1.1. Using approximate derivative and mean continuous ACG function Ellis introduced the mean continuous integral, $\text{GM}_1$-integral \[19\], which is based on the descriptive definition of the general Denjoy integral. Replacing the mean continuity in the definition of $\text{GM}_1$-integral, by the approximate mean continuity-called $D_1$-continuity—we introduce, in this chapter, an integral which is called $D_1$-integral. This integral which is an approximate extension of $\text{GM}_1$-integral, is shown to possess various properties of Denjoy integrals including integration by parts and the Cauchy and the Harnack properties.
I.2. Preliminaries

**Definition 1.2.1:** A function $f : E \to \mathbb{R}$, where $\mathbb{R}$ is the set of reals and $E \subseteq \mathbb{R}$, is said to be generalized absolutely continuous on $E$ if $E$ can be expressed as countable union of closed sets on each of which $f$ is absolutely continuous and is written $f \in \text{ACG}(E)$.

Note that this definition of $\text{ACG}$ differs from that in [41, p.223] in that we are not using continuity of $f$. Since a continuous function $f$ is absolutely continuous on the closure of a set on which $f$ is absolutely continuous it follows that if $f$ is $\text{ACG}$ in the sense of [41, p.223] then $f$ is also $\text{ACG}$ in our sense. The converse is not true.

It is clear that if $f \in \text{ACG}(E)$ then $f$ is $\text{VBG}$ on $E$ in the sense of [41, p.221] and is measurable and so by [41, Theorem 4.3, p.222] the approximate derivative $f'_\text{ap}$ exists almost everywhere on $E$. It can be verified that if $f, g \in \text{ACG}(E)$ then $\alpha f + \beta g \in \text{ACG}(E)$ and $fg \in \text{ACG}(E)$, where $\alpha$ and $\beta$ are constants.

**Lemma 1.2.1:** If $F \in \text{ACG}(E)$ then every closed subset of $E$ contains a portion on which $F$ is absolutely continuous.

**Proof:** Let $E = \bigcup_{k} E_k$ where each $E_k$ is closed and $F$ is absolutely continuous on $E_k$. Let $Q$ be any closed subset of $E$. By Baire's theorem there is a portion $P$ of $Q$ which is contained in some $E_k$ and hence $F$ is absolutely continuous on $P$.

**Lemma 1.2.2:** If $F \in \text{ACG}(E)$ then $F$ fulfils the Lusin condition $(N)$ on $E$. 

Proof: The proof given in [41, p. 225, Theorem 6.1] will suffice.

Theorem 1.2.1: Let \( F \in ACG([a, b]) \) and let \( F \) have Darboux property in \([a, b]\). If

\[
(2.1) \quad \limsup_{h \to 0} \frac{F(x+h) - F(x)}{h} \geq 0
\]

for almost all \( x \in [a, b] \) then \( F \) is continuous and nondecreasing in \([a, b]\).

Proof: Let \( G \) be the set of all points \( x \) in \([a, b]\) such that there is a neighbourhood of \( x \) in which \( F \) is nondecreasing (for the endpoints \( a \) and \( b \) we consider one sided neighbourhoods). Then \( G \) is open. Let \( H = [a, b] \setminus G \). Then \( H \) is closed. If possible let \( H \) be nonvoid. If \((c, d)\) is a contiguous interval of \( H \) then \( F \) is nondecreasing in \((c, d)\) and so by the Darboux property \( F \) is continuous and nondecreasing in \([c, d]\). Hence \( H \) cannot have isolated points. So \( H \) is perfect. By Lemma 1.2.1, there is a portion \((p, q)\) of \( H \) on which \( F \) is absolutely continuous. Let \( p < \alpha < \beta < q \) and \((\alpha, \beta) \cap H \neq \emptyset\). Then \( F \) is absolutely continuous in \([\alpha, \beta] \setminus H\). Since \( F \) is continuous and nondecreasing in the closure of the complementary intervals of \([\alpha, \beta] \cap H\), \( F \) is continuous and of bounded variation in \([\alpha, \beta]\). Since by Lemma 1.2.2, \( F \) fulfils the Lusin condition (N), \( F \) is absolutely continuous on \([\alpha, \beta]\). The condition \((2.1)\) almost everywhere then ensures that \( F \) is nondecreasing in \([\alpha, \beta]\). But this is a contradiction, since \((\alpha, \beta) \cap H \neq \emptyset\). Thus \( H \) is void. Hence \( F \) is nondecreasing in \([a, b]\).
Corollary 1.2.1: If \( F \) has Darboux property on \([a, b]\) and \( F \in ACG([a, b]) \) and \( F'_{ap} = 0 \) almost everywhere in \([a, b]\), then \( F \) is constant.

1.3. The \( D_1 \)-integral

Definition 1.3.1: Let \( f \) be a real valued function defined on \([a, b]\) and let \( c \in [a, b] \). Let \( f \) be \( D \)-integrable in some neighbourhood of \( c \). If there is a finite real number \( L \) and a measurable set \( E_c \subseteq [a, b] \) having \( c \) as a point of density (onesided point of density if \( c = a \) or \( c = b \)) such that for \( \varepsilon > 0 \) there is \( S = S(\varepsilon) > 0 \) such that

\[
\left| \frac{1}{x-c} \int_c^x f(t) \, dt - L \right| < \varepsilon \quad \text{whenever} \quad x \in E_c \quad \text{and} \quad 0 < |x-c| < S,
\]

then \( L \) is said to be \( D_1 \)-limit of \( f \) at \( c \) and we write

\[
D_1 \lim_{t \to c} f(t) = L.
\]

The function \( f \) is said to be \( D_1 \)-continuous at \( c \) if

\[
D_1 \lim_{t \to c} f(t) = f(c).
\]

In otherwords, \( f \) is \( D_1 \)-continuous at \( c \in [a, b] \) if \( f \) is \( D \)-integrable in some neighbourhood of \( c \) and \( f(c) \) is the approximate derivative at \( c \) of its indefinite \( D \)-integral; \( f \) is said to be \( D_1 \)-continuous on \([a, b]\) if it is \( D_1 \)-continuous at every point of \([a, b]\). (If \( x = a \) or \( x = b \) then appropriate onesided neighbourhood and onesided limit are to be considered in the above definition).
Clearly if \( f \) is continuous in \([a,b]\) then \( f \) is the derivative of its indefinite integral and so \( f \) is \( D_1 \)-continuous in \([a,b]\). The converse is not true. In fact, there exists a function \( f \) and there is a set \( E_0 \) of positive measure in its domain such that \( f \) is \( D_1 \)-continuous at each point on \( E_0 \) but nowhere continuous on \( E_0 \). Let \( F \) be an ACG function on an interval which is not differentiable at the points of a set \( E \) of positive measure (cf. [41, p.224]). The approximate derivative \( F'_{ap} \) exists almost everywhere (cf. [41, p.222, Theorem 4.3]). Let \( f = F'_{ap} \) where \( F'_{ap} \) exists and \( f = 0 \) otherwise. Clearly \( f \) is \( D_1 \)-continuous almost everywhere on \( E \) but \( f \) is not continuous on \( E \).

