APPENDIX AND BIBLIOGRAPHY
APPENDIX-I
RECENT LITERATURE ON d-FUNCTION THEORY

In this section we present in brief a schematic interpretation of outstanding facts dispersed in recent standard literature in d-function theories dealing on density continuity and derivatives. Some other facts that help characterising real functions on the density topology are also presented.

Unless otherwise stated, all functions treated here are assumed to be the real valued functions of a real variable.

A.I.O. INTRODUCTION: For any function \( f: \mathbb{R} \to \mathbb{R} \), four types of continuity can be discussed with two topologies \( D \) and \( T_u \) on \( \mathbb{R} \) such as

(i) \( D - T_u \) continuity (ii) \( D-D \) continuity
(iii) \( T_u - D \) continuity (iv) \( T_u - T_u \) continuity

The corresponding classes for these functions will be denoted by \( \mathcal{C}_{DU}, \mathcal{C}_D, \mathcal{C}_{UD} \) and \( \mathcal{C} \) respectively. The structure of the class \( \mathcal{C} \) is well-known and that of the \( \mathcal{C}_{DU} \) is well-understood which has been discussed already under the caption of approximate continuity. \( \mathcal{C}_{UD} \) is only the class of constant functions. Thus we are here interested only with \( \mathcal{C}_D \) which is termed as the class of density - continuous (D-D continuous) functions.

A.1.1. DENSITY CONTINUOUS FUNCTIONS:

Definition: A function \( f: \mathbb{R} \to \mathbb{R} \) is said to be
D-continuous if it is continuous with respect to the density topology D both on domain and on its range. (The term 'd-continuous' or 'd-continuity' will also be used to mean 'density continuous' or 'density continuity' where the topology D need not be in picture.

As $\mathcal{D} \supset \mathcal{T}_0$, the relation $\mathcal{D} \subset \mathcal{D}_U \supset \mathcal{C}$ is evident.

It is known that [CL-6] the class of all locally convex functions and the class of all real analytic functions are d-continuous [CL].

Every d-continuous function is app.c. and thus belong to the Baire Class I, and has the Darboux property. If $M \subset \Sigma$, then $f^{-1}(M) = U \cup V$ where $U$ is a D-open set and $V$ is a D-closed sets in $(\mathbb{R}, D)$. So $f^{-1}(M) \in \Sigma$ for each D-continuous function. The converse of this is not true. On the other hand, none of the classes of usual continuous functions and the d-continuous functions are inclusive in one another. In fact it is not difficult to verify that $\mathcal{C} \not\subset \mathcal{D} \not\supset \mathcal{C}$. Excellent examples for all these remarks are available in [Ost] where also many other characterisations for being d-continuous function are deduced.

A.1.2. Investigating the local behaviour of density continuous functions Ostaszewski [Ost] has shown that if $f: A \to \mathbb{R}$ (A is any interval in $\mathbb{R}$) preserves upper outer density at $x$ (this means if for every set $E \subset \mathbb{R}$, $\bar{d}(E, x) > 0 \Rightarrow \bar{d}(f(E), f(x)) > 0$), then $f$ is d-continuous at $x$. If $f: A \to \mathbb{R}$ be d-continuous and $x \in A$ then $f$ preserves upper outer density at $x$ iff there exists a continuous function $\tilde{f}$ which is not d-continuous and such that
A consequence of the above is the fact that a 1-1 function is \(d\)-continuous iff it preserves the upper outer density.

A.1.3. The category theorems concerning the classes of continuous and density continuous functions are seen in some recent literature. In [CLO-1] it has been shown that the subclass \(\mathcal{C}_D'\) of \(\mathcal{C}\), the set of all continuous functions \(f:[0,1]\to\mathbb{R}\) which have at least one point of density continuity is a first category subset of \(\mathcal{C}\). From this and the fact that every \(d\)-continuous function is continuous as a dense open set, it can be seen that if \(\mathcal{C}_D\) be given the topology of uniform convergence, then it is a first category subset of itself. Similarly if \(\mathcal{H}\) denotes the space of all automorphisms of \([0,1]\) with the metric \(\rho(g,h) = ||g-h|| + ||g^{-1} - h^{-1}||\) and \(\mathcal{H}_D \subseteq \mathcal{H}\) such that if \(f \in \mathcal{H}_D\) and \(f\) has at least one point of density continuity, then \(\mathcal{H}_D\) is of 1-category. Moreover a subset \(A \subseteq \mathbb{R}\) is the set of points of discontinuity of a density continuous function \(f\) (respectively an app.c. function \(f\)) iff \(A\) is a nowhere dense (resp. a 1-category) \(F_\sigma\) set.

