CHAPTER V

DYNAMICAL PROBLEMS OF YIELDING AND EXPANSION IN
NON-HOMOGENEOUS MATERIALS
V.1. EFFECT OF NON-HOMOGENEITY ON PLASTIC YIELDING DUE TO SUDDENLY APPLIED FORCES ON A SPHERICAL CAVITY

**Introduction:**

The theory of spherically symmetric elastic-plastic disturbances produced by loading in a cavity in an infinite medium is of interest in plasticity and seismology in general and in particular in the study of the formation of cavity in metals.

Hopkins (1960) gave a review of works in this subject. He also considered the problem of spherically symmetric elastic-plastic disturbances produced by the action of suddenly applied pressure on a spherical cavity in a homogeneous isotropic infinite medium. He obtained the solution with the use of Fourier integral transform technique. Hunter (1954) has given the maximum allowable pressure without causing yielding of an infinite medium due to sudden loading on a spherical cavity. Eason (1963) applied Laplace transform technique to solve wave propagation from spherical and cylindrical cavities. The effects of non-homogeneity on the propagation of elastic and plastic waves were reported in the symposium (Olszak Ed. 1958) on 'Non-homogeneity in elasticity and plasticity'. Ganguly (1980) studied a particular type of non-homogeneity and studied the problem numerically.
The problem considered here is rather general and here we obtain the maximum allowable pressure without causing yielding for an anisotropic non-homogeneous medium, under a suddenly applied pressure on the surface of the spherical cavity. The medium is radially non-homogeneous and the laws of variation of density and elastic parameters are different.

**Basic equations of the problem:**

Let \((\gamma, \Theta, \Phi)\) be the spherical polar co-ordinates referred to the centre of the cavity as origin. Let 'a' be the radius of the cavity and 't' be the time measured from the instant of application of pressure. A radial time dependent pressure is applied on the spherical surface of the cavity.

The strain displacement relations are

\[
\begin{align*}
\varepsilon_{rr} &= \frac{\partial u_{r}}{\partial r} = \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u}{r} \quad \text{and} \\
\varepsilon_{\gamma\theta} &= \varepsilon_{r\phi} = \varepsilon_{\phi\theta} = 0 \quad \ldots (5.1)
\end{align*}
\]

where \(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}\) are the normal components of strain and \(u\) is the displacement. The stress-strain relations are taken in the form:

\[
\begin{align*}
\sigma_{rr} &= c_{11} \gamma \frac{\partial u_{r}}{\partial r} + 2 c_{12} \gamma \frac{u}{r} \quad \ldots (5.2) \\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = c_{12} \gamma \frac{\partial u}{\partial r} + (c_{22} + c_{23}) \gamma \frac{u}{r} \quad \ldots (5.3)
\end{align*}
\]
where $C_{11}, C_{12}, C_{22}, C_{23}$ are constants and $\sigma_\theta, \sigma_\phi, \sigma_r$ are normal components of stresses.

The equation of motion not identically satisfied is

$$\frac{\partial \sigma_\theta}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_\theta) = \rho(r) \frac{\partial^2 u}{\partial t^2} \quad \ldots \ (5.4)$$

where $\rho(r)$ is the density of the medium. The density is assumed to be of the form

$$\rho(r) = \rho_0 r^m \quad \ldots \ (5.5)$$

where $\rho_0$ is constant.

In order to ensure finiteness at infinity, the parameters $m$ and $n$ will be considered negative or zero.

The yield condition is

$$|\sigma_\theta - \sigma_r| = \sigma_0 \quad .$$

**Elastic solution:**

Substituting (5.2) and (5.3) in (5.4) and using (5.5) we get the second order differential equation in

$$\frac{\partial^2 u}{\partial r^2} + \frac{n+\rho}{r} \frac{\partial u}{\partial r} + 2 \left[ \frac{(n+1)C_{12} - C_{22} - C_{23}}{C_{11}} \right] \frac{u}{r^2} + \frac{\rho_0 \gamma^{m-n}}{C_{11}} \frac{\partial^2 u}{\partial t^2} = \ldots \ (5.6)$$
To obtain the solution of the above equation we introduce the Laplace transform defined by

\[ \tilde{f}(\tau, s) = \int_0^\infty f(\tau, t) e^{-st} \, dt. \]

Applying Laplace transform on (5.2), (5.3) and (5.6) we get

\[ \tilde{\sigma}_\tau = c_{11} \tau^n \frac{d\bar{u}}{d\tau} + 2 c_{12} \tau^{n-1} \bar{u}, \quad \ldots \text{(5.7)} \]

\[ \tilde{\sigma}_\phi = \tilde{\sigma}_\phi = c_{12} \tau^n \frac{d\bar{u}}{d\tau} + (c_{22} + c_{23}) \tau^{n-1} \bar{u}, \quad \ldots \text{(5.8)} \]

and

\[ \frac{d^2 \bar{u}}{d\tau^2} + \frac{n+2}{\tau} \frac{d\bar{u}}{d\tau} + \left[ 2 \left\{ \frac{(n+1) c_{12} - c_{22} - c_{23}}{c_{11}} \right\} \right] \bar{u} \]

\[ - \frac{S^2}{c^2} \tau^{m-n} \bar{u} = 0, \quad \ldots \text{(5.9)} \]

where

\[ c^2 = \frac{c_{11}}{\rho_0}. \quad \ldots \text{(5.10)} \]

