In the last chapter we have seen how the wave function approach could be used to calculate the matrix elements of the $K$ operator. Two slightly different approaches are needed when the potential is regular and when the potential has a $1/r$ singularity at the origin. Based on the approach developed we shall construct exact expressions for off-shell $K$ matrices for the exponential and Morse potentials.

(a) Exponential potential

In the nonrelativistic limit the potential model helps greatly in explaining the two-nucleon interaction either in the bound state or in the scattering state. Every potential requires a number of parameters for its specification. The success of effective range theory implies that any reasonable two-parameter potential can fit the low energy data. An exponential potential

$$v(r) = -\frac{2}{4a^2} \frac{z_0}{r/a} e^{-r/a}$$

(7.1)

represents the simplest two-parameter potential which tends to fit the $N-N$ phase shift below the pion production threshold.
350 Mev. With the appropriately chosen values of the parameters \( z_c \) and \( 'a' \), the potential in (7.1) yields a reasonable value for the deuteron binding energy.

For the potential in (7.1) the s-wave part of the off-shell Jost solution satisfies

\[
\left[ k^2 + \frac{d^2}{dr^2} + \frac{z^2}{4a^2} e^{-r/a} \right] f(k,q,r) = (k^2 - q^2) e^{iqr} \quad (7.2)
\]

In writing out (7.2) we omitted the subscript \( \ell = 0 \). We now change the variable by substituting

\[
z = z_c e^{-r/2a} \quad (7.3)
\]

\[
\left[ \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz} \left( 1 - \frac{v^2}{z^2} \right) \right] f(k,q,z) = 4a^2(k^2 - q^2) z_c^{-2} e^{-2iaq} \quad (7.4)
\]

The particular solution of (7.4) is given by [46]

\[
f(k,q,z) = 4a^2(k^2 - q^2) z_c^{-2} e^{-2iaq} s_{\mu\nu}(z) \quad (7.5)
\]

where \( s_{\mu\nu}(z) \) is the Lommel function written as

\[
s_{\mu\nu}(z) = \frac{\mu+1}{(\mu+\nu+1)(\kappa-\nu+\ell)} \Gamma_2 \left( \frac{1}{2} \mu - \frac{1}{2} \nu + \frac{3}{2}, \frac{1}{2} \mu + \frac{1}{2} \nu + \frac{3}{2} \mid -\frac{1}{4} \right) \quad (7.6)
\]
with \[
\mu = -1 - 2iqa \\
\nu = 2ika .
\] (7.7)

The function \( \textsc{f}_2 \) \( \left( \begin{array}{c} \alpha \cr \beta \end{array} \right| z \) is a special case of the generalized hypergeometric function defined by Luke [47]. Inserting (7.6) in (7.5) we have

\[
f(k,q,z) = z_0 z \textsc{f}_2 \left( \begin{array}{c} \alpha \\
\beta \end{array} \left| \frac{\nu}{2} \right. \right) (z).
\] (7.8)

It can be easily shown that in the asymptotic limit

\[
f(k,q,z) \sim e^{i\nu}. \quad \nu \to \infty
\] (7.9)

Equation (7.9) represents the correct asymptotic behavior prescribed for the off-shell Jost solution. The on-shell Jost solution \( f(k,z) \) is given by

\[
f(k,z) = \lim_{q \to k} f(k,q,z) = \left( \frac{1}{2} z_0 \right)^{2i\nu} \Gamma(1-2i\nu) J_{-2i\nu} (z).
\] (7.10)

In writing (7.10) we have used

\[
\textsc{o}_1 \left( \alpha + 1 ; - \frac{1}{4} z^2 \right) = \Gamma(\alpha+1) \left( \frac{\alpha}{2} \right)^{-\alpha} J_{\alpha} (z).
\] (7.11)