It may be recalled that a function \( f \) is said to be \( C_1 \)-continuous at \( x \) if \( f \) is \( D^* \)-integrable in some neighbourhood of \( x \) and if \( F'(x) = f(x) \) where \( F \) is an indefinite \( D^* \)-integral of \( f \) (see [6]). Replacing \( D^* \)-integral by \( D \)-integral, Ellis [19] introduced the concept of \( M_1 \)-continuity. Clearly \( C_1 \)-continuity implies \( M_1 \)-continuity and \( M_1 \)-continuity implies \( D_1 \)-continuity.

**Definition 1.3.2**: A function \( f : [a,b] \to \mathbb{R} \) is said to be \( D_1 \)-integrable on \([a,b]\) if there is a \( D_1 \)-continuous, ACG function \( \Phi : [a,b] \to \mathbb{R} \) such that \( \phi'_{ap} = f \) almost everywhere in \([a,b]\).

Then the function \( \Phi \) is said to be an indefinite \( D_1 \)-integral of \( f \) and \( \Phi(b) - \Phi(a) \) is the definite integral of \( f \) on \([a,b]\).

Since a \( D_1 \)-continuous function is an approximate derivative, it has Darboux property [21] and so by the Corollary 1.2.1, \( \Phi \) is unique up to an additive constant and so the definite integral is unique. The definite integral is denoted by

\[
(D_1) \int_a^b f(t) dt \quad \text{or simply } (D_1) \int_a^b f .
\]
Recall that a function \( f : [a,b] \rightarrow \mathbb{R} \) is \( GM_1 \)-integrable on \([a,b]\) if there is an \( M_1 \)-continuous, ACG function \( \Phi : [a,b] \rightarrow \mathbb{R} \) such that \( \Phi^\prime = f \) almost everywhere in \([a,b]\). Since \( M_1 \)-continuity implies \( D_1 \)-continuity, it follows that if \( f \) is \( GM_1 \)-integrable then it is \( D_1 \)-integrable and the integrals are equal. In Example 1.6.1, we shall show that the \( D_1 \)-integral is strictly more general than the \( GM_1 \)-integral [19]. Since the \( GM_1 \)-integral includes the CP-integral [6] the \( D_1 \)-integral is more general than the CP-integra and hence more general than the \( D_- \) and \( D^* \)-integrals. In Examples 1.6.2, 1.6.3 and 1.6.4 we shall show that the \( D_1 \)-integral and the AP-integral [5] (and also the AD-integral [23]) are not comparable and even not compatible. It may be noted that the AP-integral and the AD-integral have the disadvantages that the indefinite integrals (which are expected to have properties nicer than the integrand) may not be integrable (cf. [11] and Example 1.6.3 below).

The function \( f \) is said to be \( D_1 \)-integrable on a measurable subset \( E \) of \([a,b]\) if \( f_E \) is \( D_1 \)-integrable on \([a,b]\) where \( f_E \) is defined by

\[
f_E(x) = f(x), \quad x \in E
= 0, \quad x \notin E
\]

and we write \( f \in D_1(E) \). We shall take

\[
(D_1) \int_E f = (D_1) \int_a^b f_E.
\]

**Theorem 1.3.1:** If \( f \) is \( D_1 \)-integrable in \([a,b]\), then \( \int_a^x f \) is \( D \)-integrable and \( D_1 \)-continuous on \([a,b]\).
Theorem 1.3.2: If \( f \) and \( g \) are \( D_1 \)-integrable in \([a, b]\) and \( \alpha, \beta \) are constants then \( \alpha f + \beta g \) is \( D_1 \)-integrable in \([a, b]\) and

\[
(D_1) \int_a^b (\alpha f + \beta g) = \alpha (D_1) \int_a^b f + \beta (D_1) \int_a^b g.
\]

The proofs of Theorems 1.3.1 and 1.3.2 follow from the definition of the \( D_1 \)-integral.

Theorem 1.3.3: If \( f \) is \( D_1 \)-integrable in \([a, b]\) and in \([b, c]\) then it is \( D_1 \)-integrable in \([a, c]\) and conversely if \( f \) is \( D_1 \)-integrable in \([a, c]\) and \( a < b < c \) then it is so in \([a, b]\) and \([b, c]\).

In either case

\[
(D_1) \int_a^c f = (D_1) \int_a^b f + (D_1) \int_b^c f. \tag{3.1}
\]

Proof: Let \( F_1 \) and \( F_2 \) be the indefinite \( D_1 \)-integrals of \( f \) in \([a, b]\) and in \([b, c]\) respectively. We may suppose that \( F_2(b) = 0 \). Let

\[
F(x) = \begin{cases} F_1(x) & \text{for } x \in [a, b] \\ F_1(b) + F_2(x) & \text{for } x \in [b, c]. \end{cases}
\]
Then $F$ is ACG in $[a,c]$ and $F'_{ap} = f$ almost everywhere in $[a,c]$. So we are to show that $F$ is $D_1$-continuous in $[a,c]$. Since $F_1$ and $F_2$ are $D_1$-continuous in $[a,b]$ and in $[b, c]$ respectively, we are only to consider the point $x = b$. Since $F_1$ and $F_2$ are $D$-integrable in $[a,b]$ and $[b, c]$ respectively, $F$ is $D$-integrable in $[a,c]$. Let $\Phi$ be an indefinite $D$-integral of $F$. Since $F_1$ is $D_1$-continuous at $b$,

$$\lim_{h \to 0^+} \frac{\Phi(b - h) - \Phi(b)}{-h} = \lim_{h \to 0^+} \frac{1}{h(D)} \int_{b-h}^{b} F(t) \, dt$$

$$= \lim_{h \to 0^+} \frac{1}{h(D)} \int_{b-h}^{b} F_1(t) \, dt = F_1(b).$$

Also since $F_2$ is $D_1$-continuous at $b$,

$$\lim_{h \to 0^+} \frac{\Phi(b + h) - \Phi(b)}{h} = \lim_{h \to 0^+} \frac{1}{h(D)} \int_{b}^{b+h} F(t) \, dt$$

$$= \lim_{h \to 0^+} \frac{1}{h(D)} \int_{b}^{b+h} [F_1(b) + F_2(t)] \, dt$$

$$= F_1(b) + F_2(b) = F_1(b).$$

Hence $F$ is $D_1$-continuous at $b$. Thus $F$ is an indefinite $D_1$-integral of $f$. Since $F(b) - F(a) + F(c) - F(b) = F(c) - F(a)$, the relation (3.1) is clear. The converse is easy.
Theorem 1.3.4: If \( f \) is \( D_1 \)-integrable and \( f \geq 0 \) almost everywhere in \([a,b]\), then \( f \) is Lebesgue integrable in \([a,b]\) and the integrals are equal.

Proof: Let

\[
F(x) = (D_1) \int_a^x f.
\]

Then \( F \) is \( D_1 \)-continuous and hence is an approximate derivative. So \( F \) has Darboux property \([21]\). Also \( F \) is in \( ACG([a,b]) \) and \( F' = f \geq 0 \) almost everywhere in \([a,b]\). Hence by Theorem 1.2.1, \( F' \) is nondecreasing in \([a,b]\). So \( F' \) exists almost everywhere and is Lebesgue integrable in \([a,b]\). Since \( F' = f \) almost everywhere in \([a,b]\), \( f \) is Lebesgue integrable in \([a,b]\). The rest is clear.