By putting \( \bar{u} = \tau^{-(n+1)} u \), we get from (5.9)

\[ \gamma^2 \frac{d^2 \bar{u}}{d\tau^2} + \gamma \frac{d\bar{u}}{d\tau} - \left[ \frac{(n+1)^2}{2} - 2 \left\{ \frac{(n+1) c_{12} - c_{22} - c_{23}}{c_{11}} \right\} \right] \bar{u} \]

\[ - \frac{S^2}{c^2} \tau^{m-n+2} \bar{u} = 0 \quad \ldots \text{(5.11)} \]

we put \( z = \frac{S}{c} \frac{2}{m-n+2} \tau^{m-n+2} \) and from (5.11) we get
where

\[ y^2 = \frac{4}{(\eta - n + 2)^2} \left[ \frac{(n+1)^2}{4} + 2 \left\{ \frac{\cdots}{c_{11}} \right\} \right] \] ...

(5.13)

The solution of (5.12) is

\[ y = A_1 I_y(\xi) + B_1 K_y(\xi) \]

i.e.

\[ \vec{u} = \left[ A_1 I_y(\xi) + B_1 K_y(\xi) \right] \xi^{-\frac{n+1}{2}} \]

(5.14)

where \( I_y(\xi) \) and \( K_y(\xi) \) are modified Bessel's functions of first and second kind and \( A_1 \) and \( B_1 \) are constants.

The boundary conditions to be satisfied are

\[ \vec{u} \rightarrow 0 \text{ as } \xi \rightarrow \infty \]

(5.15a)

and

\[ \sigma_{\xi} = -P \mathcal{H}(t) \text{ on } \xi = \alpha \]

(5.15b)

where \( \mathcal{H}(t) \) is the prescribed loading on the cavity wall, where \( \mathcal{H}(t) \) is Heaviside function defined by \( \mathcal{H}(t) = 1 \) for \( t > 0 \) and \( \mathcal{H}(t) = 0 \) for \( t < 0 \).

Condition (5.15a) gives \( A_1 = 0 \).

(5.16)

On applying Laplace transform to condition (5.15b),

\[ \bar{\sigma}_{\xi} = -\frac{P}{s} \text{ on } \xi = \alpha \]

(5.17)
Let us now take the case $\nu = \frac{\alpha}{2}$. Then, on using (5.16) in (5.14) and substituting in (5.7) and (5.8) we get

$$\bar{\sigma}_\nu = - B_1 \gamma \frac{n-3}{2} c_{12} \left[ \alpha_1 k_{3/2}(z) + \frac{s}{c} \gamma \frac{m-n+2}{2} k_{1/2}(z) \right], \ldots (5.18)$$

$$\bar{\sigma}_\theta = - B_1 \gamma \frac{n-3}{2} c_{12} \left[ \alpha_3 k_{3/2}(z) + \frac{s}{c} \gamma \frac{m-n+2}{2} k_{1/2}(z) \right], \ldots (5.19)$$

where

$$\alpha_1 = \left[ \left( \frac{n+1}{2} + \frac{3}{4} (m-n+2) \right) \alpha_2 \right],$$

$$\alpha_2 = \frac{c_{11}}{c_{12}},$$

$$\alpha_3 = \frac{\alpha_2 + 2}{\alpha_2} - \frac{c_{22} + c_{23}}{c_{12}}.$$

From (5.17) and (5.18) we get

$$B_1 = \frac{P c^{3-n}}{c_{12} \left[ \alpha_1 k_{3/2}(a) + \frac{s}{c} \alpha_2 \gamma \frac{m-n+2}{2} k_{1/2}(a) \right]}, \ldots (5.20)$$

where

$$a = \frac{2}{c} \gamma \frac{m-n+2}{2}.$$

Substituting (5.20) in (5.18) and (5.19) and applying the inversion of the Laplace transform, we have
\[
\sigma_p = - \frac{1}{2\pi i} P\left(\frac{a}{\pi}\right) \frac{3-n+3\beta}{2} \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\alpha_2 \rho^2 \rho^3 + \alpha_1 (\beta \rho + \rho)}{\lambda \left[ \alpha_2 \rho^2 \rho^3 + \alpha_1 (\beta \rho + \rho) \right]} e^{\lambda s} ds \quad \ldots \quad (5.21)
\]

\[
- \frac{1}{2\pi i} P\left(\frac{a}{\pi}\right) \frac{3-n+3\beta}{2} \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\lambda^2 \rho^2 \rho^3 + \alpha_3 (\beta \rho + \rho)}{\lambda \left[ \alpha_2 \rho^2 \rho^3 + \alpha_1 (\beta \rho + \rho) \right]} e^{\lambda s} ds \quad \ldots \quad (5.22)
\]

where

\[
p = \frac{m - \eta + \frac{\beta}{2}}{2},
\]

\[
\lambda = \frac{S}{e},
\]

\[
\tau = c t - \frac{p \cdot \alpha}{\beta},
\]

\(\lambda\) being a real constant chosen such that all the poles of the integrand lie to the left of the line \(\lambda = \lambda\) in the complex s-plane.

