation for the $K$ operator does not have a Lippmann-Schwinger iteration.

In chapter 4 we review the $K$ matrix theory of Kouri and Levin to develop a feeling for a possible wave function approach to the problem.

In chapter 5 we develop the DEA (also called the wave function approach) to the $T$ and $K$ matrix theory and derive various solutions and half-off-shell potential matrix elements. Having obtained the off-shell wave solutions with different boundary conditions, we establish relations between them and compare the results with the corresponding on-energy-shell relations given in chapter 2. Based on our results we also
present a proof of the off-shell unitarity relation [22] and show that the $K$ matrix theory of Kouri and Levin [12] is only a step forward of this basic relation.

Expressions for off-shell $T$ and $K$ matrices are derived in chapter 6 in terms of wave solutions obtained in chapter 5. The relations between $T$ and $K$ matrices are then established, which agree with the results of Kouri and Levin. It is also shown that the formulation which applies for potentials regular at the origin needs certain modification for dealing with potentials which have $1/r$ singularity at the origin.

In chapters 7 and 8 we obtain exact analytic expressions for fully off-shell two-body $K$ matrices for exponential, Morse and Hulthen potentials in terms of tabulated transcendental functions. In close analogy with the treatment of Hulthen potential we also derive result for the Yukawa potential by using the Ecker-Weizel approximation [23].

In chapter 9 we introduce the basic equations of the VPA as given by Calogero[13]. We also present two slightly different derivations of these equations based on

(i) the principle of invariant imbedding

(ii) the Gell-Mann-Goldberger two-potential theorem.

The main idea here is to sketch the background for possible off-energy-shell generalization of the VPA. We combine the DEA described in chapter 5 with the VPA by the powerful mathematical technique of invariant imbedding to develop a variable-phase-
off-shell-scattering theory. We present equations for interpolating off-shell T and K matrix functions. These equations can be used to study the spatial behavior of the T and K matrices. A statement of the off-energy-shell unitarity relation[22] is presented within the framework of this approach. Equations of Sobel and of Calogero are obtained as special cases. The spatial behavior of the T matrix for the combination of two Yukawa and for the Morse soft core potential is studied and some concluding remarks are presented with regard to the usefulness of the method developed.

And finally, in chapter 10 we try to summarize our outlook on the DEA to off-shell scattering with special emphasis on the uniqueness of the DEA to off-shell scattering. In particular, we employ works of Pasquier and Pasquier[24] and of Sasakawa and Sawada[25] to establish the uniqueness of the DEA. We also show that the wave function approach introduced in chapter 5 provides a natural framework for a possible off-energy-shell generalization of the Gell-Mann-Goldberger-Two-potential theorem (TPF). We point out that this generalized TPF can be used to rederive the new equations introduced in chapter 9.

For reasons of pedagogical interest we describe in appendix A a method for constructing an expression for the Coulomb distorted nuclear T matrix without the use of TPF.