ABSTRACT

While working on differentiation theory Denjoy (1915) introduced the notion of density points for a measurable set. Half a century later Goffman and Waterman (1961) introduced the density topology (abbreviated d.T.) on $\mathbb{R}^n$. In fact, defining a topology on a measure space in a considerable abstract approach can be attributed to a much earlier work of Haupt and Pauc (1952). Thereafter works on this topology on Euclidean and general measure spaces appeared in some works of Goffman et al (1961), Martin (1964), Troyer and Ziemer (1963) etc.


Our work consists of selecting some areas for the study of density topologies mainly in measure-theoretic aspects. This includes the aspects of density topological methods, characterisations for different behaviours and properties and also the study of the implications of these topologies in function theories.

Objectives of the study:

Our initial effort is to systematize the knowledge of density topologies in selected areas which are scattered haphazardly in different corners of topology and analysis. We
adm at their study and to add new facts or throw new lights as a part of development and the extension of many key results.

It is also the purpose of this track to compile, organise and to interprete critically to assess the facts known to date in these selected areas and present them in a compact form in which it is not seen so far to be included in any work. We expect it to open avenues for new researches.

Chapterisation

Before entering into the exposition of our principal object we wish to keep the introductory chapter (chapter 0) exclusively for a preliminary introduction to the background, related references, results and definitions from dispersed literature and need to recall while going through the body of the thesis. The importance of the role played by different differentiation bases and the density base operators in determination and shaping the d-topological characters are also discussed here.

The principal goal of chapter 1 is to study the aspects of general construction of density topologies along with their abstract properties. Global characterisation of different aspects of abstract density topologies are investigated here and many results concerning both measure and categorical aspects are obtained. The latest tool and technique of base operator methods is found fruitful in obtaining the topological facts.

In many aspects of density topological methods fine
topological modifications discussed in chapter II are found highly useful even though they do not belong to the nucleus of our study promised in the title. They are, in fact, indispensable tools in most of the d-topological treatments. We have compiled them here to study their crucial role and to show in density topological treatments.

Chapter-III is primarily concerned with some aspects of density topological properties. In addition to deriving some results on some general and special properties (e.g. countable chain condition, quasi-Lindelof, Blumberg etc.) here we have characterized a d-space in terms of its derivatives, isolated points, category measurable spaces etc. One of our primary results is that -- 'if there exists a locally finite Borel measure $\mu$ on a quasi-Lindelof completely regular $T_2$ space $(X,\tau)$, then under certain conditions, $(X,\tau)$ is not a Borel subset of its stone-Cech compactification.'

In Chapter IV we assert characterisations on abstract density topologies in terms of their pre-topological structures. We prove some assertions concerning their refinements induced by liftings. It is shown that density topologies having the Lusin-Menchoff properties can be expressed as a weak topology generated by a family of semi-continuous functions. Properties like normality, connectedness etc. are characterised in terms of a fine topology.

Chapter V deals with characterisation of various real-valued functions on d-topological spaces. These include semi-continuous functions, measurable functions, functions
having the Baire property along with their inter-relations with respect to a density space. Beginning with the proof of a category analogue of Lusin's theorem to hold good for abstract density space, we extend and prove a result of Sion and that one of Goffman et al. on approximate continuity.

An analysis of the results on different continuities and derivatives on ordinary $d$-topologies, $I$-density topologies and deep-$I$-density topologies from recent literature has been included in the Appendix-1 as a complementary to this chapter.

In Chapter VI we motivate to concentrate on exploring the situations under which $d$-topological methods play crucial roles in solving some age old difficult problems in real analysis. We have shown here how these methods are used to prove the existence theorem in lifting theory, their implications in solving the problems of construction (e.g. the construction of nowhere monotone differentiable functions) and characterisations of many functions in the theories of the differentiation and of the integral. As for example, it has been asserted that the use of Lusin-Menchoff property of the $d$-topology and an application of the generalised form of the Stone-Weirstrass property can remove the natural conjecture as whether any approximately continuous function $f$ can be uniformly approximated by the differences of lower-semi-continuous and approximately continuous functions.

In the last chapter VII we describe our experiences and feelings during our work when we encountered some
interesting open problems. A miscellany of problems has been described here with the aim of stimulating further investigation and research in the field of our interest. Also here we express our limitations, laxities and preventions observed during the execution of this work.

At the end, this discourse contains a bibliography which is closely related to our research enterprise.