The poles of the integrand are given by \(s = 0\),

\[
\mathcal{A} \mathcal{S} = -\beta_1 \pm i\beta_2
\]

where

\[
\beta_1 = \frac{\alpha_1}{2 \alpha_2}, \quad \beta_2 = \sqrt{4p\alpha_1 \alpha_2 - \alpha_1^2} \quad \ldots \quad (5.23)
\]

On integration round the suitable contour, we get the normal stress as
\[ \sigma_\gamma = - \rho \left( \frac{a}{\rho} \right)^{\frac{\gamma}{2}} \frac{2-\gamma+3p}{2} \left[ 1 + \frac{e^{-\beta_1 \gamma'}}{(\beta_1^2 + \beta_2^2) \beta_2} \left\{ \alpha_2 \beta_2 \left( \frac{\gamma}{\alpha} \right)^2 \right\} \right. \\
\left. - \rho \alpha_1 \beta_1 \cos \beta_2 \gamma' - \left[ \alpha_2 \left( \frac{\gamma}{\alpha} \right)^2 \beta_1 \right] \right\} \right] \\
\ldots (5.24) \]

\[ \sigma_\theta = - \rho \left( \frac{a}{\rho} \right)^{\frac{\gamma}{2}} \left[ \frac{\alpha_3}{\alpha_1} \frac{e^{\beta_1 \gamma'}}{\beta_2 (\beta_1^2 + \beta_2^2)} \left\{ \beta_2 \left( \frac{\gamma}{\alpha} \right)^2 \right\} \right. \\
\left. - \rho \alpha_2 \beta_2 \cos \beta_2 \gamma' - \left[ \beta_1 \left( \frac{\gamma}{\alpha} \right)^2 \right] \right\} \right] \\
\ldots \]

where \[ \gamma' = \frac{1}{\alpha \rho} \left[ c t - \frac{\gamma - a \rho}{\rho} \right] \].

\[ \frac{\sigma_\theta - \sigma_\gamma}{\rho} = \left( \frac{a}{\rho} \right)^{\frac{\gamma}{2}} \left[ \frac{\alpha_3 - \alpha_2}{\alpha_1} \frac{e^{-\beta_1 \gamma'}}{\beta_2 (\beta_1^2 + \beta_2^2)} \right. \]

\[ \left\{ \left[ (\alpha_2 - 1) \beta_2 \left( \frac{\gamma}{\alpha} \right)^2 - \rho \beta_2 (\alpha_1 - \alpha_3) \right] \cos \beta_2 \gamma' \\
\left. - \left[ (\beta_1 + \beta_2) \beta_1 \left( \frac{\gamma}{\alpha} \right)^2 \right] \right\} \right\} \]

\[ \ldots \]

\[ \left[ (\alpha_2 - 1) \beta_2 \left( \frac{\gamma}{\alpha} \right)^2 \beta_1 \left( \alpha_1 - \alpha_3 \right) \right] \cos \beta_2 \gamma' \\
- \left[ (\beta_1 + \beta_2) \beta_1 \left( \frac{\gamma}{\alpha} \right)^2 \right] \left( \alpha_2 - 1 \right) - (\alpha_1 - \alpha_3) \left( \beta_1 + \beta_2 \right) \left( \frac{\gamma}{\alpha} \right)^2 \\
+ (\alpha_1 - \alpha_3) \rho \beta_1 \sin \beta_2 \gamma' \right\} \].
Discussion and Numerical results:

The phase velocity is obtained by differentiating

\[ \frac{1}{\alpha^p} \left( c \tau - \frac{\gamma^p - \alpha^p}{p} \right) = \text{constant with respect to time.} \]

Therefore we get

\[ \frac{d\gamma}{d\tau} = \frac{c}{\gamma^{p-1}}. \]

This is the velocity with which the wave front is moving radially. The velocity depends obviously on \( p \) and it decreases as \( r \) increases if \( p > 1 \). If \( p < 1 \), the velocity increases as \( r \) increases.

Numerical values of \( (c_\theta - c_\tau)/p \) for different radial distances from the centre have been plotted in Figures 5.1.1, 5.1.2 and 5.1.3 for \( r/a = 1, 1.1, 1.2 \) considering the following values of the non-homogeneity parameters \( m \) and \( n \) of the medium: \( m = 0, n = 0; m = 0, n = -.1; m = -.5, n = -1; m = -.1, n = 0; m = 1, n = 0; m = .5, n = 0; m = 1, n = .5; m = .5, n = 1. \)

From the figures it is seen that the time to attain the maximum value of \( (c_\theta - c_\tau)/p \) at any point after the arrival of the wave varies with different values of the parameters and also with the distance from the centre. However in all the cases studied here \( (c_\theta - c_\tau)_{\text{max.}} \) is greatest at \( r = a \). This
result is of course to be expected. But the time to attain the maximum of \( \frac{(\sigma_{\theta} - \sigma_{\tau})}{P} \) varies with different models of non-homogeneity. Thus the time interval after the load is applied and the minimum load necessary for the starting of plastic yielding are dependent of the nature of the non-homogeneity. Hence the minimum pressure for yielding corresponds to that for yielding at the inner surface of the cavity.
Fig. 5.1.1:
Fig. 5.1.2:
Fig. 5.1.3:
V.2. DYNAMICAL EXPANSION OF A SPHERICAL CAVITY
IN A NON HOMOGENEOUS MEDIUM

Introduction:

The problem of dynamic expansion of a spherical cavity in an elastic material has been treated by Hopkins and Cox (1957) and Hunter (1958). In the present paper we discuss the dynamic expansion of a spherical cavity in a non-homogeneous elastic solid. The problem has been reduced to the solution of an integral equation.

