CHAPTER II

STATICAL PROBLEMS OF PLASTIC YIELDING IN NON-HOMOGENEOUS MEDIA
II.1. ON ELASTIC-PLASTIC DEFORMATION OF NON-HOMOGENEOUS CYLINDRICAL AND SPHERICAL SHELLS

1. Introduction:

The problem of elasto-plastic deformation of spherical shells when the material of the shell is elastically and plastically non-homogeneous is of importance for applications in engineering designs. Olszak (1958) assumed the plastic region to start either from the inner surface or from the outer surface. Chakraborty and Rakshit (1972), Chaudhury (1975) and Mukhopadhyay (1982) considered some particular types of non-homogeneity. Chakraborty and Rakshit considered the possibility of starting of yielding at an interior surface in between the two boundary surfaces of the shell.

It is well known that in the case of homogeneous shells the plastic yielding starts at the inner surface. Depending on the nature of nonhomogeneity and on the plastic yielding functions which may contain parameters involving radius \( r \), it has been found that the starting of yielding is a complex phenomenon needing careful study and consideration.

2. Starting of yielding - General considerations:

The elastic solution corresponding to any radial non-homogeneity when a pressure \( P(P > 0) \) is applied at the
inner surface of a spherical shell gives rise to principal stresses $\sigma_\phi$, $\sigma_\theta$, $\sigma_\rho (= \sigma_\theta)$ which are functions of the radial distance $r$ from the centre of the shell.

We write

$$\sigma_\theta - \sigma_\rho = Pf(r). \quad \ldots \quad (2.1)$$

We suppose that $f(r) > 0$ throughout. The yield condition in general may be put in the form

$$\sigma_\theta - \sigma_\rho = K(r), \quad \ldots \quad (2.2)$$

where plastic nonhomogeneity implies that $K$ is a function of $r$ and is greater than zero. From (2.1) and (2.2) we see that the spherical shell remains in the elastic state so long as

$$Pf(r) < K(r) \quad \ldots \quad (2.3)$$

throughout the shell.

Now the condition (2.3) is valid for all $r$ in $(a < r < b)$ provided $P$ is less than the least value of $K(r)/f(r)$.

The zone of starting of yielding depends essentially on the nature of the function $K(r)/f(r)$. If it attains its lowest value either at the inner or at the outer surface of the shell, then yielding starts there and the
subsequent elastic-plastic problem is simple. If however it starts at an interior point, a plastic region may be formed in between two elastic regions.

In the case of long cylindrical shells under internal pressure, the stresses developed are \( \sigma_\rho \), \( \sigma_\theta \) and \( \sigma_z \left\{ = \gamma (\sigma_\rho + \sigma_\theta) \right\} \). Therefore the equations (2.1) and (2.2) are valid and similar considerations are true.

3. Elastic solution:

Stress-strain relations in the spherical and cylindrical shells may be written as

\[
\sigma_\rho = \frac{E(\gamma)}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \frac{du}{d\rho} + \nu \frac{\rho}{\rho} \right],
\]

\[
\sigma_\theta - \sigma_\rho = \frac{E(\gamma)}{1+\nu} \left[ \frac{u}{\rho} - \frac{du}{d\rho} \right],
\]

where \( u \) is the only non-zero component of displacement in the radial direction, \( \rho = 1, 2 \) corresponding to the cylindrical and spherical shells respectively.

The equation of equilibrium is

\[
\frac{d}{d\rho} \left( \frac{\sigma_\rho}{\rho} \right) + \frac{\rho}{\rho} (\sigma_\rho - \sigma_\theta) = 0
\]

Now from (2.4) and (2.5) we get

\[
\frac{d}{d\rho} \left[ \frac{E(\gamma)}{(1+\nu)(1-2\nu)} \left\{ (1-\nu) \frac{du}{d\rho} + \nu \frac{\rho}{\rho} \right\} \right] + \frac{\rho}{\rho} \frac{E(\gamma)}{1+\nu} \left\{ \frac{du}{d\rho} - \frac{u}{\rho} \right\} = 0
\]
Finally we get
\[
\frac{d^2 u}{dr^2} + \frac{p+m}{r} \frac{du}{dr} + \frac{b+\mu}{r^2} \left( \frac{p\mu}{1-\nu} - p \right) \frac{u}{r^2} = 0.
\]
For non-homogeneity we take \( E = E_0 r^m \)

