CHAPTER 10

CONCLUSION

We have tried in this dissertation to look at certain aspects of the two-body off-energy-shell potential scattering within the framework of a differential equation approach to the problem. We have clearly indicated in the text that in this approach T and K matrices are obtained by using solutions of an inhomogeneous form of the Schrödinger equation with appropriate boundary conditions. These solutions are referred to as the off-shell wave functions. To obtain off-shell wave function we proceeded as follows.

Suppose the asymptotic wave is known, we solve the wave equation inward to obtain the whole solution. This represents the state of the art both for physical (outgoing wave) and standing wave solutions. In this procedure it is not quite evident that the solution should be unique. We shall presently take up this question and summarize the situation with regard to this. Meanwhile, we note that it is of some interest to work with the differential equation approach because one can always derive certain calculational advantages thereby. For example, using this approach we could obtain exact analytical expressions for the exponential, Morse, and Hulthén K matrices in terms
The Morse and exponential potentials are regular at the origin. K matrices for these potentials could therefore be determined by writing the operator equation $K = V \mathcal{N}$ in a mixed representation.

The situation is little different for the Hulthén potential. The Hulthén potential has a $1/r$ singularity at the origin. This tends to pose certain problems. In fact, slight modification of the wave function approach is needed for the purpose. This modification amounts to obtaining an expression for the K matrix which does not involve the potential explicitly, thereby avoiding certain integrals which are difficult to perform because of $1/r$ singularity in the Hulthén potential. Similar treatment is also expected to work for the Yukawa potential. But unfortunately for this potential neither the Schrödinger equation nor the inhomogeneous differential equation relating to the off-shell scattering problem can be solved in closed form. However, working within the framework of the Ecker - Weizel approximation we could solve for the off-shell wave function and construct an expression for the Yukawa K matrix in close analogy with our work on the Hulthén potential.

Studies taken up in this dissertation are tailored to be appropriate for possible usage of our expressions for K matrices of tabulated transcendental functions.
in nuclear scattering reactions. Thus physical conclusions can be drawn from the mathematical constructs presented only through extensive investigations. While such an attempt by our limited endeavour we take this opportunity to comment on the validity of the inward extrapolation of asymptotic waves. Such an analysis is necessary for and often bound up with the type of problems considered here. To that end we look at a recent work of Pasquier and Pasquier [24]. We begin with the Møller wave operator,

\[ \Omega = (1 + G^+ \psi - \Omega) \]  
(10.1)

and rewrite it in terms of the relation

\[ G^+ = -|w><w| + G^I \]  
(10.2)

to get

\[ \Omega = I + (-|w><w| + G^I) \psi \Omega . \]  
(10.3)

The symbols appearing in (10.1) and (10.2) have been explained in the text. Expressing (10.3) in a mixed representation we obtain for the s-wave case

\[ \psi^+(k,q,r) = \sin qr - \frac{1}{k} \int_{0}^{\infty} \sin k(r-r') v(r') \psi^+(k,q,r') \ dr' \]

\[ - \frac{1}{k} e^{ikr} \int_{0}^{\infty} \sin kr' v(r') \psi^+(k,q,r') \ dr'. \]  
(10.4)
As usual \( k \) and \( q \) refer to the on and off-shell momenta. The superscript ' + ' on \( \Psi \) refer to the outgoing wave boundary condition. For clarity of presentation we deal with the s-wave case only and suppress the subscript zero. It is of interest to note that (10.4) is equivalent to two integral equations one of which is Fredholm-like and the other, Volterra-like.