Formulation of the problem and Equation of motion:

The motion being spherically symmetric the only non-zero displacement component is radial and the stresses, displacements and velocities are functions of r and t only. As the radius 'a' of the cavity at time 't' is supposed to increase with time, the motion may be studied using 'a' as parameter. The equation of conservation of mass is

\[ \frac{1}{\rho^2} \frac{\partial}{\partial \tau} (f \gamma^2 v) + \frac{\partial \rho}{\partial \tau} = 0 \]  \( \cdots (5.26) \)

where \( f \) is the density, and \( v \) is the particle velocity (radial) at any time \( t \).

For a non-homogeneous solid for which the density at a point \((r, \theta, \phi)\) is given by

\[ f = f_o \frac{r^n}{a_o^n} \]  \( \cdots (5.27) \)
where $f_0$ is constant and $a_0$ is the initial radius of the spherical cavity, the equation (5.26) then reduces to
\[
\frac{1}{\rho^2} \frac{\partial}{\partial r} \left( f_0 \frac{r^{n+2}}{a_0^{n+1}} \psi \right) = 0,
\]
\[
\frac{\partial}{\partial r} \left( \rho^{n+2} \psi \right) = 0.
\]

Integrating with respect to $r$ we get
\[
\rho^{n+2} \psi = \text{function of time only},
\]
\[
= \text{value at the cavity surface},
\]
\[
= a^{n+2} \dot{a}.
\]
\[
\therefore \psi = \frac{a^{n+2}}{\rho^{n+2}} \dot{a},
\]
or
\[
\frac{\partial u}{\partial t} = -\frac{a^{n+2}}{\rho^{n+2}} \frac{\partial a}{\partial t}. \quad \ldots (5.28)
\]

Integrating (5.28) we get
\[
U = \frac{a^{n+3}}{(n+2) \rho^{n+2}} - \frac{a_0^{n+3}}{(n+2) \rho^{n+2}} \quad \text{(approximately)} \quad \ldots (5.29)
\]

where $U$ is the particle displacement which is assumed to be zero throughout initially.

The equation of motion is
\[
\frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_\theta) = \rho \frac{\partial^2 u}{\partial t^2}, \quad \ldots (5.30)
\]

where $\rho$ is the density, $\frac{\partial^2 u}{\partial t^2}$ is the acceleration, $\sigma_r$, $\sigma_\theta$, $\sigma_r' (= \sigma_\theta)$ are the normal stresses.
The stress-strain relations give the strain components 
\[ \varepsilon_{rr} = \varepsilon_{0 \theta} \]

\[ \varepsilon_{rr} = \frac{1}{E} \left[ \sigma_r - \nu (\sigma_\theta + \sigma_\phi) \right] \]
\[ \varepsilon_{\theta \theta} = \frac{1}{E} \left[ \sigma_\theta - \nu (\sigma_\phi + \sigma_r) \right] \]

... (5.31)

where \( E \) is Young's modulus, and \( \nu \), Poisson's ratio.

For spherical symmetry \( \sigma_\phi = \sigma_\theta \) and for incompressibility, Poisson's ratio \( \nu = \frac{1}{2} \), therefore from equations (5.31)

\[ \varepsilon_{rr} = \frac{\partial u}{\partial r} = \frac{1}{E} (\sigma_r - \sigma_\theta) \]

or

\[ E \frac{\partial u}{\partial r} = (\sigma_r - \sigma_\theta) . \]

... (5.32)

Also,

\[ \varepsilon_{\theta \theta} = \frac{u}{r} = \frac{1}{2E} (\sigma_\theta - \sigma_r) \gamma \]

\[ 2E \frac{u}{r} = \sigma_\theta - \sigma_r . \]

... (5.33)

We consider the non-homogeneity of the elastic parameter \( E \) be of the type

\[ E = E_0 \frac{r^n}{\alpha^n} \]

Therefore from (5.33) and (5.29) we get

\[ \sigma_\theta - \sigma_r = 2E \frac{u}{r} \]

\[ = 2E_0 \frac{r^n}{\alpha^n} \frac{\alpha^{n+3} - \alpha_0^{n+3}}{(n+3) \gamma^{n+3}} , \quad n \neq -3 . \]
substituting the above in equation (5.30) we get

\[ \frac{\partial \sigma_r}{\partial r} - \frac{4E_0}{(n+3)a_n^3} \frac{a^{n+3} - a_o^{n+3}}{r^4} = \frac{p_o}{a_n^3} \left\{ \begin{array}{l} a^{n+2} \ddot{a} \\ + (n+3)a^{n+1} \dot{a}^2 \end{array} \right\} / r^2 - (n+2) \frac{a^2(n+2)}{\gamma^{n+5}} \ddot{a}^2 \]

Integrating with respect to \( r \) we get

\[ \sigma_r + 4E_0 \frac{a^{n+3} - a_o^{n+3}}{3(n+3)a_o^3} r^{-3} = \frac{p_o}{a_n^3} \left\{ \begin{array}{l} a^{n+2} \ddot{a} \\ + (n+2)a^{n+1} \dot{a}^2 \end{array} \right\} \frac{r^{-1}}{-1} + \frac{(n+2)a^2(n+2)}{(n+4)\gamma^{n+4}} + f(t) . \]

The condition at infinity gives \( \sigma_r \to 0 \) as \( r \to \infty \).