From (7.8) and (7.10) we obtain the off-shell and on-shell
Jost functions in the forms

\[ f(k, q) = \frac{1}{F^2} \left( 1 - i\kappa - i\beta \right) \left( \frac{-i\kappa^2}{4} \right) \]  \hspace{1cm} (7.12)

and

\[ f(k) = i^{2\kappa} \left( \frac{1}{2} \right) \Gamma \left( i - 2\kappa \right) J_{-2i\kappa} \left( Z_0 \right). \]  \hspace{1cm} (7.13)

In terms of the Jost solution and Jost function, the s-wave off-shell principal value wave function for the exponential potential is given by

\[ \psi^P(k, q, \gamma) = A(k, q) \left[ C(k) J_{-2i\kappa} (Z) + C^*(k) J_{2i\kappa} (Z) \right] \]

\[ + B(k, q) \left[ Z_0 \left( Z_0 \right) - Z_0 \left( Z_0 \right) \right] \]

\[ - 2i\kappa \left( Z_0 \right) - 2i\kappa \left( Z_0 \right) \]  \hspace{1cm} (7.14)

where

\[ A(k, q) = -\frac{1}{4} \pi q \left\langle k \mid K(k) \mid q \right\rangle \]  \hspace{1cm} (7.15a)

\[ C(k) = \left( \frac{1}{2} \right) \frac{2i\kappa}{2i\kappa} \Gamma \left( 1 - 2i\kappa \right) \]  \hspace{1cm} (7.15b)

\[ C^*(k) = \left( \frac{1}{2} \right) \frac{1}{2i\kappa} \Gamma \left( 1 + 2i\kappa \right) \]  \hspace{1cm} (7.15c)

and

\[ B(k, q) = -2i\kappa \left( k^2 - q^2 \right) \]

with

\[ \left\langle k \mid K(k) \mid q \right\rangle = \frac{8e^{2i\kappa q^2}}{i\pi q} \left[ 2i\kappa \left( Z_0 \right) - 2i\kappa \left( Z_0 \right) \right] \]

\[ \times \left[ C(k) J_{-2i\kappa} (Z_0) + C^*(k) J_{2i\kappa} (Z_0) \right] \]  \hspace{1cm} (7.16)
obtained from the behavior of the principal value wave function \( \psi(k,q,r) \) near the origin (i.e., \( r \to 0 \)). The s-wave fully off-shell \( K \) matrix is given by

\[
\langle p | K(k^2) | q \rangle = \frac{2}{\pi pq} \int_0^\infty u(pr) v(r) \psi^p(k,q,r) \, dr
\]

where

\[
u(pr) = \sin pr.
\]

Using (7.3), (7.18) we get

\[
\sin pr = \frac{i}{2i} \left[ \left( \frac{Z}{Z_0} \right)^{2iap} - \left( \frac{Z}{Z_0} \right)^{-2iap} \right],
\]

Combining (7.1), (7.14) and (7.17) we can write

\[
\langle p | K(k^2) | q \rangle = \frac{Z_0}{2i\pi a \phi pq} \int_0^{\frac{Z}{Z_0}} \left[ \left( \frac{Z}{Z_0} \right)^{1+2ia\phi} - \left( \frac{Z}{Z_0} \right)^{-1-2ia\phi} \right]
\]

\[
\left\{ A(k,q) \int_{-2i\phi k_a}^{2i\phi k_a} \left( \frac{Z}{Z_0} \right)^{i2iap} \, dZ + B(k,q) \int_{-2i\phi k_a}^{2i\phi k_a} \left( \frac{Z}{Z_0} \right)^{-i2iap} \, dZ \right\}
\]

\[
= \frac{Z_0}{2i\pi a \phi pq} \left[ A(k,q) I_1 + B(k,q) I_2 \right]
\]

where

\[
I_1 = \int_0^{\frac{Z}{Z_0}} \left[ \left( \frac{Z}{Z_0} \right)^{1+2ia\phi} - \left( \frac{Z}{Z_0} \right)^{-1-2ia\phi} \right] \left[ c(k) J_{-2i\phi k_a} (z) + c^*(k) J_{2i\phi k_a} (z) \right] \, dZ
\]