In particular we have

\[
\psi^+(k,q,r) = \frac{1}{2\text{i}} \left[ f(k,q,r) - f(k,-q,r) \right]
- \frac{f(k,r)}{k} \int_0^\infty \sin kr' \Psi^+(k,q,r') \, dr'
\]

(10.5)

and

\[
f(k,q,r) = e^{\text{i}qr} + \int_r^\infty G(r,r')v(r')f(k,q,r') \, dr'.
\]

(10.6)

The function \( f(k,q,r) \) in (10.6) satisfies the differential equation and boundary condition [14] prescribed for the off-shell irregular solution. Thus we see that the decomposition carried out above introduces in a rather natural way the so-called generalized or off-shell Jost solutions. The off-shell Jost functions and their on-shell analogues can now be introduced as in Chapter 5. The integral \( \int_0^\infty \sin kr' v(r') \Psi^+(k,q,r') \, dr' \) in (10.5) is related to the half-off-shell T matrix and can be determined in terms of Jost functions by the behavior of \( \psi^+(k,q,r) \).
near the origin. We therefore have

\[ \psi^+(k,q,r) = \frac{1}{2i} \left[ f(k,q,r) - f(-k,q,r) \right] - \frac{1}{2} \pi q T(k,q,k^2) f(k,r) \quad (10.7) \]

where \( T(k,q,k^2) \) is the half-off-shell T matrix given in (5.46).

In a similar way rewriting the Møller wave operator for the standing wave boundary condition in terms of the relation

\[ G^p = -\langle v \mid u \rangle^T G^T \], \quad (10.8) \]

it can be shown that

\[ \psi^+(k,q,r) = \frac{1}{2i} \left[ f(k,q,r) - f(-k,q,r) \right] - \frac{1}{4} \pi q K(k,q,k^2) \left[ f(k,r) + f(-k,r) \right] \quad (10.9) \]

where \( K(k,q,k^2) \) has been given in (5.47).

The expression in (10.7) and (10.9) form the basis for calculating off-shell T and K matrices by the wave function approach. As we have already noted, these results were obtained with particular attention to the corresponding asymptotic waves. Here we have arrived at the same results without reference to any explicit dynamical framework. Thus the expressions for K matrices derived in this work should be regarded as reliable to
the extent of validity of a non-relativistic potential model.

The representation for the off-shell wave matrix introduced by Sasakawa and Sawada [25] also provides a general framework to develop the wave function approach. We have seen that in the wave function approach the two-body amplitude is studied by expressing the Møller wave operator $\mathcal{M}$ in a mixed representation $\langle \varphi | \mathcal{M} | \psi \rangle$. The operator $\mathcal{M}$ satisfies the equation

$$\mathcal{M} = 1 + G V \mathcal{M},$$

where $V$ and $G$ stand for the potential and free particle Green's function operator. Boundary conditions on $G$ are incorporated by writing

$$G = -|A\rangle \langle B| + G^I. \quad (10.11)$$

In (10.11) $G$ is the Green's function associated with the Jost boundary condition. The $l$-th projected component of $G^I$ is given as

$$G^I_{\ell} (r, r') = \frac{1}{k} \left[ u_{\ell} (kr') v_{\ell} (kr) - u_{\ell} (kr) v_{\ell} (kr') \right] \theta (r-r'), \quad (10.12)$$
where $\theta(x)$ is a unit step function and $k$ the on-shell momentum related to the two-body centre-of-mass energy. The Riccati-Bessel functions $u_\ell, v_\ell$ etc. used here follow the phase convention of Messiah[26] such that $w_\ell = v_\ell + i u_\ell$. The choice of $|A\rangle \langle B|$ is crucial to the determination of boundary conditions. For example, $|A\rangle \langle B| = |w\rangle \langle u|$ and $|A\rangle \langle B| = |v\rangle \langle u|$ relate to the out-going and standing wave boundary conditions.

In terms of (10.11), the off-shell wave matrix of Sasakawa and Swada [25] is given by

$$\mathcal{N} = \tilde{\mathcal{N}} - \tilde{\mathcal{N}} |A\rangle \frac{1}{1 + \langle B| v \tilde{\mathcal{N}} |A\rangle} \langle B| v \tilde{\mathcal{N}}$$

(10.13)

with

$$\tilde{\mathcal{N}} = \frac{1}{1 - G^I v}$$

(10.14)

Introducing an off-shell momentum $q$ we can write (10.14) in a mixed representation. For a spherically symmetric local potential the physical wave function is obtained in the form

$$\Psi(k, q, r) = \text{Im} f(k, q, r) - \frac{1}{2} \tilde{\lambda} q \langle k | T(k^2) | q \rangle f(k, r),$$

(10.15)

where $\langle k | T(k^2) | q \rangle$ refer to the half off-shell T-matrix and are determined by the behaviour of (10.15) near the origin.