Utilising this condition, we get \( f(t) = 0 \), where we suppose that \( n > -4 \).

We get therefore

\[ \sigma_r = -4E_0 \frac{(a^{n+3} - a_o^{n+3})}{3(n+3)a_o^3} \frac{1}{r^3} + \frac{p_o}{a_n^3} \left\{ \frac{(n+2)a^2(n+2)}{(n+4)\gamma^{n+4}} \ddot{a}^2 \right\} - \left\{ \begin{array}{l} a^{n+2} \ddot{a} \\ + (n+2)a^{n+1} \dot{a}^2 \end{array} \right\} / r^2 \right\} . \]

... (5.34)

Boundary condition at the surface of the cavity is

\[ \left( \sigma_r \right)_{r=a} = -p , \]

... (5.35)
where \( p \) is the cavity pressure. From (5.34) and boundary condition (5.35) we get

\[
-p = -\frac{4E_0}{3(n+3)a_0^3} \left( \frac{a^{n+3} - a_0^{n+3}}{a_0^{n+3}} \right) + \frac{\rho_0}{a_0^n} \left\{ \frac{(n+2)}{(n+4)} \frac{a^{n+2} \dot{a}^2}{a_0^{n+4}} \right\}.
\]

Multiplying both sides by \( a^2 \, da \) and integrating from \( a_0 \) to \( a \),

\[
\int_{a_0}^{a} p \, a^2 \, da = \frac{4E_0}{3(n+3)a_0^3} \int_{a_0}^{a} \left( \frac{a^{n+3} - a_0^{n+3}}{a_0^{n+3}} \right) \, da
\]

\[
+ \frac{\rho_0}{a_0^n} \left\{ \frac{(n+2)(n+3)}{n+4} \int_{a_0}^{a} a^{n+3} \dot{a}^2 \, da + \int_{a_0}^{a} \frac{a^{n+3} \ddot{a}}{a_0} \, da \right\}.
\]

Supposing \( p \) to be practically constant, say \( p_0 \), we get approximately

\[
p_0 \frac{a^3 - a_0^3}{3} = \frac{4E_0}{3(n+3)a_0^3} \left[ \frac{a^{n+3} - a_0^{n+3}}{n+3} - a_0^{n+3} \log \frac{a}{a_0} \right]
\]

\[
+ \frac{\rho_0}{a_0^n} \left\{ \frac{(n+2)(n+3)}{n+4} \int_{a_0}^{a} a^{n+2} \dot{a}^2 \, da + \int_{a_0}^{a} \frac{a^{n+3} \ddot{a}}{2} \, da \right\}.
\]
Retaining only up to second order terms in expansion of 
\( \log \left( \frac{a}{a_0} \right) \), the above gives

\[
\rho_0 \frac{a^3 - a_0^3}{3} = \frac{4 E_0}{3 (n+3) a_0^n} \left[ \frac{a^{n+3} - a_0^{n+3}}{n+3} - \frac{a_0^{n+3}}{n+3} \right] - \frac{1}{2} \left( \frac{a^{n+3} - a_0^{n+3}}{a_0^{n+3}} \right)^2 
+ \frac{\rho_0}{a_0^n} \left[ \frac{a^{n+3} - a_0^{n+3}}{2} + \frac{n (n+3)}{2 (n+4)} \int a^{n+2} \, d a \right],
\]

or

\[
\rho_0 \frac{a^3 - a_0^3}{3} = \frac{2 E_0 a_0^{n+3}}{3 (n+3)^2 a_0^n} \left[ \frac{a^{n+3} - a_0^{n+3}}{a_0^{n+3}} \right]^2 + \frac{\rho_0}{a_0^n} \frac{a_0^{n+3}}{2} \frac{a^2}{2} 
+ \frac{n (n+3) \rho_0}{2 (n+4) a_0^n} \int a^{n+2} \, d a.
\]

or

\[
\frac{\rho_0}{2} a_0^{n+3} \frac{a^2}{a_0^n} = \frac{\rho_0 (a^3 - a_0^3)}{3} - \frac{2 E_0}{3 (n+3)^2 a_0^{2n+3}} \left( a^{n+3} - a_0^{n+3} \right)^2 
- \frac{\rho_0 n (n+3)}{2 a_0^n (n+4)} \int a^{n+2} \, d a.
\]  \( \ldots (5.36) \)

Let us write the equation (5.36) in the form

\[
C_1 a^{n+3} f(a) = d_1 (a^3 - a_0^3) + d_2 (a^{n+3} - a_0^{n+3})^2 
+ C_2 \int_{a_0}^{a} t^{n+2} f(t) \, d t.
\]  \( \ldots (5.37) \)

where
\[ f(a) = a^2, \quad c_1 = \frac{f_0}{2a_0^n}, \quad d_1 = \frac{b_0}{3} \]

\[ c_2 = -\frac{f_0 n (n+3)}{2a_0^n (n+4)}, \quad d_2 = \frac{\alpha_0^2 n^3}{3(n+3)^2} \frac{a_0}{a^n+3}. \]

For the solution of (5.36) let us assume \( a^2 \) to be the function \( f(a) \) given by

\[ f(a) = \frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n}, \quad \ldots \ (5.38) \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are constants to be determined from equation (5.36)