Therefore the equation becomes
\[
\frac{d^2 u}{dr^2} + \frac{b+m}{r} \frac{du}{dr} + \left( \frac{p\mu}{1-\nu} - p \right) \frac{u}{r^2} = 0. \quad \ldots \ (2.6)
\]

The solution of the equation is
\[
U = A r^{\frac{\alpha-(b+m-1)}{2}} + B r^{\frac{(b+m-1)+\nu}{2}}, \quad \ldots \ (2.7)
\]

where
\[
\alpha^2 = \frac{(p+m-1)^2(1-\nu) - 4p\mu + 4b - 4\nu}{1-\nu}, \quad \ldots \ (2.8)
\]
and \( A, B \) are arbitrary constants.

Therefore
\[
\sigma_r = \frac{E_0}{(1+\nu)(1-\nu)} \left[ A \left( \frac{1-\nu}{2} \right) r^{\alpha-m-b+1} + B \nu \right] r^{\frac{\alpha+m-b-1}{2}}
\]
\[
+ B \left[ (1-\nu) \left( -\frac{\alpha-b+m-1}{2} + \nu \right) r^{\frac{\alpha-b+m-1}{2}} \right], \quad \ldots \ (2.9)
\]

\[
\sigma_\theta - \sigma_r = \frac{E_0 \rho^m}{1+\nu} \left\{ A \left( \frac{1-\alpha+b+m}{2} \right) r^{\frac{\alpha-b+m-1}{2}}
\]
\[
+ B \left( \frac{\alpha+b+m+1}{2} \right) r^{\frac{\alpha+m+b+1}{2}} \right\}, \quad \ldots \ (2.10)
\]
Boundary conditions are

\[ \sigma_\gamma = -p \quad \text{at} \quad \gamma = \alpha \]

\[ = 0 \quad \text{at} \quad \gamma = \beta . \]

Using boundary conditions we get the constants as

\[
\frac{E_0 A}{(1+\nu)(1-2\nu)} = \frac{pl_{-\alpha+\nu-b-1}}{[(1-\nu)(\frac{\alpha+m-b+1}{2})+p\nu]}.D
\]

\[
\frac{E_0 B}{(1+\nu)(1-2\nu)} = \frac{pl_{\alpha+\nu-b-1}}{[(1-\nu)(\frac{\alpha+m-b+1}{2})-p\nu]}.D
\]

where

\[
D = \begin{bmatrix}
\alpha -\frac{\alpha+\nu-b-1}{2} & \frac{\alpha+\nu-b-1}{2} \\
\frac{\alpha+\nu-b-1}{2} & \frac{\alpha+\nu-b-1}{2}
\end{bmatrix}
\]

\[
\sigma_\gamma = \frac{p}{D} \left[ \frac{-\alpha+\nu-b-1}{2} \gamma -\frac{\alpha+\nu-b-1}{2} - \frac{\alpha+\nu-b-1}{2} \gamma -\frac{\alpha+\nu-b-1}{2} \right] \ldots (2.12)
\]

\[
\sigma_\theta - \sigma_\gamma = \frac{1-2\nu}{(\frac{\beta}{\alpha})^{d+m-b-1}} \left[ \frac{-\alpha+\nu-b-1}{2} \frac{\alpha+\nu-b-1}{2} \frac{-\alpha+\nu-b-1}{2} \frac{-\alpha+\nu-b-1}{2} \frac{\alpha+\nu-b-1}{2} \frac{\alpha+\nu-b-1}{2} \frac{\alpha+\nu-b-1}{2} \frac{\alpha+\nu-b-1}{2} \right]
\]

\[
+ \frac{(\beta+m+\nu+1)(\nu)}{2[(1-\nu)(\frac{\alpha+m+b+1}{2})+p\nu]} = f(\gamma) \quad \text{say}
\]
4. Numerical Results:

Numerical values of \( \frac{\sigma_0 - \sigma_r}{P} \), i.e. \( f(r) \) in case of a spherical shell have been calculated and shown in table 1, for some particular values of \( \frac{b}{a} \), \( \alpha \) while \( m \) the elastic nonhomogeneity parameter has been taken to be \(-2, -1, 1, 2, 3, 4\) and \(5\).