(7.22)
and
\[ I_2 = \int_0^\infty \left[ \left( \frac{Z}{Z_c} \right)^{1+\lambda} - \left( \frac{Z}{Z_c} \right)^{1-\lambda} \right] Z_c \delta_{-1-2iq\alpha, 2i\kappa} \]
\[ \left( 1 + z \right) \int_{-1}^{2i\kappa} \frac{dZ}{Z_c} . \]  

(7.23)

Fortunately, the integrals in (7.22) and (7.23) can be related to the tabulated integrals [48] given by

\[ y(p,k) = \int_0^{Z_c} \left( \frac{Z}{Z_c} \right)^{1+\lambda} \delta_{-1+2iq\alpha, 2i\kappa} \frac{dZ}{Z_c} \]

\[ = \frac{(2/Z_c)^\lambda}{(\lambda-\nu+2) \Gamma(1-\nu)} \left[ F_2 \left( \frac{1}{2} \left( \lambda-\nu+2 \right), \frac{1}{2} (\lambda-\nu+4) \right) \left[ \frac{1}{4} Z_c^2 \right] \right] \]  

(7.24)

and

\[ x(p,q,k) = \int_0^{Z_c} Z \delta_{-1+2iq\alpha, 2i\kappa} \frac{dZ}{Z_c} \]

\[ = \frac{1}{2} \left( \frac{2(\alpha+1) \Gamma(1+\alpha)}{\Gamma(1+\alpha)} \right) \left[ x_2 F_3 \left( \frac{1}{2} \left( \mu-\nu+3 \right), \frac{1}{2} (\mu+\nu+3), \alpha+2 \right) \frac{1}{4} Z_c^2 \right] \]  

(7.25)

where \( \lambda = 2ipa, \alpha = ia (p-q) \) and \( \mu, \nu \) are given by (7.7). Combining (7.21), (7.22), (7.23), (7.24) and (7.25) we obtain the s-wave part of the off-shell K matrix for the exponential potential in the form
\[
\langle p | K(k^2) | q \rangle = \frac{Z_0}{2i \pi \alpha pq} \left( A(k, q) Z_0 \left\{ c(k) \left[ y(p, -k) - y(-p, k) \right] + c^*(k) \left[ y(p, k) - y(-p, k) \right] \right\} + B(k, q) \left\{ x(p, q, k) - x(p, -q, k) - x(-p, q, k) + x(-p, -q, k) \right\} \right) . \tag{7.26}
\]

Thus we see that the results for the off-shell K matrix elements for the exponential potential can be expressed in closed form involving functions whose series representations have infinite radii of convergence. It should therefore be possible to sum the series on a computer and use it as a check on programmes which evaluate K matrix elements by numerical methods.

(b) Morse potential

The study of the nucleon-nucleon interaction is a very interesting topic of nuclear research. In N-N interaction many body forces are expected to play a role. This is one of the interesting features of the pion exchange theory of nuclear force. We use the results of two-body forces to interpret the many-body experimental data. But this does not assume that we would expect the nuclear forces in a many-body system to be identical to the one determined from two-body experiments. The many-body problem depends crucially not on the two-body scattering amplitudes but on the two-body off-shell scattering amplitudes.
We have noted in section (a) that the exponential potential is a purely attractive potential which can fit the N-N phase shift below 350 MeV [49, 50]. To account for the higher energy phase shift, the existence of a short range repulsive core is necessary. In particular, it was pointed out by Jastrow [51] that the s-wave phase shift changes sign at energies above 350 MeV. Further he proposed to the introduction of a hard core repulsion to explain the observed isotropic p-p scattering at 350 MeV and also to interpret the saturation character of nuclear forces with regard to density as well as binding energy.