Thus

\[ c_1 a \frac{n+2}{n+3} \left[ \frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n} \right] = d_1 (a^2 - a_0^2) \]

\[ + d_2 (a^{n+3} - a_0^{n+3})^2 + c_2 \int_{a_0}^{a} \frac{1}{t} \left( \frac{\alpha_1}{t} + \alpha_2 t^{n+3} + \alpha_3 + \frac{\alpha_4}{t^n} \right) dt. \]

which gives

\[ c_1 a^{n+3} \left[ \frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n} \right] = d_1 (a^2 - a_0^2) + d_2 (a^{n+3} - a_0^{n+3})^2 \]

\[ + c_2 \left[ \frac{\alpha_1}{n+2} - \frac{a^n}{n+2} + \frac{\alpha_2}{2n+6} - \frac{a_0^{n+6}}{2n+6} + \frac{\alpha_3}{n+3} - \frac{a_0^{n+3}}{n+3} \right] \]

\[ + \alpha_4 \frac{a^3 - a_0^3}{3}. \]
On the assumption that $3, n+3, n+2, 2n+6$ are all different, i.e., for all values of $n$ other than $0, -3, -2, -3/2$.

Equation coefficients of $a^3, a^{n+2}, a^{n+3}, a^{2n+6}$ we get

\[ c_i \alpha_i = d_i + \frac{c_i \alpha_i}{3} \rightarrow \alpha_i = \frac{2b_0 a_0^{n+3}(n+4)}{E_o (n^2 + 6n + 12)} \]

\[ c_1 \alpha_1 = \frac{c_2 \alpha_1}{n+2} \rightarrow \alpha_1 = 0 \]

\[ \alpha_2 = \frac{4E_o (n+4)}{3(n+3)^2 (n+2) f_0} \]

Therefore from (5.38) and (5.39) we get

\[ f(\alpha) = \alpha^2 = \frac{-8E_o (n+4)}{3f_0(n+3)^2(3n+8)} \cdot \frac{a^{n+3}}{a_0^{n+3}} + \frac{4E_o (n+4)}{3f_0(n+3)^2(n+2)} \]

\[ + \frac{2b_0 (n+4)}{f_0(n^2 + 6n + 12)} \cdot \frac{a^n}{a_0^n} \]

Or, finally,

\[ \frac{f_0 \alpha^2}{E_o} = \frac{4(n+4)}{3(n+3)^2(n+2)} + \frac{2b_0 (n+4)}{E_o (n^2 + 6n + 12)} \cdot \frac{a^n}{a_0^n} \]

\[ - \frac{8(n+4)}{3(n+3)^2(3n+8)} \cdot \frac{a^{n+3}}{a_0^{n+3}} \]
The values of $a^2$ for different $n$ have been evaluated and shown as follows:

For $n = 0.5$, $\frac{b_o}{E_o} = 0.5$

\begin{align*}
\frac{a}{a_o} &\rightarrow 1.02 \quad 1.04 \quad 1.06 \quad 1.08 \quad 1.10 \\
\int_0^\infty \frac{da}{E_o} &\rightarrow 0.377577 \quad 0.366983 \quad 0.356085 \quad 0.34487 \quad 0.333323
\end{align*}

$n = 0.75$, $\frac{b_o}{E_o} = 1.0$

\begin{align*}
\frac{a}{a_o} &\rightarrow 1.02 \quad 1.04 \quad 1.06 \quad 1.08 \quad 1.10 \\
\int_0^\infty \frac{da}{E_o} &\rightarrow 0.617688 \quad 0.602507 \quad 0.587401 \quad 0.572042 \quad 0.556505
\end{align*}

$n = 1.5$, $\frac{b_o}{E_o} = 2.0$

\begin{align*}
\frac{a}{a_o} &\rightarrow 1.02 \quad 1.04 \quad 1.06 \quad 1.08 \quad 1.10 \\
\int_0^\infty \frac{da}{E_o} &\rightarrow 0.958668 \quad 0.926516 \quad 0.895198 \quad 0.864616 \quad 0.834677
\end{align*}

$n = 2.0$, $\frac{b_o}{E_o} = 0.5$

\begin{align*}
\frac{a}{a_o} &\rightarrow 1.02 \quad 1.04 \quad 1.06 \quad 1.08 \quad 1.10 \\
\int_0^\infty \frac{da}{E_o} &\rightarrow 0.235492 \quad 0.222501 \quad 0.209537 \quad 0.196546 \quad 0.183472
\end{align*}

$n = 3.0$, $\frac{b_o}{E_o} = 2.5$

\begin{align*}
\frac{a}{a_o} &\rightarrow 1.02 \quad 1.04 \quad 1.06 \quad 1.08 \quad 1.10 \\
\int_0^\infty \frac{da}{E_o} &\rightarrow 0.863176 \quad 0.811075 \quad 0.762089 \quad 0.715864 \quad 0.672074
\end{align*}

Thus the velocity of expansion diminishes with expansion of the cavity.
V.3. ON THE DYNAMIC PROBLEM OF EXPANSION OF A SPHERICAL CAVITY IN A TRANSVERSELY ISOTROPIC MEDIUM

Introduction:

The problem of expansion of a spherical cavity inside a material is important for understanding of onset of plastic yielding and its spread with time in the medium. Problems of this type have been considered by Hopkins (1960), Morland (1969) and others. We consider here the problem of expansion of a spherical cavity in an incompressible transversely isotropic elastic medium. The rate of expansion of the cavity surface is obtained when the interval is on the point of yielding due to the pressure.