<table>
<thead>
<tr>
<th>( \frac{x}{a} )</th>
<th>( \frac{\sigma_0 - \sigma_r}{P} )</th>
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</table>

The above calculations show that \( f(r) \) is greatest at the inner surfaces for the cases \( m = -2, -1, 1, 2, 3 \) while \( f(r) \) is least at the inner surface and greatest at the outer surface when \( m \) is \(4\) and \(5\). Hence assuming \( K(r) \) to be constant, we have from the general consideration in Art. 2, that the yielding will start at the outer surface for \( m = 4 \) or \(5\) and at the inner surface for the other values.
Particular cases of Non-homogeneity: We consider here several cases:

First case: Plastically homogeneous and elastically nonhomogeneous with yielding starting from the inner surface. Let \( \sigma_\theta - \sigma_\rho = \text{constant} = \kappa_0 \) (say) be the yield condition and yielding starts at the inner surface i.e. \( \sigma_\theta - \sigma_\rho \) must have least values at \( r = a \).

From (2.9) and (2.10) we get

\[
\sigma_\rho = \frac{E_0}{(1+\nu)(1-2\nu)} \left[ A \left( 1 - \nu \frac{(\alpha - m - p + 1)}{2} + p \nu \right) \gamma^\frac{\alpha + m - p - 1}{2} 
+ B \left( - \frac{(1-\nu)(\alpha - b - m - 1) + p \nu}{2} \right) \gamma^\frac{-\alpha - b + m - 1}{2} \right]
\]

\[
\sigma_\theta - \sigma_\rho = \frac{E_0}{1+\nu} \left[ A \left( \frac{1-\alpha + p + m}{2} \right) \gamma^\frac{\alpha - b + m + 1}{2} 
+ B \left( \frac{\alpha + p + m + 1}{2} \right) \gamma^\frac{-\alpha - b + m - 1}{2} \right]
\]

Here \( a < r < c \) is the plastic region and \( c < r < b \) is the elastic region of the elasto-plastic cylinder or sphere.

Now for yielding at \( r = c \), boundary conditions are

\[
(\sigma_\rho)_{r=c} = 0
\]

and

\[
\sigma_\theta - \sigma_\rho = \kappa_0 \ \text{at} \ \ r = c
\]
From this we get

\[
A = \frac{2(1+\nu)k_0\frac{E}{2}\left\{ (1-\nu)(1-\nu) + 2\nu \right\}}{E_0\left[\frac{E}{2}\left\{ (1-\nu)(1-\nu) + 2\nu \right\} - \frac{E}{2}\left\{ (1-\nu)(1-\nu) + 2\nu \right\}(1+\nu+m)}{A}
\]

\[
B = -\frac{2E^{\alpha}(1+\nu)k_0E^{\frac{\alpha-1}{2}}}{E_0\left[\frac{E}{2}\left\{ (1-\nu)(1-\nu) + 2\nu \right\} - \frac{E}{2}\left\{ (1-\nu)(1-\nu) + 2\nu \right\}(1+\nu+m)}{A}
\]

\[
\sigma_P = \frac{k_0E^{\frac{\alpha-1}{2}}}{(1-\nu)(1-\nu)}\left\{ (1-\nu)(1-\nu) + 2\nu \right\}\left[ \frac{E^{\frac{\alpha-1}{2}}}{2} - \frac{E^{\frac{\alpha-1}{2}}}{2}\right]
\]

In the plastic region using the yield condition in the equation of equilibrium

\[
\frac{d\sigma_p}{d\gamma} - \frac{p}{\gamma}k_0 = 0
\]

and on integrating and applying the boundary condition

\[
(\sigma_P)_{\gamma=0} = -p, \quad \text{we have}
\]

\[
\sigma_P = -p + \frac{p}{k_0} \log \frac{\gamma}{\alpha}
\]