Theorem 1.3.5: If both \( f \) and \( g \) are \( D_1 \)-integrable on \([a,b]\) and if \( f \geq g \) almost everywhere then

\[
(D_1) \int_a^b f \geq (D_1) \int_a^b g.
\]
Proof: Let $F$ and $G$ be indefinite integrals of $f$ and $g$ respectively. Then $F$ and $G$ both are $D_1$-continuous and ACG on $[a,b]$ and $F'_\text{ap} = f$, $G'_\text{ap} = g$ almost everywhere on $[a,b]$. Hence $F'_\text{ap} \geq G'_\text{ap}$ almost everywhere on $[a,b]$. If $\phi = F - G$, then $\phi$ is $D_1$-continuous and ACG on $[a,b]$ and $\phi'_\text{ap} \geq 0$ almost everywhere on $[a,b]$ and hence by Theorem 1.2.1, we have

$$\phi(b) - \phi(a) \geq 0.$$ 

So

$$(D_1) \int_a^b f \geq (D_1) \int_a^b g.$$

Theorem 1.3.6: If $f$ is $D_1$-integrable then $f$ is measurable and finite almost everywhere.

Proof: Let $F$ be an indefinite $D_1$-integral of $f$ in $[a,b]$. Then, since $F$ is ACG in $[a,b]$, $[a,b] = \bigcup \nolimits\limits_{n=1}^{\infty} E_n$, $E_n$ closed and $F$ is absolutely continuous on each $E_n$. For each $n$, let $F_n = F$ on $E_n$ and $F_n$ is linear in the closure of each contiguous interval. Then $F_n$ is of bounded variation in $[a,b]$. Since $f = F'_\text{ap} = F'_n$ almost everywhere on $E_n$ and since the derivative of a function of bounded variation is finite almost everywhere and measurable, the result follows.
Theorem 1.3.7 (Dominated convergence theorem): If (i) for each n, \( g \leq f_n \leq h \) almost everywhere in \([a,b]\) where \( g, f_n, h \) are \( D_1 \)-integrable and

\[
(ii) \lim_{n \to \infty} f_n(x) = f(x) \text{ almost everywhere on } [a,b] \text{ then } f \text{ is } D_1 \text{-integrable and}
\]

\[
\lim_{n \to \infty} (D_1) \int_{a}^{b} f_n = (D_1) \int_{a}^{b} f.
\]

Proof: Write \( \phi_n = f_n - g, \phi = f - g, \psi = h - g \). Then \( \phi_n \) and \( \psi \) are non-negative almost everywhere and \( D_1 \)-integrable in \([a,b]\). By Theorem 1.3.4 they are Lebesgue integrable in \([a,b]\). Since \( 0 \leq \phi_n \leq \psi \), the Lebesgue theory of limits under the integral sign shows that \( \phi \) is Lebesgue integrable and

\[
\lim_{n \to \infty} (L) \int_{a}^{b} \phi_n = (L) \int_{a}^{b} \phi.
\]

That is

\[
\lim_{n \to \infty} [(D_1) \int_{a}^{b} f_n - (D_1) \int_{a}^{b} g] = (D_1) \int_{a}^{b} f - (D_1) \int_{a}^{b} g.
\]

Hence the result.

Theorem 1.3.8 (Monotone convergence theorem): If \( \{f_n\} \) is a non-decreasing sequence of \( D_1 \)-integrable functions on \([a,b]\) and if the sequence \( \{ (D_1) \int_{a}^{b} f_n \} \) is bounded above then the function

\[
f(x) = \lim_{n \to \infty} f_n(x) \text{ is } D_1 \text{-integrable on } [a,b] \text{ and}
\]

\[
(3.2) \quad \lim_{n \to \infty} (D_1) \int_{a}^{b} f_n = (D_1) \int_{a}^{b} f.
\]
Proof: Since \( f_n - f_1 \) is \( D_1 \)-integrable and nonnegative, by Theorem 1.3.4 it is Lebesgue integrable. Since \( f_n - f_1 \to f - f_1 \), by the Lebesgue theory

\[
\lim_{n \to \infty} (\int_a^b (f_n - f_1)) = (\int_a^b (f - f_1)) \tag{3.3}
\]

Since the sequence of integrals \( \left\{ (\int_a^b f_n) \right\} \) is bounded above, the sequence \( \left\{ (\int_a^b (f_n - f_1)) \right\} \) is also bounded above and therefore

\[
0 \leq (\int_a^b (f - f_1)) < \infty
\]

Hence \( f - f_1 \) is Lebesgue integrable and a fortiori, is \( D_1 \)-integrable and \( f_1 \) being \( D_1 \)-integrable, \( f \) is also so. Hence (3.2) follows from (3.3).

**Theorem 1.3.9:** If \( f \) is \( D_1 \)-integrable on \([a, b]\) then for every closed set \( E \subset [a, b] \) there is a closed interval \( J \subset [a, b] \) containing points of \( E \) in its interior such that

(1) \( f \) is Lebesgue integrable on \( J \cap E \)

(2) if \( \left\{ I_k \right\} \) is the sequence of contiguous closed intervals of \( J \cap E \) then

\[
\sum_{k} |(D_1)_{I_k} f| < \infty
\]
Proof: Let \( F(x) = (D_1) \int_a^x f \).

Then since \( F \) is ACG on \([a,b]\) by Lemma 1.2.1 there is a closed interval \( J \subset [a,b] \) containing points of \( E \) in its interior such that \( F \) is absolutely continuous on \( J \cap E \). Let \( G \) be the function on \( J \) which coincides with \( F \) on \( J \cap E \) and is linear on each contiguous closed intervals of \( J \cap E \). Then \( G \) is continuous and of bounded variation on \( J \) and \( F \) satisfies Lusin condition on \( J \). So \( G \) is absolutely continuous on \( J \) and hence \( G' \) is Lebesgue integrable on \( J \). Since \( G' = F_{ap} = f \) almost everywhere on \( J \cap E \), \( f \) is Lebesgue integrable on \( J \cap E \), proving (i).

To prove (ii), note that since \( F \) is absolutely continuous on \( J \cap E \), \( F \) is also of bounded variation on \( J \cap E \) and hence

\[
\sum_k \left| (D_1) \int_{I_k} f \right| = \sum_k \left| F(b_k) - F(a_k) \right| < \infty
\]

where \( I_k = [a_k, b_k] \). This proves (ii).

1.4. Integration by Parts

Lemma 1.4.1: Let \( \Phi \) be \( D_1 \)-continuous at \( x_0 \in [a,b] \) and \( F \) be an indefinite \( L \)-integral of a function \( f \) of bounded variation in \([a,b] \), then \( \Phi F \) is \( D_1 \)-continuous at \( x_0 \).