Basic Equations:

A spherical cavity situated in a transversely isotropic elastic medium is subjected to pressure. The resulting motion is purely radial. Taking $\rho$, $\theta$, $\phi$ the spherical polar coordinates with origin at the centre of the cavity, the displacement is radial and is written as $u(\rho, t)$. The velocity $v$ at a point is also radial and

$$v = \frac{\partial u}{\partial t}.$$  \hfill (5.40)

Since the problem considered is one of expansion, and there is a one-one correspondence between time and radius of the
cavity, one may consider the radius 'a' of the cavity to be
a single-valued function of time.

The stress-strain relations in a transversely elastic
medium are

\[
\begin{align*}
\sigma_r &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} + C_{12} \varepsilon_{\phi\phi} \\
\sigma_{\theta} &= C_{12} \varepsilon_{rr} + C_{22} \varepsilon_{\theta\theta} + C_{23} \varepsilon_{\phi\phi} \\
\sigma_{\phi} &= C_{12} \varepsilon_{rr} + C_{23} \varepsilon_{\theta\theta} + C_{22} \varepsilon_{\phi\phi} \\
\sigma_{r\theta} &= 2C_{44} \varepsilon_{r\theta} \\
\sigma_{r\phi} &= 2C_{44} \varepsilon_{r\phi} \\
\sigma_{\theta\phi} &= (C_{\text{II}} - C_{12}) \varepsilon_{\theta\phi}
\end{align*}
\]

where \( C_{11}, C_{12}, C_{22}, C_{23}, C_{44} \) are the elastic constants of
the medium.

The medium is incompressible, hence the equation of con-
servation of mass gives

\[
\frac{\partial}{\partial \varphi} (r^2 \psi) = 0 , \quad \ldots (5.42)
\]

at every point of the medium. In our problem

\[
\begin{align*}
\sigma_{r\theta} &= \sigma_{r\phi} = \sigma_{\phi\phi} = 0 , \\
\varepsilon_{rr} &= \frac{\partial u}{\partial r} , \\
\varepsilon_{\theta\theta} &= \varepsilon_{\phi\phi} = \frac{\partial \psi}{\partial \varphi} , \\
\varepsilon_{r\phi} &= \varepsilon_{\theta\phi} = \varepsilon_{\phi\theta} = 0 ,
\end{align*}
\]

and the only equations of motion not identically satisfied
reduces to
\[ \frac{\partial \sigma_r}{\partial r} + \frac{\rho}{r} (\sigma_r - \sigma_\phi) = \rho f_r, \quad \ldots \quad (5.43) \]

where \( \rho \) is the density and \( f_r \) is the radial acceleration.

**Solution:**

Equation (5.42) on integration gives

\[ r^2 v = \text{function of time only} \]

\[ = \text{value at the cavity surface} \]

\[ = \frac{a^2}{2} \frac{\partial a}{\partial t} \]

\[ v = \frac{\partial u}{\partial t} = \frac{a^2 \dot{a}}{r^2}. \quad \ldots (5.44) \]

Integrating (5.44) and neglecting second order terms in displacement we get approximately

\[ u = \frac{a^3 - a_o^3}{3r^2}, \quad \ldots (5.45) \]

where \( a_o \) is the initial radius of the cavity.

(This is justifiable on the assumption that displacement is small over a small interval of time although velocity is not small.) Therefore the strain components are

\[ \varepsilon_{rr} = -2 \frac{(a^3 - a_o^3)}{3r^3}, \]

\[ \varepsilon_{\phi\phi} = \frac{(a^3 - a_o^3)}{3r^3}. \quad \ldots (5.46) \]
The equation of motion (5.40) on substituting from (5.47) and (5.45) becomes

\[
\sigma_r - \sigma_\theta = -2 \left( c_{11} - c_{12} \right) \frac{(a^3 - a_o^3)}{3 \pi r^3}
\]

\[
- \left( c_{22} - 2 c_{12} + c_{23} \right) \frac{(a^3 - a_o^3)}{3 \pi r^3}
\]

\[
= -2 \left( c_{11} - 4 c_{12} + c_{22} + c_{23} \right) \frac{(a^3 - a_o^3)}{3 \pi r^3}.
\]...

(5.47)

Integrating with respect to 'r' we get

\[
\frac{2 \sigma_r}{\gamma r} + \frac{2}{\gamma^3} \frac{a^3 - a_o^3}{3 \pi r^3} \left( 2 c_{11} - 4 (c_{12} + c_{22} + c_{23}) \right)
\] = \[ p \left\{ \left( \frac{a^3}{\gamma r} + 2 a \dot{a}^2 \right) / r^3 - \frac{a^4}{2 \gamma^4} \right\} + f(t),
\]

where

\[
A = 2 c_{11} - 4 c_{12} + c_{22} + c_{23}
\]...

(5.48)

(5.49)

and \( f \) is an arbitrary function of \( t \).

However if the radial stress is taken to vanish at infinity, then \( f(t) = 0 \) and (5.48) becomes

\[
\sigma_r = - \frac{2}{q} A \frac{a^3 - a_o^3}{\gamma^3} - p \left\{ \left( \frac{a^3}{\gamma r} + 2 a \dot{a}^2 \right) / r^3 - \frac{a^4}{2 \gamma^4} \right\}.
\]...