The condition of continuity of stress at the plastic-elastic boundary \( r = c \) gives the equation
\[ Pk_o \log \frac{c}{\alpha} = P + \frac{k_o}{1-2\nu} \cdot \frac{L}{M} \cdot \left\{ 1 - \left( \frac{r}{b} \right)^4 \right\} \]

where
\[ L = \left[ (1-\nu)(-\alpha - b - m - 1) + 2P \right] \left[ (1-\nu)(\alpha - b - m + 1) + 2P \right], \]

\[ M = (1-\alpha + b + m) \left[ (1-\nu)(-\alpha - b - m - 1) + 2P \right] \left( \frac{l}{c_0} \right)^{(\alpha + b + m + 1)} \left[ (1-\nu)(\alpha - b - m + 1) + 2P \right], \]

which determines \( c \) for a given \( P \).

**Second case**: Plastically homogeneous and elastically non-homogeneous with yielding starting at the outer surface. In this case yielding starts at the outer surface and the yield condition is

\[ \sigma_{\theta} - \sigma_{\rho} = k_o \text{(constant)}. \]

Here \( \sigma_{\theta} - \sigma_{\rho} \) must have least value at \( r = b \).

From (2.9) and (2.10) we get

\[ \sigma_r = \frac{E_o}{(1+\nu)(1-2\nu)} \left[ A \left\{ (1-\nu)(\alpha - m - 1) + 2P \right\} \gamma^{\frac{d + m - b - 1}{2}} + B \left\{ (1-\nu)(-\alpha - b - m - 1) + 2P \right\} \gamma^{-\frac{\alpha - b + m - 1}{2}} \right], \]

\[ \sigma_{\theta} - \sigma_{\rho} = \frac{E_o \gamma^m}{1+\nu} \left[ A \left( \frac{1-x+p+m}{2} \right) \gamma^{\frac{d + b - m - 1}{2}} + B \left( \frac{1+d+b+m}{2} \right) \gamma^{-(\frac{d+m+b}{2})} \right]. \]
Here \( c \lesssim r \lesssim b \) is the plastic region while \( a \lesssim r \lesssim c \) is the elastic region of the elasto plastic cylinder or sphere. Now for yielding at \( r = c \), the boundary conditions are

\[
(\sigma_r)_r = -P
\]

\[
\sigma_r - \sigma_\theta = k_0 \text{ at } r = c
\]

The constants are

\[
A = -\frac{(1+\nu)[(1-\nu)(\frac{-\alpha-b-p+1}{2})+b\nu]k_0\alpha^{\frac{\alpha-b-p+1}{2}} + P(1-2\nu)(\frac{\alpha+b+m+1}{2})e^{\frac{\alpha-b-m+1}{2}}}{E_0 \cdot D_1}
\]

\[
B = \frac{(1+\nu)[(1-\nu)(\frac{-\alpha-m-b+1}{2})+b\nu]k_0\alpha^{\frac{\alpha-m-b+1}{2}} + P(1-2\nu)(\frac{1-\alpha+p+m}{2})e^{\frac{\alpha-p+m-1}{2}}}{E_0 \cdot D_1}
\]

where

\[
D_1 = \left\{ (1-\nu)(\frac{-\alpha-m-b+1}{2})+b\nu \right\} \left( \frac{\alpha+m+b+1}{2} \right) \alpha^{\frac{\alpha+m+b+1}{2}} \cdot e^{\frac{\alpha-b-m+1}{2}}
\]

\[
- \left\{ (1-\nu)(\frac{-\alpha-m-b+1}{2})+b\nu \right\} \left( 1-\alpha+p+m \right) \alpha^{\frac{\alpha-b-m+1}{2}} \cdot e^{\frac{\alpha-p+m-1}{2}}
\]

\[
\sigma_r = \frac{1}{(1-2\nu)D_1} \left[ \left( (1-\nu)(\frac{\alpha+b+m+1}{2}) + b\nu \right) k_0 \alpha^{\frac{\alpha-b-m+1}{2}}
\right.
\]