It has been observed in [Ost-2] that the class \(\mathcal{C}_D\) is not closed under uniform convergence. His query whether polynomials are \(D\)-continuous was answered affirmatively in [CL-6] by showing that real analytic functions are \(d\)-continuous (Cor-3, p.293). Another result there is that \(\mathcal{C}_D\) is not closed under pointwise addition by showing that there exists a \(d\)-continuous function \(f: \mathbb{R} \to \mathbb{R}\) such that \(f(x) + x\) is not \(d\)-continuous. A lot of interesting results are also given
in [CL-4] such as that the level sets of d-continuous functions are precisely the density closed sets which are in \( F_\sigma \cap G_\delta \) (for definitions etc cf. [CL-4]).

**A.1.4. I-DENSITY CONTINUOUS FUNCTIONS**

**Definition:** A function \( f: \mathbb{R} \to \mathbb{R} \) is said to be **I-density continuous** (deep-I-density continuous) if it is continuous with respect to I-density (deep-I-Density) topology on both the domain and the range.

A continuous function from \(( \mathbb{R}, T_I) \) to \(( \mathbb{R}, T_u) \) will simply be called as **I-continuous**.

We denote the class of all I-density (deep-I-density) continuous functions as \( \mathcal{C}_I \) (\( \mathcal{C}_d \))

If \( J, K \) represent any of the four topologies \( T_u, T_I, T_d, \) and \( D \) (ref Ch-2) on \( \mathbb{R} \), then we define \( \mathcal{E}_{JK} \) as the class of all continuous functions from \(( \mathbb{R}, J) \) to \(( \mathbb{R}, K) \). Then we have sixteen different classes on varying \( J, K \) in both domain and range. Thus \( \mathcal{C}_{II} = \mathcal{C}_I = \text{class of I-density continuous functions} \)

\( \mathcal{C}_{T_d T_I} = \mathcal{C}_d = \text{class of deep-I-density continuous functions} \)

\( \mathcal{C}_{T_u T_u} = \mathcal{C} = \text{Class of ordinary continuous functions} \)

\( \mathcal{C}_{DT_u} = \mathcal{C}_{DU} = \text{class of app.c.functions} \)

\( \mathcal{C}_{DD} = \mathcal{C}_D = \text{Class of density continuous functions} \)

\( \mathcal{C}_{T_I T_u} = \mathcal{C}_{IU} = \text{Class of I-app.c functions or I-continuous functions}. \)

Except the above six and the classes \( \mathcal{C}_{T_d T_u} \) and \( \mathcal{C}_{T_I T_d} \) (to be denoted by \( \mathcal{E}_{dU} \) and \( \mathcal{E}_{dI} \) respectively) all other classes
are trivial and coincide with the classes of constant functions \((\mathcal{LCL}-l\])\). However \(\mathcal{C}_d \cup \mathcal{C}_d\) coincides with \(\mathcal{C}_I \cup \mathcal{C}_d\) and \(\mathcal{C}_d\) coincides with \(\mathcal{C}_I\).

For every \(f: \mathbb{R} \to \mathbb{R}\), \(f \in \mathcal{C}_d\) iff \(gof \in \mathcal{C}_I \cup \mathcal{C}_d\) for every \(g \in \mathcal{C}_I\). From this it can be seen that every \(I\)-density continuous function is deep-\(I\)-density continuous. Thus \(\mathcal{C}_I \subseteq \mathcal{C}_d \subseteq \mathcal{C}_I \cup \mathcal{C}_d\).

In \([\text{CL}]\) it has been shown that if \(f: \mathbb{R} \to \mathbb{R}\) be such that \(f^{-1}(E) \subseteq \mathcal{I}_I\), for all \(E \in \mathcal{I}_I\), where \(\mathcal{I}_I\) is the ideal of \(1\)-category subsets of \(\mathbb{R}\), then \(f\) is deep-\(I\)-density continuous iff \(f\) is an \(I\)-density continuous function.

Linear (piecewise linear) functions are density as well as \(I\)-density continuous. However there exists a continuous, density continuous and \(I\)-density continuous function which does not belong to \(\mathcal{C}_\infty\), or there exists a density continuous and \(I\)-density continuous function which is not continuous. On the other hand there is a deep-\(I\)-density continuous function \(f\) which is not \(I\)-density continuous such that \(f \in \mathcal{C}_d \cap \mathcal{C}_\infty\), and there is an \(f \in (\mathcal{C}_d \cap \mathcal{C}_\infty) \setminus \mathcal{C}_d\). Including these many other examples and counterexamples are given in \([\text{CL}-l]\) in support of more relations among the above classes (cf. below).

Firstly we have that \(\mathcal{C}_d \subseteq \mathcal{C}_I \cup \mathcal{C}_d\) (we note \(\mathcal{T}_U \subseteq \mathcal{T}_d \subseteq \mathcal{T}_I\)).