Proof: Let \( a < x_0 < b \). Since \( \Phi \) is \( D_1 \)-continuous at \( x_0 \), it is \( D \)-integrable in some neighbourhood of \( x_0 \). Let \( \Phi_1 = (D) \int_{x_0}^x \Phi \). Then \( \Phi_1 \)
is continuous in that neighbourhood of \( x_0 \). Hence for \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
(4.1) \quad |\Phi_1(t)| < \varepsilon \quad \text{whenever} \quad |t - x_0| < \delta.
\]

Since \( f \) is bounded in \([a, b]\), there is \( M > 0 \) such that

\[
(4.2) \quad |f(t)| \leq M, \quad \text{for all} \quad t \in [a, b].
\]

Since \( \Phi \) is D-integrable in a neighbourhood of \( x_0 \) and \( F \) is absolutely continuous in \([a, b]\), by \([41, \text{p.}246, \text{Theorem} \ 2.5]\),

\( \Phi F \) is D-integrable in that neighbourhood of \( x_0 \) and

\[
(D)\int_{x_0}^{x_0+h} \Phi F = \Phi_1(x_0 + h) F(x_0 + h) - (S)\int_{x_0}^{x_0+h} \Phi_1 dF.
\]

Let

\[
H(x) = (D)\int_{x_0}^{x} \Phi F.
\]

Let \( f = g_1 - g_2 \) where \( g_1 \) and \( g_2 \) are nonnegative nondecreasing functions and let \( F = F_1 - F_2 \) where \( F_1 \) and \( F_2 \) are indefinite integrals of \( g_1 \) and \( g_2 \) respectively.

Then, if \( 0 < |h| < \delta \)

\[
(4.3) \quad \left[ \frac{H(x_0 + h) - H(x_0)}{h} - F(x_0) \Phi(x_0) \right] = \left[ \frac{1}{h} (D)\int_{x_0}^{x_0+h} \Phi F - F(x_0) \Phi(x_0) \right]
\]
\[
\begin{align*}
= \left[ \frac{1}{h} F(x_0 + h) \tilde{\phi}_1(x_0 + h) - \frac{1}{h} \int_{x_0}^{x_0 + h} \tilde{\phi}_1 dF - F(x_0) \tilde{\phi}(x_0) \right].
\end{align*}
\]

If \( S(x) = (S) \int_{x_0}^{x} \tilde{\phi}_1 dF \), then since \( F \) is continuous and non-decreasing, \( S(x) \) is continuous. Also by [41, p.244, Theorem 2.1(ii)], \( S'(x) = \tilde{\phi}_1(x) g_1(x) \) except enumerable set and hence by [41, p.235, Theorem 10.5], \( S \) is ACG*.

Since \( \tilde{\phi}_1 g_1 \) is Riemann integrable

\[
\begin{align*}
(S) \int_{x_0}^{x} \tilde{\phi}_1 dF &= S(x) = (R) \int_{x_0}^{x} \tilde{\phi}_1 g_1.
\end{align*}
\]

Considering similarly for \( g_2 \) and taking the difference, we have

\[
(S) \int_{x_0}^{x} \tilde{\phi}_1 dF = (R) \int_{x_0}^{x} \tilde{\phi}_1 f.
\]

Hence from (4.3), (4.1) and (4.2)

\[
\begin{align*}
\left| \frac{H(x_0 + h) - H(x_0)}{h} - F(x_0) \tilde{\phi}(x_0) \right| &= \left| \frac{1}{h} F(x_0 + h) \tilde{\phi}_1(x_0 + h) - F(x_0) \tilde{\phi}(x_0) - \frac{1}{h} \int_{x_0}^{x_0 + h} \tilde{\phi}_1 f \right| \\
&\leq \left| \frac{1}{h} F(x_0 + h) \tilde{\phi}_1(x_0 + h) - F(x_0 + h) \tilde{\phi}(x_0) \right| \\
&\quad + \left| F(x_0 + h) \tilde{\phi}(x_0) - F(x_0) \tilde{\phi}(x_0) \right| + \frac{1}{h} \int_{x_0}^{x_0 + h} |\tilde{\phi}_1| |f| \\
&\leq \left| F(x_0 + h) \right| \left| \frac{\tilde{\phi}_1(x_0 + h)}{h} - \tilde{\phi}(x_0) \right| + |\tilde{\phi}(x_0)| \left| F(x_0 + h) - F(x_0) \right| \\
&\quad + \frac{M}{h} \cdot \varepsilon \cdot h.
\end{align*}
\]
Therefore, since \( \xi \) is arbitrary,
\[
\lim_{h \to 0^+} \left[ \frac{H(x_0 + h) - H(x_0)}{h} - \frac{F(x_0)}{h} \right] = 0.
\]

Similarly if \( a < x_0 < b \) then
\[
\lim_{h \to 0^+} \left[ \frac{H(x_0) - H(x_0 - h)}{h} - \frac{F(x_0)}{h} \right] = 0.
\]

Hence the result.

Lemma 1.4.2: Let \( \varphi \) be \( D_1 \)-integrable and \( f \) be of bounded variation in \([a, b] \). Let
\[
\hat{\varphi}(x) = (D_1) \int_a^x \varphi, \quad F(x) = (R) \int_a^x f, \quad a \leq x \leq b.
\]
Then the function \( \psi \) defined by
\[
\psi(x) = F(x) \hat{\varphi}(x) - (D) \int_a^x \hat{\varphi} f, \quad a \leq x \leq b,
\]
is \( D_1 \)-continuous, ACG on \([a, b] \).

Proof: By Theorem 1.3.1, \( \hat{\varphi} \) is \( D \)-integrable and so by [41, p. 246, Theorem 2.5], \( \hat{\varphi} f \) is \( D \)-integrable and so \( \psi \) is well defined. Since \( \hat{\varphi} \) is \( D_1 \)-continuous, by Lemma 1.4.1, \( F \hat{\varphi} \) is \( D_1 \)-continuous. Also since \( \hat{\varphi} \) is ACG and \( F \) is absolutely continuous, \( F \hat{\varphi} \) is ACG in \([a, b] \). Since a continuous function is \( D_1 \)-continuous, \((D) \int_a^x \hat{\varphi} f \) is \( D_1 \)-continuous and ACG. So \( \psi \) is \( D_1 \)-continuous and ACG in \([a, b] \).

Theorem 1.4.1 (Integration by parts): If \( \varphi \) is \( D_1 \)-integrable and \( f \) is of bounded variation in \([a, b] \) and if \( \hat{\varphi}(x) = (D_1) \int_a^x \varphi \), \( F(x) = (R) \int_a^x f, \ a \leq x \leq b \) then \( \varphi F \) is \( D_1 \)-integrable in \([a, b] \) and
\[
(D_1) \int_a^b \varphi F = [\hat{\varphi} F]_a^b - (D) \int_a^b \hat{\varphi} f.
\]
Proof: By Lemma 1.4.2, the function $\psi$ defined by

$$\psi(x) = F(x) \phi(x) - (D) \int_a^x \phi f, \quad a \leq x \leq b,$$

is $D_1$-continuous and ACG in $[a, b]$. Also almost everywhere in $[a, b]$,

$$F' = f, \quad \phi'_{ap} = \phi' \text{ and } ((D) \int_a^x \phi f)'_{ap} = \phi f.$$

Hence almost everywhere in $[a, b]$,

$$\psi'_{ap} = f \phi + F \phi - \phi f = F \phi.$$

So $F \phi$ is $D_1$-integrable and $\psi$ is an indefinite $D_1$-integral of $F \phi$.