\[
- P(1-2\nu)(\frac{\alpha+b+m+1}{2})e^{\frac{\alpha-b-m+1}{2}} \left[ \frac{1}{(1-2\nu)D_1} \right] \left( \frac{\alpha+b+m+1}{2} \right) \cdot e^{\frac{\alpha-b-m+1}{2}}
\]

\[
+ k_0 \left( (1-\nu)(\frac{\alpha-b-m+1}{2}) + b\nu \right) \alpha^{\frac{\alpha-b-m+1}{2}} \left( \frac{\alpha-b-m+1}{2} \right) \cdot e^{\frac{\alpha-b-m+1}{2}}
\].
Therefore in the plastic region, applying the boundary condition

\[
(\sigma_\nu)_{\nu=0} = 0
\]

and integrating the equation of equilibrium we get the solution as

\[
\sigma_\nu = P \cdot k_0 \cdot \log \frac{\nu}{\alpha}
\]

Therefore by the condition of continuity of stress at the plastic-elastic boundary \( r = c \) we get

\[
P \cdot k_0 \cdot \log \frac{c}{b} = \frac{1}{(1-2\nu) \cdot D_1} \left\{ \left[ \left( \frac{1-\nu}{2} \right) \left( \frac{\alpha + \beta + m + 1}{2} + \nu \right) \right] k_0 \cdot \alpha^{\frac{\alpha - \beta + m - 1}{2}} - P \left( \frac{1-\nu}{2} \right) \left( \frac{\alpha + \beta + m + 1}{2} \right) e^{-\frac{\alpha - \beta + m - 1}{2}} \right\} \cdot c^{\frac{\alpha + m - b - 1}{2}}
\]

\[
+ \left[ P \left( \frac{1-\nu}{2} \right) e^{-\frac{\alpha - \beta + m + 1}{2}} + k_0 \left( \frac{1-\nu}{2} \left( \frac{\alpha - \beta + m + 1}{2} + \nu \right) \right) \right] e^{-\frac{\alpha - \beta + m - 1}{2}} \cdot \frac{\alpha}{2}
\]

which determines \( c \) for a given \( P \).

Third case: Plastically and elastically nonhomogeneous:

Let the yield condition be nonhomogeneous and of the form

\[
\sigma_\epsilon - \sigma_\nu = k_0 \left( \frac{\nu}{\alpha} \right)^2
\]
Assuming the parameter \( m \) to be such that \( f(r) \) has greatest value at the inner surface we conclude that yielding starts at the inner surface.

For solution of the elasto-plastic problem we consider the region \( a < r < c \) plastic, while \( c < r < b \) is elastic.

In the plastic zone \( a < r < c \) the equation of equilibrium reduces to

\[
\frac{d\sigma_r}{dr} - \frac{p}{r} k_0 \left( \frac{r}{a} \right)^2 = 0.
\]

On integration and on applying boundary condition

\[
(\sigma_r)_{r=a} = -p,
\]

the stress

\[
\sigma_r = \frac{p k_0}{2} \left( \frac{r^2}{a^2} - 1 \right) - p.
\]

The solution in the elastic zone \( c < r < b \) is obtained from the solution of the 1st case.

\[
\sigma_r = \frac{k_0 \left( \frac{r}{a} \right)^2 \epsilon_{a+b-m} \left\{ (1-v)(-\alpha-p-m-1+2\nu) \right\} \left[ (1-v)(\alpha-p-m+1+2\nu) \right] \left[ \frac{p^{a+m-b+1}}{2} - \epsilon \frac{\rho^a-m+b-1}{2} \right]}{(1-2\nu) \left[ (1-\alpha+p+m) \left[ (1-v)(-\alpha+m-p-1+2\nu) \right] - \epsilon \left[ (1-v)(\alpha-p+m+1+2\nu) \right] \right]}
\]