The following table summarizes some interrelations by inclusions among the important classes of functions. We denote by \(P \to Q\) if \(P \subseteq Q\). For different classes \(\mathcal{C}_i, \mathcal{C}_j\) we denote \(\mathcal{C}_i \cap \mathcal{C}_j\) by \(\mathcal{C}_i \cdot \mathcal{C}_j\).

Thus \(\mathcal{C}_d \cdot \mathcal{C}_i\) means the set of all ordinary continuous functions which are deep-\(I\)-density continuous.
From table I it can be observed that $I$-density continuous and deep-$I$-density continuous functions bear the same relationship to the ordinary continuous functions as do the density continuous functions. In particular,
\[ c_I \neq c \quad \text{and} \quad c_I \subseteq c \quad \text{implies} \quad c_d \neq c \]

To argue for the above inclusion charts Ciesielski and Larson ([CL-1]) have produced examples of functions belonging to the following sets.

\[
\begin{align*}
&c_I \subseteq c_{DU}, \quad c_D \subseteq c_{IU}, \quad c_I \subseteq c_D, \quad c_D \subseteq c_D \\
&c_I \subseteq c_{DU}, \quad c_D \subseteq c_{IU}, \quad c_I \subseteq c_D, \quad c_D \subseteq c_D \\
&c_I \subseteq c_{DU}, \quad c_D \subseteq c_{IU}, \quad c_I \subseteq c_D, \quad c_D \subseteq c_D \\
&c_I \subseteq c_{DU}, \quad c_D \subseteq c_{IU}, \quad c_I \subseteq c_D, \quad c_D \subseteq c_D \\
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&c_I \subseteq c_{DU}, \quad c_D \subseteq c_{IU}, \quad c_I \subseteq c_D, \quad c_D \subseteq c_D \\
\end{align*}
\]

A1.5. It is natural to enquire if the known properties of density continuous functions have their identicals with those of I-density or deep-I-density continuous functions. Results of such efforts can be seen in many works such as [CLO], [CL-3], [CL-4], [Ci-1], [BLJ], [Laz], [PWW] etc.

Study of I-density continuous transformation through the preservation of I-density points can be observed in [AW], [Nie], [Ci] etc. Aversa and Wilczynski ([AW]) have shown that if h is a homeomorphism such that h and \( h^{-1} \) fulfill a local Lipschitz condition then h and \( h^{-1} \) preserve density and I-density points and they are density and I-density continuous (also see [CL-1]). Denoting by \( H_D \) and \( H_I \) for the classes of increasing homeomorphisms that preserve density and I-density points respectively and denoting by \( h^{-1} \) and \( h^{-1} \) for their inverses, it has been shown ([Ci], [Nie], [AW]) that the classes \( H_D \) and \( H_I \) that are density and I-density continuous respectively are closed under addition. Counterexamples proved that none of the above assertions is true if the condition of 'increasing homeomorphism' is relaxed. A homeomorphism preserves deep-I-density iff it preserves I-density ([Wil-1]).
A.1.6. In discussing some category theorems analogous to those of density continuity (5.2.) it is found in [CL-3] that if $C'_I$ be the class of all ordinary continuous functions $f: [0,1] \to \mathbb{R}$ which have at least one point of $I$-density continuity, then $C'_I$ is a 1-category subset of $C'[0,1] = \{ f: [0,1] \to \mathbb{R} : f \text{ is continuous} \}$.

A function $f: \mathbb{R} \to \mathbb{R}$ is in the class Baire* 1 if for each non-empty perfect set $P$ there is a portion $Q$ of $P$ such that $f|_Q$ is continuous ([Om-2]). It is clear that any function $f$ belonging to Baire*-1 class must be continuous at each point of a dense open set. The class $C_{d}$ of constant functions is a proper subset of the Baire*-1 class ([Ci-1]). Same is true for the class $C_d$. The spaces $C'_I$ and $C_d$ equipped with the topology of uniform convergence are of the 1-category in themselves ([CL-3]).

A.1.7. I-APPROXIMATE CONTINUITY:

Definition-1: A function $f: \mathbb{R} \to \mathbb{R}$ is I-app.c iff $f$ is a continuous function from $(\mathbb{R}, T^1)$ to $(\mathbb{R}, T_u)$.

Definition-2: A function $f: \mathbb{R} \to \mathbb{R}$ having the Baire property is said to be I-approximately continuous at $\alpha$ iff $\forall \epsilon > 0$, the set $f^{-1}((f(\alpha) - \epsilon, f(\alpha) + \epsilon))$ has as an $I$-density point.