Hence

$$(D_1) \int_a^b F \phi = \psi(b) - \psi(a)$$

$$= [F(x) \phi(x)]_a^b - (D) \int_a^b \phi f.$$

1.5. Cauchy and Harnack property

Theorem 1.5.1 (Cauchy property): If $f$ is $D_1$-integrable in $[a, b]$ for every $\beta$, $a < \beta < b$, and if

$$\lim_{\beta \to b^-} (D_1) \int_a^\beta f = L$$

then $f$ is $D_1$-integrable in $[a, b]$ and

$$(D_1) \int_a^b f = L.$$
Proof: Let \( b_1, b_2, \ldots, b_n, \ldots \) be an increasing sequence which converges to \( b \) with \( b_1 = a \). Then \( f \) is \( D_1 \)-integrable on each \( I_n = [b_n, b_{n+1}] \) and so there is a function \( F_n \) which is \( D_1 \)-continuous and ACG on \( I_n \) and \( (F_n)'_\text{ap} = f \) almost everywhere on \( I_n \). We may suppose \( F_n(b_n) = 0 \) for all \( n \). Let

\[
F(x) = F_1(x), \quad x \in I_1
\]

\[
= F_n(x) + \sum_{k=1}^{n-1} F_k(b_{k+1}), \quad x \in I_n, \quad n \geq 2
\]

\[
= L, \quad x = b.
\]

Then since \( F \) is ACG on each \( I_n \), \( F \) is ACG on \([a, b]\). Also \( F'_\text{ap} = f \) almost everywhere in \([a, b]\). Since \( F(x) = (D_1)\int_a^x f, \quad a \leq x < b \), we have from the given condition \( D_1-\lim_{\beta \to b^-} F(\beta) = L \) and so \( F \) is \( D \)-integrable in some neighbourhood of \( b \) and hence \( F \) is \( D \)-integrable in \([a, b]\). Let

\[
\phi(x) = (D)\int_a^x F, \quad a \leq x \leq b.
\]

Then

\[
\phi'_{\text{ap}}(b) = \lim_{x \to b^+} \frac{1}{x-b} \int_b^x F = D_1-\lim_{\beta \to b^-} F(\beta) = L = F(b).
\]

Thus \( F \) is \( D_1 \)-continuous at \( x = b \). Also \( F \) is \( D_1 \)-continuous on each \( I_n \). So \( f \) is \( D_1 \)-integrable in \([a, b]\) and \( F \) is an indefinite \( D_1 \)-integral in \([a, b]\). Thus

\[
(D_1)\int_a^b f = F(b) - F(a) = F(b) = L.
\]
Theorem 1.5.2 (Harnack Property): Let \( E \subseteq [a, b] \) be a closed set with complementary intervals \( I_k = (a_k, b_k), k = 1, 2, \ldots \). Let \( f \in D_1(E) \) and \( f \in D_1([a_k, b_k]) \) for each \( k \) with

\[
F_k(x) = (D_1) \int_{a_k}^{x} f, \quad a_k \leq x \leq b_k.
\]

Let (if there are infinite number of intervals \( I_k \))

(i) \[
\sum_{k=1}^{\infty} |(D_1) \int_{a_k}^{b_k} f| < \infty.
\]

(ii) \[
\lim_{k \to \infty} \sup_{x \in (a_k, b_k]} x - a_k \int_{a_k}^{x} F_k(t) dt = 0.
\]

Then \( f \) is \( D_1 \)-integrable in \([a, b]\) and

\[
(D_1) \int_{a}^{b} f = (D_1) \int_{E} f + \sum_{k} (D_1) \int_{a_k}^{b_k} f.
\]

Remark: It may be noted that Sargent [42] has obtained the Harnack property for the \( C_1 D \)-integral with the conditions (i), (ii) replaced by

(\( \alpha \)) \[
\sum_{k=1}^{\infty} \sup_{a_k < x < b_k} \left| \frac{1}{x - a_k} \int_{a_k}^{x} F_k(t) dt \right| < \infty,
\]

(\( \beta \)) \[
\sum_{k=1}^{\infty} \sup_{a_k < x < b_k} \left| \frac{1}{b_k - x} \int_{x}^{b_k} F_k(t) dt - F_k(b_k) \right| < \infty.
\]

(see [42, property B]). But (\( \alpha \)) and (\( \beta \)) together imply (i) and (ii) and so our conditions (i) and (ii) are more relaxed. In fact, we get from [42, Lemma III] that

\[
\sum_{k=1}^{\infty} \left| \int_{a_k}^{b_k} f(t) dt \right| = \sum_{k=1}^{\infty} \left| F_k(b_k) - F_k(a_k) \right| \leq \sum \omega_k(a_k, b_k)
\]
where $H$ is a constant and

$$\omega_k(a_k, b_k) = \max \left[ \sup_{a_k < x < b_k} \left| \frac{1}{x-a_k} \int_{a_k}^{x} F_k(t) dt - F_k(a_k) \right|, \right.$$

$$\left. \sup_{a_k < x < b_k} \left| \frac{1}{b_k-x} \int_{x}^{b_k} F_k(t) dt - F_k(b_k) \right| \right].$$

Since $F_k(a_k) = 0$, (α) and (β) imply

$$\sum_{k=1}^{\infty} \int_{a_k}^{b_k} |f(t) dt| < \infty$$

implying (i). Also convergence of the series in (α) implies (ii).

This result is also known for the D-integral [41, p.257, Theorem 5.1] with the condition (ii) replaced by

$$(ii)' \lim_{k \to \infty} O(F_k; a_k, b_k) = 0$$

where $O(F_k; a_k, b_k)$ denotes oscillation of $F_k$ in $[a_k, b_k]$. Note that for the D-integral the condition $(ii)'$ implies (ii). In fact, for the D-integral $F_k$ is continuous and

$$\left| \frac{1}{x-a_k} \int_{a_k}^{x} F_k(t) dt \right| \leq \frac{1}{x-a_k} (R) \int_{a_k}^{x} |F_k(t)| dt \leq O(F_k; a_k, b_k)$$

and so if $(ii)'$ holds then (ii) holds. So in this case also our conditions are more relaxed.
Let

\[ \psi_k(x) = F_k(x), \quad a_k \leq x \leq b_k \]

\[ = F_k(b_k), \quad x > b_k \]

\[ = 0, \quad x < a_k \]

and

\[ F(x) = \sum_{k=1}^{\infty} \psi_k(x). \]

We shall show that \( F \) is \( D_1 \)-continuous in \([a, b]\).