A number of scientists, namely, Bressel and Kerman [52] Tabakin and Davies [53] raised an interesting question as to what extent is the conventional hard-core necessary to reproduce the observed nuclear properties? They pointed out that the repulsive core appeared to be necessary but the very hard nature was unphysical. The approximation of an infinite hard-core was often made for mathematical convenience in solving the two-body problem. To obtain correct binding energies and other scattering data for either static or nonstatic potentials consistent with two-body scattering, they argued, one must deal with strong if not infinite repulsion. It was observed by sprung and Srivastava [54] that many-body calculations are sensitive to this simplification. Hence a number of soft core potentials have been developed. Several proposals by Reid [55] and Wong [56]
were made to replace the repulsive hard-core with a repulsive short-range Yukawa potential. It was suggested by Breit [57] that the exchange of heavy neutral vector mesons such as the $\rho$ and the $\omega$, can well account for both the hard-core and spin-orbit interaction. His model suggests a static potential with a soft core than a phenomenological hard-core potential. Later on, observations made by Brown et al. [58] that the phenomenological potentials reproduced the properties of nuclear matter worse than that was expected, gave impetus to the suggestion of Breit. In particular, the calculations of Brueckner and Masternson [59] and of Rezavy [60] reproduced the binding energy of about 8 MeV/nucleon instead of the experimental value of about 16 MeV/nucleon [61].

In an attempt to overcome this inconsistency there are three suggestions. The first one proposes to change the two-nucleon forces between the nucleons so that the binding energy of nuclear matter/nucleon increases without disturbing the N-N scattering data. In the light of the Wong's suggestion one replaces the completely hard-core interaction by a somewhat soft repulsion like $e^{-\mu r}/r$. A theoretical justification of this model was given by Bethe [62]. The repulsion was justified by the exchange of a heavy meson, particularly the $\omega$ meson as pointed out by Scotti and Wong [63] and Bryan et al. [64]. The range $1/\mu$ is then the Compton wavelength of the $\omega$ meson.
which is about 0.2 fm. With this modification the binding energy comes out to be about 12 MeV/nucleon which is encouraging.

A potential model may be regarded as some kind of an extrapolation of the off-shell behavior from the on-shell information because of the fact that nuclear forces are usually computed from on-the-energy-shell scattering data, while in nuclear matter calculations one is interested in off-shell amplitudes. Thus we arrive at the second suggestion that such an extrapolation may not be permissible. Before reaching to either of these conclusions, however, a third possibility should be explored namely whether the traditional method of nuclear matter calculations is reliable.

Bethe [62] studied the properties of nuclear matter and suggested the replacement of the hard-core by a soft-core without appreciably affecting the phase shifts. Soft-core potentials can equally well account for the behavior of s-wave phase shifts at high energies, where these phase shifts become negative. The numerical calculations of Bystritskii et al [65] with a mixture of attractive and repulsive Yukawa potentials, the latter having a shorter range, provides an example of a static soft-core potential which fits experimental s-wave phase-shifts throughout the energy range. On the other hand, the investigations made by Green [66] and by Rezavy et al [67]
reveals that phenomenological velocity dependent potentials can also account for the behavior of s-wave phase-shifts.

The Morse potential is an analytic soft-core potential of which the behavior can be specified conveniently by means of three parameters. Its characteristic behavior to reproduce the basic features of the s-wave N-N interaction justifies its use in the N-N interaction. The Morse potential has been used to incorporate the effect of a repulsive core in atom-atom collisions and studies of the bound states of diatomic molecules. As observed by Morse [68, 69], the phase shifts and bound states of the Morse function are expressible in closed form. Its wave function can be written in terms of the hypergeometric function. The Morse potential can also be used to study the problem of a nucleon bound in a velocity-dependent nuclear potential. Using the theoretical value of the phase shift obtained from the Morse potential one can study the s-wave problem of a phenomenological velocity-dependent potential as well as a two-meson-exchange velocity dependent potential. As pointed out by Green et al [70] and by Lodhi [71, 72] the Morse potential can be used to approximate the velocity dependent N-N potential. From their studies it was clear that the soft repulsive core of the Morse potential could serve as a good substitute for the infinite hard core.