We have

$$\langle k | T(k^2) | q \rangle = \frac{2 \text{Im} f(k, q)}{\tilde{\lambda} q f(k)}.$$

(10.16)
Here $f(k,q)$ and $f(k)$ stand for the off- and on-shell Jost function, the properties of which have been discussed in some detail in [14]. As $q \to k$, the off-shell wave functions go over to the corresponding solutions of the Schrödinger equation. For example, \[ \lim_{q \to k} \psi^+(k,q,r) = \psi^+(k,r). \] We have followed this convention for all kinds of solutions. Also \[ \lim_{q \to \pm k} f(k,q) = f(\pm k). \] In writing (10.15) we have used the following.

The operator $\Omega$ in the mixed representation is expanded as
\[ \langle \eta | \Omega | \eta \rangle = \frac{1}{\sqrt{2\pi}} (q \cdot r)^{-1} \psi^+(k,r,q). \]

The off- and on-shell Jost solutions $f(k,q,r)$ and $f(k,r)$ satisfy the operator equations
\begin{align*}
   f_q &= \omega_q + G^I \psi^+ f_q 
   \tag{10.17a} \\
   f_k &= \omega_k + G^I \psi^+ f_k. 
   \tag{10.17b}
\end{align*}

Obviously, the projections of $w_q$ and $w_k$ in the representation space stand for Riccati-Hankel functions with arguments $q \cdot r$ and $k \cdot r$ respectively. Also $u_q = \text{Im} w_q$. Note that (10.17a) is in complete agreement with the equation for off-shell Jost solution introduced by Fuda and Whiting. To incorporate the outgoing wave boundary condition we have used $|A\rangle \langle B| = |w\rangle \langle u|$. 
Throughout this article, nonsubscripted Riccati-Bessel functions will refer to the on-shell momentum. Finally, we have also used the result

\[
\frac{1}{1 + \langle u | \mathcal{L} | \psi \rangle} \langle u | \mathcal{L} | \psi \rangle = \frac{1}{2} \pi q \langle k | T(k^2) | \psi \rangle. \tag{10.18}
\]

Equation (10.18) represents the half-off-shell generalization of (34) in [25]. We have fixed the normalization

\[
\langle k | T(k^2) | k \rangle = -\frac{2}{\Lambda} e^{i \delta(k)} \sin \delta(k) \tag{10.19}
\]

for the on-shell $T$-matrix. Here $\delta(k)$ refers to the s-wave phase shift. The fully off-shell $T$-matrix is given by

\[
\langle p | T(k^2) | q \rangle = \frac{2}{\Lambda p q} \int_0^\infty dr \sin pr V(r) \; \Psi(k,q,r) \tag{10.20}
\]

As noted earlier the Green's function appropriate to the standing wave boundary condition is obtained with the choice $|A\rangle \langle \beta | = | \nu \rangle \langle u |$. Proceeding as above, we get the principal value solution as

\[
\Psi(k,q,r) = \text{Im} f(k,q,r) - \frac{1}{2} \Lambda q \langle k | K(k^2) | q \rangle \text{Re} f(k,r), \tag{10.21}
\]
where the half-off-shell $K$ matrix

$$\langle k | \frac{K(k^2)}{V(q)} | q \rangle = \frac{2}{\pi q} \frac{1}{1 + \langle u | V \tilde{a} | v \rangle} \langle u | V \tilde{a} | v \rangle$$

$$= \frac{2}{i \pi q} \frac{f(k, q) - f(k, -q)}{f(k) + f(-k)}.$$  \hspace{1cm} (10.22)