If \( x \) is an interior point of some \( I_k \), say \( I_m \), then since

\[ F(t) = \sum \psi_k(b_k) + F_m(t), \quad \text{for} \quad a_m < t < b_m \]

where \( \sum \) is taken for those \( k \) for which \( I_k \subset [a, a_m) \) and since \( F_m \) is \( D_1 \)-continuous, \( F \) is \( D_1 \)-continuous at \( x \). If \( x \) is an isolated point of \( E \) then \( x \) is the common endpoint of two intervals \( I_k \), say \( I_p \) and \( I_m \) where

\[ b_p = x = a_m. \]

Then for small \( h > 0 \), \( F(x+h) - F(x) = F_m(x+h) \) and

\[ F(x) - F(x-h) = F_p(x) - F_p(x-h). \]

Since \( F_m \) and \( F_p \) are \( D_1 \)-continuous in \([a_m, b_m]\) and in \([a_p, b_p]\) respectively,

\[ D_1-lim \left[ F(x+h) - F(x) \right] = F_m(a_m) = 0 \]

\[ h \to 0^+ \]

and

\[ D_1-lim \left[ F(x) - F(x-h) \right] = F_p(x) - F_p(b_p) = 0. \]

\[ h \to 0^+ \]

So \( F \) is \( D_1 \)-continuous at \( x \). If \( E \) has component intervals then clearly these intervals are closed and \( F \) is constant in these intervals and hence is \( D_1 \)-continuous there with one sided
$D_1$-continuity at the endpoints. Also the endpoints of the component intervals are the endpoints of suitable contiguous intervals $(a_k, b_k)$ and so, applying the second case above, $F$ is both sided $D_1$-continuous at the endpoints of the component intervals of $E$. The only remaining case we are to consider is that $x$ is a limit point of endpoints of the contiguous intervals. Let $x$ be such a limit point from the right. Let $\varepsilon > 0$ be arbitrary. Then from the condition (i) and (ii) there is $k_0$ such that

\[(5.1) \quad \sum_{k > k_0} |F_k(b_k)| < \varepsilon\]

\[(5.2) \quad \sup_{x \in (a_k, b_k)} \int_{x-a_k}^{x} F_k \, dt < \varepsilon, \text{ for all } k > k_0.\]

Let $\delta > 0$ be such that $(x, x+\delta)$ does not contain the intervals $I_k, 1 \leq k \leq k_0$. Let $t \in (x, x+\delta) \cap E$. Then denoting by $\Sigma_1$ the summation taken over those $k$ for which $I_k \subseteq [x, t]$, we have from (5.1)

\[(5.3) \quad |F(t) - F(x)| = |\Sigma_1 F_k(b_k)| \leq \Sigma_1 |F_k(b_k)| \leq \sum_{k > k_0} |F_k(b_k)| < \varepsilon.

Since the functions $F_k$ are $N$-integrable in $[a_k, b_k]$, $\nu_k$ is measurable in $[a, b]$ for each $k$ by [41, p. 243, Theorem 1.3] and so $F$ is measurable. So by (5.3), $F(t) - F(x)$ is Lebesgue integrable in $(x, x + \delta) \cap E$ and

\[(5.4) \quad (L) \quad \int_{(x, x+h) \cap E} |F(t) - F(x)| \, dt \leq \varepsilon \mu((x, x+h) \cap E) \text{ for } 0 < h < \delta,

where $\mu$ is the Lebesgue measure.
Let \( t \in (x, x + \delta) \sim E \). Then \( t \in I_k \) for some \( k \). Let \( t \in I_m \).

Denoting by \( \Sigma_2 \) the summation taken over these \( k \) for which \( I_k \subset [x, a_m] \), we have

\[
F(t) - F(x) = \Sigma_2 F_k(b_k) + F_m(t)
\]

That is

\[
(5.5) \quad |F(t) - F(x) - F_m(t)| \leq \Sigma_2 |F_k(b_k)| \leq \Sigma_k (b_k) < E
\]

Hence as above \( F(t) - F(x) - F_m(t) \) is Lebesgue integrable and so it is D-integrable in \( I_m \). The function \( F_m \) which is an indefinite \( D_1 \)-integral is D-integrable and hence \( F(t) - F(x) \) is D-integrable in \( I_m \).

Taking D-integral in \( (5.5) \)

\[
\frac{1}{t - a_m} \int_{a_m}^{t} [F(\xi) - F(x)] d\xi \leq E + \frac{1}{t - a_m} \int_{a_m}^{t} F_m(\xi) d\xi,
\]

for \( a_m \leq t \leq b_m \).

Hence by \((5.2)\), since \( m > k_0 \)

\[
(5.6) \quad -2E(t - a_m) < \int_{a_m}^{t} [F(\xi) - F(x)] d\xi < 2E(t - a_m),
\]

for \( a_m \leq t \leq b_m \).

Hence

\[
(5.7) \quad \int_{a_m}^{b_m} F(\xi) d\xi \leq (2E + |F(x)|)(b_m - a_m).
\]

Since \( I_m \) is any interval \( I_k \subset (x, x + \delta) \sim E \), \((5.7)\) is true for all such intervals and so adding for these intervals
where \( \Sigma_3 \) denotes summation over those \( k \) for which \( I_k \subset (x, x + \delta) \sim E \).

Also if

\[
H_k(t) = (D) \int_{a_k}^{t} F(\xi) d\xi, \quad a_k \leq t \leq b_k,
\]

then

\[
(5.9) \quad H_k(t) = \Sigma_4 F_p(b_p) \ast (t - a_k) + G_k(t)
\]

where \( \Sigma_4 \) denotes summation over those \( p \) for which \( I_p \subset (a, a_k) \)
and \( G_k \) is defined by

\[
G_k(t) = (D) \int_{a_k}^{t} F_k(\xi) d\xi, \quad a_k \leq t \leq b_k
\]

Then for \( u, v \in [a_k, b_k] \)

\[
|G_k(u) - G_k(v)| \leq |G_k(u) - G_k(a_k)| + |G_k(v) - G_k(a_k)|
\]

\[
\leq \sup_{z \in (a_k, b_k]} \left| \int_{z-a_k}^{b_k} \left| \frac{1}{z-a_k} \int \xi \right| d\xi \right| \left( |u-a_k| + |v-a_k| \right)
\]

\[
\leq 2(b_k - a_k) \sup_{z \in (a_k, b_k]} \left| \frac{1}{z-a_k} \int_{a_k}^{b_k} \xi \right| d\xi
\]

\[
\leq 2(b_k - a_k) \sup_{z \in (a_k, b_k]} \left| \frac{1}{z-a_k} \int_{a_k}^{b_k} F_k(\xi) d\xi \right|
\]
and hence
\[ \lim_{k \to \infty} 0 \left( G_k, a_k, b_k \right) = 0. \]

Therefore from (5.9)
\[ (5.10) \lim_{k \to \infty} 0 \left( H_k, a_k, b_k \right) = 0. \]

The relations (5.8) and (5.10) show that \( F \) satisfies the hypothesis of the corresponding theorem for \( D \)-integral [41, p.257, Theorem 5.1]. Hence by [41, p.257, Theorem 5.1], \( F \) is \( D \)-integrable in \([x, x + h]\) and
\[ (5.11) \quad (D) \int_{x}^{x+h} F = (D) \int_{x}^{x+h} F + \sum_{k}^{b_k} (D) \int_{a_k}^{b_k} F, \]

where \( \sum_{k} \) is the summation over all \( k \) for which \( I_k \subset [x, x+h] \sim E \).