Thus the Morse potential can fairly represent the physical situation and serve as a far more realistic point
of departure than other analytic potentials such as the square well, exponential or Gaussian used in earlier studies.

We now proceed to obtain analytic expressions for the s-wave part of the off-shell two-particle K matrix.

The Morse potential is given by

$$V(r) = V_0 \left( e^{\frac{-2(r-d)}{b}} - e^{\frac{-(r-d)}{b}} \right).$$

(7.27)

It is clear that $V$ approaches zero exponentially for large $r$, has a minimum value $-V_0$ at $r = d$ and becomes large and positive as $r$ approaches zero if the breadth 'b' of the attractive region is somewhat smaller than the equilibrium distance $d$.

For the Morse potential the s-wave off-shell Jost solution satisfies the inhomogeneous equation

$$\left\{ k^2 + \frac{d^2}{dr^2} - V_0 \left[ e^{\frac{-2(r-d)}{b}} - e^{\frac{-(r-d)}{b}} \right] \right\} f(k,q,r)$$

$$= (k^2-q^2) e^{iqr}.$$  

(7.28)

We now change the variable by substituting

$$Z = 2V_0^{-\frac{1}{2}} e^{\frac{-(r-d)}{b}}$$

(7.29)

to reduce (7.28) to a well known inhomogeneous hypergeometric differential equation. We thus obtain
The homogeneous equation corresponding to (7.30) has the asymptotic solution \( e^{-\gamma z} \) and the solution for small \( Z \) is \( Z^{-ikb} \). This suggests that we look for an exact solution of (7.30) of the form

\[
    f(k,q,r) = Z^{ikb} e^{-\gamma z} F(Z). \tag{7.31}
\]

Substitution of (7.31) in (7.30) yields

\[
    ZF'' + \left\{ (1-2ikb) - Z \right\} F' - \left( \frac{1}{2} - ikb \right) \gamma V_0 b F
\]

\[
    = b^2 (k^2 - q^2) (2V_0 \gamma V_2 b) \frac{q^2}{2} e^q \frac{d}{dz} \frac{d}{dz} \frac{1}{e^{ikb-iqb-1}} e^{-\gamma z}, \tag{7.32}
\]

where the prime on \( F \) denotes differentiation with respect to \( Z \). Interestingly, the solution of (7.32) can be obtained from an inhomogeneous confluent hypergeometric differential equation studied by Babister [46] namely

\[
    Z \frac{d^2 y}{dz^2} + (c-z) \frac{dy}{dz} - ay = e^{\rho z} \sigma^{-1} \tag{7.33}
\]
Comparing (7.32) and (7.33) we have

\[
\begin{align*}
C &= 1 - 2ibk \\
\alpha &= \frac{1}{2} - ikb - Vo \gamma_2 b \\
\rho &= \frac{1}{2} \\
\sigma &= ikb - iq b
\end{align*}
\]

(7.34)

and \( y = F \left[ b^2(k^2 - q^2)(2Vo \frac{1}{2} b)^{iq b} \right]. \)

A particular integral of (7.33) is the function given by

\[
\Lambda_{\rho \sigma}(\alpha, c, x) = \sum_{n'=0}^{\infty} \theta_{\sigma+n'} \frac{\rho^{n'}}{n'!},
\]

(7.35)

where

\[
\theta_{\sigma+n'} = \frac{x^{\sigma+n'}}{(\sigma+n')(\sigma+n'+c-1)} _2F_2 \left( 1, \sigma+n'+a \left| \frac{x}{\sigma+n'+1}, \sigma+n'+c \right. \right)
\]

(7.36)

Here \( _2F_2 \) is the special case of the generalized hypergeometric function defined by Luke [47]. Using (7.34), (7.35) and (7.36), \( F \) of (7.32) is given by
\[ F = (2\nu_0^2 b)^{\text{i}qb} e^{\text{i}qd} \sigma^{(\sigma+c-1)} \]
\[
\sum_{n'=0}^{\infty} \frac{\rho}{\Gamma} \frac{\sigma+n'}{2F_2 \left( \left. \begin{array}{c} 1, \sigma+n'+a \\ \sigma+n'+1, \sigma+n'+c \end{array} \right| z \right)} (\sigma+n'! (\sigma+n'+c-1) \]  