when \( c < r < b \).
By the condition of continuity of stress we get

\[
\frac{\chi K_0}{2} \left( \frac{c^2}{a^2} - 1 \right) - P = \frac{K_0}{(1-2\nu)} \cdot \frac{L_1}{M_1} \cdot \left[ 1 - \left( \frac{b}{c} \right)^m \right]
\]

where

\[
L_1 = \left[ (1-\nu)(-\alpha + \beta + m - 1) + 2\nu \right] \left[ (1-\nu)(\alpha + \beta - m + 1) + 2\nu \right]
\]

\[
M_1 = (1-\alpha + \beta + m) \left[ (1-\nu)(-\alpha + \beta + m - 1) + 2\nu \right] - \left( \frac{b}{c} \right)^m \left[ (1-\nu)(\alpha + \beta - m + 1) + 2\nu \right] (\alpha + \beta + m)
\]

which determines \( c \) for a given \( P \).
II.2. EFFECT OF OVER ELASTICITY ON THE YIELD CONDITION OF A SPHERICAL SHELL

1. Introduction:

Experiments on test specimens by Campus, F. (1963), and Maiden, C. and Campbell, J.D. (1958) and others have shown that the yield condition depends not only on the stress components at a point, but also on their first order partial derivatives as well. This property exhibited in problems of non-constant states of stresses is known as 'over-elasticity' in which the yield limit is increased if the state of stress is spatially nonhomogeneous. This is not surprising since the transition of a material into a plastic state is connected with a change of its structure and the notion of structure is related to a finite region and not a single point. Actually Prandtl (1925) took the fact of increase of yield limit while presenting the model of ideally plastic uniaxial stress-strain model. König and Olszak (1974) investigated the form of the yield criterion in the general case of nonhomogeneous stress and deformation fields, assuming the yield condition to depend on stress, strain, first order partial derivatives of stress and strain components with respect to space and time coordinates. They have considered a modification of Mises' yield condition and have applied the same to the case of pure bending of a rectangular beam.
In this paper we have considered the effect of a particular type of 'over-elasticity' on the yield point in the case of a spherical shell. It is found that over-elasticity does not alter the physical situation i.e. the yielding starts from the inner surface.

2. Elastic solution:

Let \((r, \theta, \phi)\) be the spherical polar coordinates referred to the centre of the spherical shell as origin. Let 'a' and 'b' be the inner and outer radius respectively of the spherical shell. The strain components for a purely radial displacement \(u(r)\) are

\[
\varepsilon_{rr} = \frac{\partial u}{\partial r}, \\
\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u}{r}, \\
\varepsilon_{r\theta} = \varepsilon_{r\phi} = \varepsilon_{\theta\phi} = 0
\]  \hspace{1cm} (2.13)

The elastic stress-strain relations for the non-zero stresses are

\[
\sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left[ (1-\nu) \frac{\partial u}{\partial r} + 2\nu \frac{u}{r} \right] \\
\sigma_{\theta} = \sigma_{\phi} = \frac{E}{(1+\nu)(1-2\nu)} \left[ \frac{u}{r} + \nu \frac{\partial u}{\partial r} \right]
\]  \hspace{1cm} (2.14)

where \(\sigma_r\), \(\sigma_{\theta}\) and \(\sigma_{\phi}\) are stress components. The equation of equilibrium to be satisfied for purely radial displacement is
\[ \frac{d \sigma_p}{d \tau} + \frac{z}{r^2} (\sigma_r - \sigma_0) = 0. \quad \ldots \ (2.15) \]

The boundary conditions are

\[
(\sigma_p)_{\tau = a} = -p, \quad (\sigma_r)_{\tau = b} = 0, \quad \ldots \ (2.16)
\]

where \( p \) is the pressure on the inner surface, the outer surface being stress free.