The relation (5.6) being true for all intervals \( I_k \subset [x, x+\delta] \sim E \), by adding all the relations in (5.6) with (5.4), we get from (5.11)
\[ (D) \int_{x}^{x+h} |F(t) - F(x)| dt \leq 2\varepsilon h \quad \text{for all} \ h, 0 < h < \delta. \]

Dividing by \( h \) and letting \( h \to 0^+ \), since \( \varepsilon \) is arbitrary,
\[ \lim_{h \to 0^+} \frac{1}{h} \int_{x}^{x+h} F(t) dt = F(x). \]

If \( x \) is a limit point of endpoints of the contiguous intervals from the left then we get similarly
\[ \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^{x} F(t) dt = F(x). \]
In fact, in this case left side of (5.5) will be

\[ |F(t) - F(x) + F_m(t)| \]

and the summation \( E_2 \) there will be taken over those \( k \) for which \( I_k \subset [a_m, x] \).

Hence \( F \) is \( D_1 \)-continuous at \( x \). Thus \( F \) is \( D_1 \)-continuous in \([a, b]\).

We shall next show that \( F \) is ACG on \([a, b]\). Clearly \( F \) is ACG on each interval \([a_k, b_k]\). So we are to show that \( F \) is absolutely continuous on \( E \). Let

\[
g(x) = 0, \ x \in E
\]

\[
= \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f, \ x \in [a_k, b_k].
\]

By the condition (i), \( g \) is Lebesgue integrable in \([a, b]\). Put

\[
G(x) = (L) \int_a^x g, \ a \leq x \leq b.
\]

If \( x \in E \), then \( G(x) = F(x) \). Since \( G \) is absolutely continuous on \([a, b]\), \( F \) is absolutely continuous on \( E \). Thus \( F \) is ACG on \([a, b]\).

Finally, since \( G = F \) on \( E \), we have \( F = G = g = 0 \) almost everywhere on \( E \). Also in \( I_k \), \( F \) and \( F_k \) differ by a constant and hence \( F' = (F_k)' = f \) almost everywhere in \( I_k \). Hence \( F' = f \) almost everywhere in \([a, b] \sim E \). So it follows that \( F \) is an indefinite \( D_1 \)-integral of \( \check{f} \) where \( \check{f}(x) = f(x) \) if \( x \in [a, b] \sim E \) and \( \check{f}(x) = 0 \) if \( x \in E \).

On the other hand if \( \psi(x) = f(x) \) if \( x \in E \) and \( \psi(x) = 0 \) if \( x \in [a, b] \sim E \), then by hypothesis \( \psi \) is \( D_1 \)-integrable in \([a, b]\). Hence \( \check{f} + \psi = f \) is \( D_1 \)-integrable in \([a, b]\) and
\[
(D_1)^{b}_{a} f = (D_1)^{b}_{a} \phi + (D_1)^{b}_{a} \psi \\
= F(b) - F(a) + (D_1)^{b}_{a} \psi \\
= \sum_k \psi_k(b) - \sum_k \psi_k(a) + (D_1)^{b}_{a} f \\
= \sum_k F_k(b_k) + (D_1)^{b}_{a} f = \sum_k (D_1)^{b_k}_{a_k} f + (D_1)^{b}_{a} f.
\]

Hence the result.

1.6. Examples

For the definitions of GM\(_1\)-integral, AP-integral and AD-integral, considered below, we refer to [19] [5] and [23] respectively. Note that AD-integral includes AP-integral.

Example 1.6.1: There is a function which is D\(_1\)-integrable but not GM\(_1\)-integrable.

Let \( \{ I_n = (a_n, b_n) \} \) be a sequence of intervals such that

(i) \( I_n \subset (0, 1) \), for all \( n \),

(ii) \( b_1 > a_1 > b_2 > a_2 > \ldots > b_n > a_n > \ldots \) and \( \lim_{n \to \infty} b_n = 0 \),

(iii) 0 is a point of dispersion of the set \( \bigcup_{n=1}^{\infty} I_n \),

(iv) \( \frac{a_n + b_n}{b_n - a_n} \to \infty \) as \( n \to \infty \).

(One may take \( a_n = \frac{1}{n} + \frac{1}{(n+1)^2} \), \( b_n = \frac{1}{n} + \frac{1}{n^2} \).
Let 
\[ F(x) = \frac{a_n + b_n}{2} \sin \frac{\pi(x - a_n)}{b_n - a_n} \quad \text{for } x \in I_n \]
\[ = 0 \quad \text{for } x \notin \bigcup_{n=1}^{\infty} I_n. \]

Then \( F(x) \) is continuous in \([0, 1]\). Also \( F' \) exists in \((0, 1]\) but \( F' \) does not exist at 0 while \( F'_\text{ap}(0) \) exists and is 0. Since \( F \) is continuous and is ACG in \([0, 1]\) and since \( F'_\text{ap} \) exists everywhere in \([0, 1]\), \( F'_\text{ap} \) is \( D_1 \)-continuous in \([0, 1]\). Since
\[ F'_\text{ap}(x) = \frac{a_n + b_n}{b_n - a_n} \cdot \pi \sin \frac{\pi(x-a_n)}{b_n-a_n} \cos \frac{\pi(x-a_n)}{b_n-a_n}, \quad \text{for } x \in I_n \]
\[ = 0 , \quad \text{for } x \notin \bigcup_{n=1}^{\infty} I_n , \]
\( F'_\text{ap} \) is ACG in \([0, 1]\). Also \( (F'_\text{ap})' \) exists almost everywhere on \([0,1]\).

Hence the function \( f \), where
\[ f(x) = (F'_\text{ap})'(x) \quad \text{if } (F'_\text{ap})'(x) \text{ exists} \]
\[ = 0 \quad \text{otherwise} \]
is \( D_1 \)-integrable and \( F'_\text{ap} \) is its indefinite \( D_1 \)-integral. But \( f \) is not \( GM_1 \)-integrable. For, if it is so then there exists an \( M_1 \)-continuous ACG function \( \tilde{\phi} \) such that \( \tilde{\phi}'_\text{ap} = f \) almost everywhere on \([0, 1]\). Since \( M_1 \)-continuity implies \( D_1 \)-continuity \( \Psi = \tilde{\phi} - F'_\text{ap} \) is \( D_1 \)-continuous, ACG in \([0, 1]\) and \( \Psi'_\text{ap} = 0 \) almost everywhere on \([0, 1]\). Hence by Corollary 1.2.1, \( \tilde{\phi} \) and \( F'_\text{ap} \) differ by a constant. Since \( \tilde{\phi} \) is \( M_1 \)-continuous at 0, \( F'_\text{ap} \) is also so at 0. Since \( F \) is
indefinite D-integral of $F'_{ap}$, the derivative $F'(0)$ exists. But this is a contradiction, since $F'(0)$ does not exist.