(7.37)

The series in (7.37) is uniformly convergent for all values of \( \rho \) and \( Z \). Substituting (7.37) in (7.31) we obtain

\[ f(k,q,r) = \sum_{n=0}^{\infty} G_n'(\nu_0, b, d, q, p) \frac{\sigma^{(\sigma+c-1)}}{(\sigma+n') (\sigma+n'+c-1)} \]
\[
\times e^{\frac{-1/n}{z}} n^{+1} e^{\frac{-1/n}{z}} 2F_2 \left( \left. \begin{array}{c} 1, \sigma+n'+a \\ \sigma+n'+1, \sigma+n'+c \end{array} \right| z \right) \]  

(7.38)

where

\[ G_n'(\nu_0, b, d, q, p) = (2\nu_0^2 b)^{\text{i}qb} e^{\text{i}qd} \frac{\rho}{\Gamma} \frac{1}{n'}! \]  

(7.39)

The off-shell Jost solution \( f(k,q,r) \) given by (7.38) satisfies the asymptotic boundary condition

\[ f(k,q,r) \sim e^{\text{iqr}} \]  

(7.40)

as \( r \to \infty \). This can be seen as follows:

Equation (7.38) can be rewritten as

\[ f(k,q,r) = G_0(\nu_0, b, d, q, p) \frac{\sigma^{(\sigma+c-1)}}{\sigma^{(\sigma+c-1)}} e^{\frac{-1/n}{z}} n^{+1} e^{\frac{-1/n}{z}} 2F_2 \left( \left. \begin{array}{c} 1, \sigma+n'+a \\ \sigma+n'+1, \sigma+n'+c \end{array} \right| z \right) \]
\[
\sum_{n'=1}^{\infty} \frac{G_n'(\nu_0, b, d, q, p)\sigma^{(\sigma+c-1)}}{(\sigma+n') (\sigma+n'+c-1)} e^{\frac{-1/n}{z}} n^{+1} e^{\frac{-1/n}{z}} 2F_2 \left( \left. \begin{array}{c} 1, \sigma+n'+a \\ \sigma+n'+1, \sigma+n'+c \end{array} \right| z \right). \]
Now as \( r \to \infty \), the first term becomes \( e^{iqr} \) and the second term goes to zero due to the presence of \( Z^n \) as a multiplicative factor. From (7.38), the off-shell Jost function is found to be

\[
\begin{align*}
f(k, q) &= \sum_{n'=0}^{\infty} G_{n'}(\nu, b, d, q, \rho) \frac{\sigma (\sigma + c - 1)}{(\sigma + n') (\sigma + n' + c - 1)} \\
& \quad \times e^{-\frac{1}{2} \xi' n' + \sigma' - ikb} \cdot e^{i \xi' q} \cdot \frac{1}{\xi'} \cdot \frac{\sigma + n' + a}{\sigma + n' + c} \cdot \text{F}_2 \left( \frac{1}{\xi}', \frac{\sigma + n' + a}{\sigma + n' + c} \mid \xi' \right)
\end{align*}
\]

(7.41)

where \( \xi' = \alpha \gamma \)

with \( \alpha = 2 V_r b \)

and \( \gamma = e^{d/b} \)  

(7.42)

We note that (7.38) and (7.41) yield the correct on-shell Jost solution and Jost function defined by

\[
f(k, r) = \alpha e^{ikd} e^{-ikb} e^{-\frac{1}{2} Z} \cdot \frac{1}{\xi'} \cdot \text{F}_1 (a, c, z) \quad (7.43)
\]

and \( f(k) = e^{-\frac{1}{2} \xi} \cdot \text{F}_1 (a, c, \xi) \) .  