In terms of the wave function $\psi(k, q, r)$, the expression for the off-shell $K$ matrix is

$$\langle p | \frac{K(k^2)}{V(q)} | q \rangle = \frac{2}{\pi p q} \int_0^\infty dr \sin pr \psi(\psi(k, q, r)$$

$$= \frac{2}{\pi p q} \int_0^\infty dr \sin pr \psi(\psi(k, q, r) \quad \text{Im} f(k, q, r)$$

$$- \frac{1}{p} \langle k | \langle k^2 | q \rangle \rangle \int_0^\infty dr \sin pr \psi(Re f(k, r).$$

The on-shell $K$ matrix is normalized as

$$\langle k | \frac{K(k^2)}{V(k)} | k \rangle = -\frac{2}{\pi k} \tan \delta(k).$$

We have used off-shell Jost functions as basic elements for building expressions for off-shell amplitudes for local potentials. A question which follows at once is to know whether such Jost type formalism can be extended to treat the case of non-local potentials. For a general non-local potential the Jost solution does not satisfy a Volterra-type integral equation.

Therefore the convergence of iteration for such an equation is not guaranteed. The convergence criterion and the method for
calculating the Jost solution has recently been derived by Sasakawa [25]. However, for a separable potential the off-shell Jost function and the T and K matrices can be obtained in a relatively noncomplicated manner [88, 89].

In this dissertation we have mainly been concerned with the problem of constructing expression for T and K matrices for scattering by a single interaction.

There exist experimental situations which involve scattering by additive interactions. The calculation of state vectors or scattering amplitudes for such problems need not start with the kinetic energy as a zero order Hamiltonian. Instead, one begins by calculating the states, Green's function etc. for a model Hamiltonian and arrives at the two-potential formula (TPF) of Gell-Mann and Goldberger [81]. Application of the immediately followed its discovery. With the availability of high-speed digital computers, there has been a surge of interest in this line of investigation. The TPF is effective in treating a wide variety of phenomena ranging from the atomic domain to GeV region [90, 91, 92, 93].

Harrington [94] suggested that the TPF should be continued off-the-energy-shell for its possible usage in nuclear three-body problems involving charged particles. This was exploited by Alessandrini et al [95], by Tandy and McCarthy [96], by Zachary [97], by Bajzer [98] and by van Haeringen
and van Wageningen [99] to calculate the off-shell proton-proton T matrix. Despite that, much advance has not been made because of rather involved algebra.

Attempts were made to construct analytic T matrices for Coulomb (distorting potential) plus rational separable potentials. For a separable kernel the Fredholm series becomes a polynomial [100]. Thus it was possible to derive results in terms of a momentum-space approach. But the method involves typical contour integrals which are difficult to perform [101].

Interestingly, the DEA described in chapter 5 can be used to derive an off-shell TBF.

The two-body Hamiltonian under consideration is

\[ H = H_D + V_1 \quad (10.24) \]

with \[ H_D = H_0 + V_D \]. \quad (10.25)

Here \( H_0 \) is the kinetic energy operator, and \( V_D \) and \( V_1 \), the distorting and residual-potential operators. We write \( V = V_D + V_1 \) for the total potential operator. In the representation space all \( V \)'s refer to spherically symmetric local potentials. Throughout this article unsubscripted variables refer to scattering by \( V \). We use subscripts \( D \) and \( 1 \) for variables relating to scattering on \( V_D \) and \( V_1 \) respectively. The angular
momentum $l$, however, will occur as subscript in all appropriate places. The $T$ matrices $T$ and $T_D$ satisfy the Lippmann-Schwinger equations

\[
T(k^2) = V + V G^+(k^2) T(k^2) \tag{10.26}
\]

and

\[
T_D(k^2) = V_D^+ V_D G^+_D(k^2) T_D(k^2) \tag{10.27}
\]