From equations (2.13), (2.14), (2.15) and (2.16) we get the solutions in the elastic region are

\[ \sigma_p = p \left( \frac{\alpha^2}{a^2} - \frac{\alpha^2}{b^2} \right), \quad \sigma_\theta = p \left( \frac{\alpha^2}{a^2} + \frac{\alpha^2}{b^2} \right). \quad \ldots \ (2.17) \]

3. **Yield condition**

We take yield condition in the invariant form as

\[ \frac{1}{2} \sigma_{ij} \sigma_{ij} = k^2 + (k^2 - k') F(\theta), \quad \ldots \ (2.18) \]

with \( 0 < F(\theta) < 1 \),
where $\sigma_{ij}$ is the deviatoric stress tensor and $\sigma_{ij,jk}$ is the covariant derivative of $\sigma_{ij}$ with respect to $x_k$ and $F(\phi)$ is a function of the scalar

$$\Theta = \sigma_{ij,jk} \sigma_{ij,jk} \phi^2,$$

depending on the space derivatives of the stresses.

The dependence on the stress derivatives are such that the values of $\frac{1}{2}\sigma_{ij} \sigma_{ij}$ lies between $K^2$ and $K^2$, where $K^2 - K^2$ is determined from experiments. Using spherical polar coordinates and utilising the fact that the stresses are independent of $\phi$ and $x$, we get

$$\Theta = \frac{1}{2} \left( \frac{\partial S}{\partial R} \right)^2 - \frac{2}{R} \frac{\partial S}{\partial R} + 3 \frac{S^2}{R^2} + S^2 + \frac{R^2 S^2}{2}, \quad \cdots (2.19)$$

where $S$ stands for $\sigma_{rr} - \sigma_\theta$ and $R$ is the dimensionless radial distance $R = \frac{r}{\alpha}$. Using the solution in § 2, we get from (2.17)

$$\begin{align*}
S &= -\frac{2b}{M^2 R^2} \phi, \\
\frac{\partial S}{\partial R} &= \frac{4b}{M^2 R^3} \phi, \\
\end{align*} \quad \cdots (2.20)$$

where $M^2 = 1 - \frac{\alpha^2}{c^2}$.

Using the value of $S$ from (2.20) in (2.19) we get

$$\Theta = \frac{b^2}{M^4 R^6} \left[ 30 + 4R^2 + R^4 \right].$$
We take the form of \( F(\phi) \) as
\[
F(\phi) = 1 - e^{-\frac{\phi}{k^2}},
\]
which is evidently positive and is less than 1.

The yield condition (2.18) thus finally takes the form
\[
\frac{1}{4}(\sigma_R - \sigma_0)^2 + (k^2 - k^2) e^{-\frac{\phi}{k^2}} = k^2,
\]
which on simplification after using (2.20), gives
\[
\frac{p^2}{k^2 M^2 R^2} + \left(\frac{k^2}{k^2} - \frac{k^2}{k^2}\right) e^{-\frac{\phi}{k^2}} = \frac{k^2}{k^2},
\]
\[
\frac{p^2}{k^2} \left(\frac{k^2}{k^2}\right) \frac{1}{M^2 R^2} + \left(\frac{k^2}{k^2} - 1\right) e^{-\frac{\phi}{k^2}} - \frac{k^2}{k^2} = 0. \quad \ldots (2.21)
\]
Equation (2.21) gives the condition of yielding at any point \((R > 1)\). For an interpretation of this condition, following numerical calculations have been made.

4. Numerical results and discussion:

On taking the yield parameter \(\frac{K^2}{k^2} = 1.05\), the value of \(\frac{p^2}{K^2}\) from (2.21) are tabulated in next page.
It is seen from the numerical results that the pressure for yielding is minimum at \( r = a \) i.e. the yielding starts at the inner surface and the critical pressure \( p^* \) for this to happen is given by

\[
\frac{p^*}{k^2} \left( \frac{k^2}{r^2} - 1 \right) \frac{1}{M_4} + \left( \frac{k^2}{r^2} - 1 \right) e^{-\frac{\theta^*}{k^2}} - \frac{k^2}{r^2} = 0,
\]

where \( \theta^* = 3.5 \frac{b^*}{M_4} \).

It is to be noted that the term \( e^{-\frac{\theta^*}{k^2}} \) is very small compared to the first term. Also the critical pressure in the absence of over-elasticity is obtained from (2.21) by putting \( K = k \). It is seen that due to over-elasticity the minimum pressure for starting of yielding is more than that in the absence of over-elasticity.

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