**Example 1.6.2**: There is a function which is $D_1$-integrable but not AD-integrable.

Let

$$f(x) = -\frac{1}{x^2} \cos \frac{1}{x}, \ x \neq 0$$

$$= 0, \ x = 0.$$

Then $f$ is $D_1$-integrable in $[0,1]$ and

$$F(x) = \sin \frac{1}{x}, \ x \neq 0$$

$$= 0, \ x = 0$$

is its indefinite $D_1$-integral. In fact $F$ is the derivative of

$$G(x) = x^2 \cos \frac{1}{x} - 2(R) \int_0^x x \cos \frac{1}{x} \ dx, \ x \neq 0$$

$$= 0, \ x = 0$$

and so $F$ is $D_1$-continuous in $[0,1]$. Moreover $F \in ACG ([0,1])$ and $F' = f$ almost everywhere. Hence $f$ is $D_1$-integrable in $[0,1]$ (in fact, $f$ is CP-integrable in $[0,1]$). But $f$ is not AD-integrable in $[0,1]$. For, if possible let $\varphi$ be an indefinite AD-integral of $f$. Let $0 < \alpha < 1$. Then since $\varphi$ is approximately continuous and ACG in $[0,1]$, $F - \varphi$ is approximately continuous and ACG in $[\alpha, 1]$. Also since $\varphi'_{ap} = f$ almost everywhere, $(F - \varphi)'_{ap} = 0$ almost everywhere in $[\alpha, 1]$. Since approximately continuous functions
possess Darboux property, by Corollary 1.2.1, there is a constant $K$ such that $\varphi(x) = \sin \frac{1}{x} + K$ for $x \in [\alpha, 1]$. The constant $K$ cannot be different for different $\alpha$ and hence $\varphi(x) = \sin \frac{1}{x} + K$ for $x \in (0, 1]$. Therefore since $\varphi$ is approximately continuous at $x = 0$, $\lim_{x \to 0} \sin \frac{1}{x}$ exists. But this is a contradiction since this approximate limit does not exist. In fact, let

$$I_n = [\frac{4}{\pi(8n+3)}, \frac{4}{\pi(8n+1)}], \quad J_n = [\frac{4}{\pi(8n+7)}, \frac{4}{\pi(8n+5)}].$$

Then

$$|I_n| = \frac{8}{\pi(8n+1)(8n+3)}, \quad |J_n| = \frac{8}{\pi(8n+5)(8n+7)}.$$

So,

$$\left| \sum_{n=N}^{\infty} \frac{I_n}{\pi(8n+1)} \right| = \frac{\pi}{4} \sum_{n=N}^{\infty} \frac{8}{\pi(8n+1)(8n+3)} \geq (8N+1) \lim_{x \to \infty} \left[ \int_{N}^{x} \frac{2dt}{(8t+1)(8t+3)} \right]
= (8N+1) \lim_{x \to \infty} \left[ \int_{N}^{x} \frac{dt}{8t+1} - \int_{N}^{x} \frac{dt}{8t+3} \right]
= \frac{8N+1}{8} \lim_{x \to \infty} \left[ \log \frac{8t+1}{8t+3} \right]_{N}^{x}
= \frac{8N+1}{8} \lim_{x \to \infty} \left[ \log \frac{8x+1}{8x+3} - \log \frac{8N+1}{8N+3} \right] = \frac{8N+1}{8} \log \frac{8N+3}{8N+1}
= \frac{8N+1}{8} \log(1 + \frac{2}{8N+1}).$$
So,
\[
\lim_{N \to \infty} \frac{\sum_{n=N}^{\infty} |I_n|}{\frac{4}{\pi(8N+1)}} \geq \frac{1}{4} \lim_{x \to 0} \frac{1}{x} \log(1 + x) = \frac{1}{4}.
\]

Similarly
\[
\lim_{N \to \infty} \frac{\sum_{n=N}^{\infty} |J_n|}{\frac{4}{\pi(8N+5)}} = \frac{7}{4} (8N+5) \sum_{n=N}^{\infty} \frac{8}{\pi(8n+5)(8n+7)} \geq (8N+5) \lim_{x \to \infty} \int_{N}^{x} \frac{2dt}{(8t+5)(8t+7)} = \frac{8N+5}{8} \log(1 + \frac{2}{8N+5})
\]

and so
\[
\lim_{N \to \infty} \frac{\sum_{n=N}^{\infty} |J_n|}{\frac{4}{\pi(8N+5)}} \geq \frac{1}{4} \lim_{x \to 0} \frac{1}{x} \log(1 + x) = 1/4.
\]

Thus the sets \( \bigcup_{n=1}^{\infty} I_n \) and \( \bigcup_{n=1}^{\infty} J_n \) have positive upper density at \( x = 0 \). Also if \( x \in I_n \), then \( 2n\pi + \frac{\pi}{4} \leq \frac{1}{x} \leq 2n\pi + \frac{3\pi}{4} \) and hence \( \sin \frac{1}{x} \geq \frac{1}{\sqrt{2}} \) for \( x \in I_n \) and if \( x \in J_n \), then \( 2n\pi + \frac{5\pi}{4} \leq \frac{1}{x} \leq 2n\pi + \frac{7\pi}{4} \) and hence \( \sin \frac{1}{x} \leq -\frac{1}{\sqrt{2}} \) for \( x \in J_n \). That is \( \sin \frac{1}{x} \geq 1/\sqrt{2} \) for \( x \in \bigcup_{n=1}^{\infty} I_n \) and \( \sin \frac{1}{x} \leq -1/\sqrt{2} \) for \( x \in \bigcup_{n=1}^{\infty} J_n \). So \( \lim_{x \to 0} \sin \frac{1}{x} \) cannot exist.
Example 1.6.3: There is a function which is AP-integrable (and hence AD-integrable) but not $D_1$-integrable.

In [11] a nonnegative function $\varphi$ has been constructed such that $\varphi'$ exists finitely everywhere in $(0, 1]$ and $\varphi_{ap}'(0)$ exists finitely but $\varphi$ is not Lebesgue integrable in $[0, 1]$. Clearly $\varphi_{ap}'$ is AP-integrable and $\varphi$ is its indefinite AP-integral. Note that $\varphi$ is not even AD-integrable in $[0, 1]$. For if $\varphi$ is so then since $\varphi \geq 0$, $\varphi$ would be Lebesgue integrable (as in Theorem 1.3.4). But $\varphi_{ap}'$ is not $D_1$-integrable in $[0, 1]$. For, if possible, let $F$ be its indefinite $D_1$-integral. Then for $0 < \alpha < 1$, $F - \varphi \in ACG ([\alpha, 1])$ and $F - \varphi$ is $D_1$-continuous in $[\alpha, 1]$ and so as in Example 1.6.2, $F(x) = \varphi(x) + K$ for $x \in (0, 1]$ where $K$ is a constant. Since $F$ is $D$-integrable in $[0, 1]$, $\varphi$ is also $D$-integrable in $[0, 1]$. Since $\varphi \geq 0$, $\varphi$ is Lebesgue-integrable in $[0, 1]$ which is a contradiction.

Example 1.6.4: The $D_1$-integral and the AP-integral (and hence the AD-integrals) are not compatible.

Ellis [20] constructed a function $f$ such that $f$ is AP-integrable and is CP-integrable but the values of the integrals are different. Since the $D_1$-integral includes CP-integral and the AD-integral includes AP-integral the result follows.