respectively. Here $k$ is the on-shell momentum related to the energy by $E = k^2 - i\epsilon$ and $G^+(k^2)$, the free particle Green's function. The superscript $+$ is used throughout for the outgoing wave boundary condition. We introduce resolvents $G^+$ and $G^+_D$ and Møller wave operators $\mathcal{M}$ and $\mathcal{M}_D$ by

\[
G^+(k^2) = (k^2 - H)^{-1}, \tag{10.28}
\]

\[
G^+_D(k^2) = (k^2 - H_D)^{-1}, \tag{10.29}
\]

\[
\mathcal{M}(k^2) = 1 + G^+_D(k^2) V \mathcal{M}(k^2), \tag{10.30}
\]

and

\[
\mathcal{M}_D(k^2) = 1 + G^+_D(k^2) V_D \mathcal{M}_D(k^2). \tag{10.31}
\]

From equations (10.26), (10.27), (10.30) and (10.31) we get

\[
T(k^2) = V \mathcal{M}(k^2) \tag{10.32}
\]

and

\[
T_D(k^2) = V_D \mathcal{M}_D(k^2). \tag{10.33}
\]

With the help of the operator identity $A^{-1} - B^{-1} = A^{-1}(B - A) B^{-1}$ it can easily be seen that
\[ G^+(k^2) - G^+(k^2)_D = G^+(k^2)_D (V-V_D) \]
\[ G^+(k^2) = G^+(k^2)_D (V-V_D) G^+(k^2)_D. \] (10.34)

Equations (10.30) and (10.31) can also be written in alternative forms:

\[ \mathcal{N}(k^2) = 1 + G^+(k^2)V = 1 + G^+_0(k^2)T(k^2) \] (10.35)

and
\[ \mathcal{N}_p(k^2) = 1 + G^+_D(k^2)V_D = 1 + G^+_0(k^2)T_D(k^2). \] (10.36)

Combining equations (10.34), (10.35) and (10.36) we arrive at

\[ \mathcal{N}(k^2) = \mathcal{N}_p(k^2) + G^+_D(k^2)(V-V_D) \mathcal{N}(k^2) \] (10.37)

Introducing an off-shell momentum \( q \) we can take equations (10.35) and (10.36) in a mixed representation to read

\[ \psi^+_\ell(k,q,r) = -\frac{1}{2} \pi q T^\ell(k,q,k^2) e^{-\frac{i\ell r}{2}} f^\ell(k,r) + \frac{1}{21} \left[ e^{-\frac{i\ell r}{2}} f^\ell(k,q,r) - e^{i\ell q/2} f^\ell(k,-q,r) \right] \] (10.38)

and

\[ \psi^+_\mu(k,q,r) = -\frac{1}{2} \pi q T^\mu(k,q,k^2) e^{-\frac{i\ell r}{2}} f^\mu(k,r) + \frac{1}{21} \left[ e^{-\frac{i\ell r}{2}} f^\mu(k,q,r) - e^{i\ell q/2} f^\mu(k,-q,r) \right] \] (10.39)
where $T_{\ell}(k,q,k^2)$ is the $\ell$-th projected half-off-shell $T$ matrix. The objects $\phi_k^{\pm}(k,r)$ and $\phi_{\ell}(k,q,r)$ stand for the on- and off-shell Jost functions. In writing equations (10.38) and (10.39) we have used the following

$$<\tilde{\tau}|\Lambda(k')|\eta m> = \left(\frac{2}{\pi}\right)^\frac{1}{2} (qr)^{-1} \psi_{\ell}(k,q,r) \gamma_{\ell m}(\hat{r}) \quad (10.40)$$

and

$$<\tilde{\tau}|\eta m> = \left(\frac{2}{\pi}\right)^\frac{1}{2} \phi_{\ell}(qr) \gamma_{\ell m}(\hat{r}) \quad (10.41)$$

in addition to the well known eigen function expansion of the free particle Green's function. Use was also made of

$$<\tilde{\tau}|\Lambda_D(k')|\eta m>.$$

In the mixed representation equation (10.37) becomes

$$\psi_{\ell}(k,q,r) = \psi_{\ell}^{+}(k,q,r) + \int G_{\ell}^{+}(k,r,r') V_{\ell}(r') \psi_{\ell}^{+}(k,q,r'), \quad (10.42)$$

where $G_{\ell}^{+}(k,q,r') = -\frac{1}{k} \psi_{\ell}^{+}(k,q') f_{\ell}^{+}(k,q) e^{-ikr/2} \quad (10.43)$