(7.44)

Combining the s-wave form of (5.38) and the equations (7.38), (7.43) we get the off-shell principal value wave function for the Morse potential as
where
\[ \sigma' = ikb + iq \]
\[ B(k, q) = -\frac{i}{4} \pi q \langle k | K(k) | q \rangle \]
\[ A(\pm k) = \alpha \pm ikb \pm ikd \]

and

\[ D_{n'}(\pm q, \alpha) = G_{n'}(V_0, k, d, \pm q, \rho) \frac{\alpha(\alpha + c - 1)}{(\alpha + n')(\alpha + n' + c - 1)} \]

with
\[ \langle k | K(k) | q \rangle = \sum_{n'=0}^{\infty} \frac{D_{n'}(q, \sigma) F_2\left( \frac{1}{2}, \sigma + n' + a \, \frac{1}{2}, \sigma + n' + c \right) - D_{n'}(q, \sigma') F_2\left( \frac{1}{2}, \sigma' + n' + a \, \frac{1}{2}, \sigma' + n' + c \right)}{i\pi q} \]

obtained from the behavior of the principal value wave function \( \psi^P(k, q, r) \) near the origin (i.e. \( z \to \xi \)). Making use of (7.17), (7.27), (7.29), (7.42) and (7.45) we can write the s-wave part of the off-shell K matrix for the Morse potential in the form
\[
\langle b | \kappa (k^2) | q \rangle = \frac{2V_0}{\pi p q} \sum_{m,n=0}^{\infty} \frac{(-1)^m m^n}{2^m m!} \left[ B(k,q) \frac{(a)_n}{(c)_n!} \left\{ A(k) q \right\} i k b \right.
\]
\[
\times \left( \gamma I(2,-k,0) - 2 \gamma I(1,-k,0) + A(-k) q \left( \gamma I(2,k,0) - 2 \gamma I(1,k,0) \right) \right)
\]
\[
\left. + \frac{1}{2i} \sum_{n'=0}^{\infty} \left[ D_{n'}(q,s) \frac{(\sigma + n'+a)_{n'}}{(\sigma + n'+1)_{n'}(\sigma + n'+c)_{n'}} \right] q \tilde{n} + i q b \right\}
\]
\[
\times \left\{ \gamma I(2,q,n') - 2 \gamma I(1,q,n') \right\},
\]
(7.50)

The functions \( \Gamma_1 \) are uniformly convergent for all values of the independent variable \([47]\). The integral in (7.49) can therefore be evaluated by integrating the series term by term to give

\[
\langle b | \kappa (k^2) | q \rangle = \frac{2V_0}{\pi p q} \sum_{m,n=0}^{\infty} \frac{(-1)^m m^n}{2^m m!} \left[ B(k,q) \frac{(a)_n}{(c)_n!} \left\{ A(k) q \right\} i k b \right.
\]
\[
\times \left( \gamma I(2,-k,0) - 2 \gamma I(1,-k,0) + A(-k) q \left( \gamma I(2,k,0) - 2 \gamma I(1,k,0) \right) \right)
\]
\[
\left. + \frac{1}{2i} \sum_{n'=0}^{\infty} \left[ D_{n'}(q,s) \frac{(\sigma + n'+a)_{n'}}{(\sigma + n'+1)_{n'}(\sigma + n'+c)_{n'}} \right] q \tilde{n} + i q b \right\}
\]
\[
\times \left\{ \gamma I(2,q,n') - 2 \gamma I(1,q,n') \right\},
\]
where \((\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\)

and \(I(j,g,n') = \frac{pb^2}{p^2b^2 + (m+n+j+n'+igb)^2}\).

The Pochhammer symbol \((\alpha)_n\) as well as the integral \(I(j,g,n')\) characterizing the off-shell \(K\) matrix in (7.50) converge uniformly as \(n\) and \(n'\) tend to \(\infty\). We therefore conclude that the expression for the off-shell \(K\) matrix element in (7.50) admits of an easy numerical evaluation.