(5.50)
Therefore the cavity pressure \( p \) is given by

\[
-p = (\sigma_r)_{r=a} 
\]

\[
-p = \frac{p_0}{qA^3} A\left(\frac{a_0^3 - a^3}{a^3} + \int (a\ddot{a} + \frac{3}{2} \dot{a}^2) \right) 
\]

\[
= \frac{p_0}{q} A \left(1 - \frac{a_0^3}{a^3}\right) + \int (a\ddot{a} + \frac{3}{2} \dot{a}^2) \right) \ldots \ldots (5.51)
\]

Equation (5.51) subject to initial condition \( a = a_0 \) and value of \( \dot{a} \) at \( t = 0 \) is \( v_0 \) determines \( a(t) \) throughout the elastic motion.

Therefore multiplying (5.50) by \( a^2 \, da \) and integrating from \( a_0 \) to \( a \), we get

\[
\int_{a_0}^{a} \frac{2}{q} A \int_{a_0}^{a} \left(1 - \frac{a_0^3}{a^3}\right) a^2 \, da + \int_{a_0}^{a} (a\ddot{a} + \frac{3}{2} \dot{a}^2) a^2 \, da 
\]

\[
= \frac{2}{q} A \left[ \frac{a_0^3 - a^3}{3} - \frac{a_0^3}{3} \log \frac{a_0^3}{a_0^3} \right] + \frac{1}{2} \int a^3 \ddot{a}^2 \right) \ldots \ldots (5.52)
\]

Supposing \( p \) to be practically constant \( p_0 \) and considering \( (a^3 - a_0^3)/a_0^3 \) to be small, we get approximately

\[
\frac{a^3 - a_0^3}{3} = \frac{2}{q} A \left[ \left(\frac{a_0^3 - a^3}{a_0^3} \right) - \frac{1}{2} \left(\frac{a_0^3 - a^3}{a_0^3} \right)^2 \right] \left(\frac{a_0^3}{3} \right)
\]

\[
+ \frac{a_0^3 - a_0^3}{3} \right] + \frac{1}{2} \int a^3 \ddot{a}^2 - \frac{1}{2} \int a_0^3 \dot{v}_0^2 \right),
\]
The cavity surface velocity $a$ is therefore determined.

Starting of yielding:

Hill (1950) has given a general form of yield condition for anisotropy. Accordingly we take the yield condition as

$$
2f_\ell (\sigma; j) = F (\sigma_j - \sigma_\phi)^2 + G (\sigma_\phi - \sigma_\rho)^2 + H (\sigma_\rho - \sigma_\phi)^2
$$

$$
+ L \sigma_\rho^2 + M \sigma_\phi^2 + N \sigma_\phi^2 = 1.
$$

where $F$, $G$, $H$, $L$, $M$, $N$ are parameters of the characteristic of the anisotropy. Therefore the yield condition reduces to

$$
\sigma_j - \sigma_\phi = \frac{1}{\sqrt{F + H}},
$$

$$
\left( c_{22} - 4 c_{12} + c_{23} \right) \frac{a^3}{\gamma^2} = \frac{2}{3} (c_{11} - c_{12}) \left( \frac{a^3}{\gamma^3} \right)
$$

$$
= \frac{1}{\sqrt{F + H}},
$$

$$
\left( c_{22} - 2 c_{12} + c_{23} \right) \frac{a^3}{\gamma^2} = \frac{2}{3} (c_{11} - c_{12}) \left( \frac{a^3}{\gamma^3} \right)
$$

$$
= \frac{1}{\sqrt{F + H}},
$$

$$
A \frac{a^3}{\gamma^3} = \frac{1}{\sqrt{F + H}},
$$

$$
A = c_{22} - 4 c_{12} + c_{23} + 2 c_{11}.
$$
Hence yielding first occurs at the cavity surface when the cavity radius is \( a_1 \) given by

\[
A \frac{a_1^3 - a_0^3}{3a_0^3} = \frac{1}{\sqrt{F + H}}
\]

\[
\frac{a_1}{a_0} = 1 + \frac{1}{A \sqrt{F + H}} \quad \ldots (5.54)
\]

The above expression does not depend on the cavity surface acceleration and therefore the result is valid for quasistatic case also.

Thus at the end of the completely elastic part of the motion, the cavity radius '\( a_1 \)' is given by (5.54).

Now from (5.53) we get

\[
\frac{1}{2} \rho \frac{A^3}{a_0^3} \dot{a}_1^2 = \frac{p_0}{3} \left( \frac{a_1^3 - 1}{a_0^3} \right) - \frac{A}{2 \gamma} \left( \frac{a_1^3 - 1}{a_0^3} \right)^2 + \frac{1}{2} f v_o^2
\]

or

\[
\frac{1}{2} f (1 + \frac{3}{A \sqrt{F + H}}) \dot{a}_1^2 = \frac{p_0}{A \sqrt{F + H}} - \frac{A}{3(A \sqrt{F + H})^2} + \frac{1}{2} f v_o^2
\]

or

\[
\int \dot{a}_1^2 = \frac{2}{A \sqrt{F + H}} \left( \frac{p_0}{3 \sqrt{F + H}} - \frac{1}{3 \sqrt{F + H}} \right) + f v_o^2
\]

provided \( p \) does not significantly diminish from its initial value \( p_0 \) during the expansion of the cavity. Thus the state of motion at the end of the purely elastic part of the expansion is determined.