The functions $\psi_{\ell}^{+}(k,q,r)$ and $f_{\ell}(k,q,r)$ are the on-shell outgoing and irregular solutions of $H_D$. For $r \to \infty$ equation (10.42) takes the form

$$\psi_{\ell}(k,q,r) \sim \psi_{\ell}^{+}(k,q,r) - \frac{e^{i\pi/2}}{k} \int_{D_{\ell}} f_{\ell}(k,r) \int d'r' \psi_{\ell}^{+}(k,r') V_{\ell}(r') x \psi_{\ell}^{+}(k,q,r'). \quad (10.44)$$

Using the asymptotic form of $\psi$'s and $f_{\ell}$ in equation
we get
\[ T_L(k,q,k^2) = T_{DL}(k,q,k^2) + \frac{2}{\pi kq} \int_{\infty}^{\infty} dr' \psi_D^+(k,r')V_1(r') \times \psi_L^+(k,q,r'). \] (10.45)

Equation (10.45) represents the half-off-shell generalization of the two-potential formula.

To look for the fully off-shell case we multiply equation (10.42) by \( U_L(kr)V_0(r) \) and integrate. We thus obtain
\[ T_L(p,q,k^2) = T_{DL}(p,q,k^2) + \frac{2}{\pi pq} \int_{\infty}^{\infty} dr' \left[ U_L(pr') + \int_{\infty}^{\infty} dr \phi_D^+(k,r,r')V_D(r) U_L(pr') V_1(r') \psi_L^+(k,q,r') \right], \] (10.46)

where \( U_L(x) \) is the Ricatti-Bessel function written as
\[ U_L(x) = x j_L(x). \] In deriving (10.46) we have used the matrix elements of (10.32) and (10.33) between states designated by the off-shell momenta \( p \) and \( q \). In view of (10.31) it is easy to see that the object in the square bracket of (10.46) stands for the off-shell wave function \( \psi_L^+(k,p,r') \).

Thus we write (10.46) in the form
\[ T_L(p,q,k^2) = T_{DL}(p,q,k^2) + \frac{2}{\pi pq} \int_{\infty}^{\infty} dr' \psi_D^+(k,p,r')V_1(r') \times \psi_L^+(k,q,r'). \] (10.47)

Equation (10.47) is clearly the off-shell two-potential formula.
Some comments on applications of the method developed are now in order. For a separable residual interaction $V_\perp = |v\rangle \langle v|$ (10.46) and (10.47) remain effectively unchanged except that one needs to replace the single quadrature occurring here by appropriate double integrals. Off-shell Coulomb functions are now available in the literature [102]. Thus one would expect to treat the Coulomb plus Yamaguchi potential in a relatively noncomplicated manner by using the present approach.

There are two important channels for deuteron which are excited with relatively large probability - the stripping and breakup channels. The r-space formalism developed here may be utilized to deal with break up channels in a realistic way. It is of interest to note that (9.31) can be derived from (10.47) in the same way as Tikochinski obtained (9.24) or (9.15) from (9.19).

Applicability of the TPF is directly related with the existence and/or completeness of wave operators for the scattering system [98]. Presence of long range forces in addition to the nuclear potential tends to pose some problems in respect of this [103, 104, 105]. With this in mind, we outline in Appendix A a co-ordinate space approach to the Coulomb- nuclear problem, which does not make explicit use of the Gell-Mann-Goldberger theorem.