$f: \mathbb{R} \to \mathbb{R}$ is I-app.c. iff in open interval $(x, y)$ the set $f^{-1}(x, y) T_1$, or iff $f$ is I-app.c. at every point ([PWW]). This implies that with respect to $T_1$ the I-app.c. functions are continuous. It is known that the classes of I-app.c. functions $C_{IU}$ and deep-I-approximately continuous functions $C_{dU}$ coincide ([Laz]) Th-2). However, the coarsest topology making all I-app.c. functions continuous is not $T_1$ but is $T_d$ ([Laz]), which is C.R.
Every sequence \( \{g_n\} \) of I-app.c. (app.c) functions converges uniformly to an I-app.c. (app.c) function (\([CL-1]\)). If a function \( f: \mathbb{R} \to \mathbb{R} \) is I-app.c., then \( f \in B_1 \) and has the Darboux property. In \([WA]\) we can find an example of an I-app.c. function having a perfect nowhere dense set as the set of points of discontinuity (w.r.t. natural topology).

Further properties of I-app.c. continuity (i.e. I-continuity and deep-I-continuity) can be seen in \([PWW]\), \([WA]\), \([Wil-1]\), \([CLO]\) etc. For example \( C_1 \) is closed under pointwise addition, multiplication and uniform convergence and hence the bounded I-app.c. functions form a Banach space. There is an I-app.c. function which is not a derivative. Using a result that there is no connection between the notions of I-density and ordinary density, a comparison between the class \( A_1 \) with other classes of real functions have been shown in \([AW]\). If \( A_1 \) be the set of points of I-app.c. functions then neither \( A_1 \subset A \) nor \( A \subset A_1 \).

Some results of approximate continuity (separately app.continuity/ strong app.continuity) for I and deep-I-density topology on a plane set are investigated in \([RW]\), \([BLW]\), \([Wil-1]\).

A.1.8. Let \( D_1 \) be the family of functions whose sets of points of discontinuity belong to I. Then \( B_1 (C) \subset D_1 \) and also

\[ C_1 \subseteq A_1 \subseteq D_1 (C) \ (\text{\([PWW]\)}) \].

Therefore \( C_1 \cap D_1 = C_1 = A_1 \). The condition necessary for a function \( f \) to belong to \( A_1 (A_1 \cap D_1) \) or \( B_1 (C_1) \) was investigated in \([BL]\). \( B_1 (C_1) \) properly contains the Baire Class-1 and is contained in the Baire Class-2 (\([BL]\)).
Moreover $\mathcal{B}_1(\mathcal{A}_1)=\mathcal{B}_2(\mathcal{E})$ ([Pr-1]). A comprehensive study on the Baire classes generated by the family of all I-continuous functions, app.c. and a.e. continuous functions is made in [Laz-2] by E. Lazarow. Among others she has shown

**Theorem:** For each $n \in \mathbb{N} \cup \{0\}$, $\mathcal{B}_n(\mathcal{E}) \subsetneq \mathcal{B}_n(\mathcal{E}_{1U}) \subsetneq \mathcal{B}_{n+1}(\mathcal{E})$ and $\mathcal{B}_{\omega_1}(\mathcal{E}) = \mathcal{B}_{\omega_1}(\mathcal{E}_{1U})$ where $\omega_1$ is the first limit ordinal. $\mathcal{B}_0(\mathcal{X})=\mathcal{X}$ and for any ordinal $\alpha (0<\alpha<\omega_1)$ $\mathcal{B}_\alpha(\mathcal{X})$ is the family of all limits of pointwise convergent sequences with terms taken from $\bigcup_{\gamma<\alpha} \mathcal{B}_\gamma(\mathcal{X})$ ($\omega_1$ = first uncountable ordinal).

A.1.9. I-APPROXIMATE DERIVATIVE

The category analogue of the approximate derivative which is called as I-approximate derivative (I-app. derivative) is investigated in [LW], [Laz-1], [LL], [CL-2], [CL-2] etc. First we give some definitions (cf.[LW] or [Laz-1]).

**Definition:** Let $F$ be any finite function defined in some neighbourhood of $x_o$ and having there the Baire property. Let

$$C(x,x_o) = \frac{F(x) - F(x_o)}{x - x_o} \text{ for } x \neq x_o.$$  

We define the I-approximate upper derivative $(F_{I-ap}^{-1}(x_o))$ as the greatest lower bound of the set

$$\{\alpha : \{x: C(x,x_o) > \alpha \} \text{ has } x_o \text{ as an } I\text{-dispersion point}\}.$$  

The I-approximate lower derivatives $(F_{I-ap}^{'}(x_o))$ are defined similarly. If these two derivatives are equal, their common value is called the I-approximate derivative of $F$ at $x_o$ $(F_{I-ap}'(x_o)).$

The classical Peano curve is I-density and
deep-I-density continuous but it is nowhere approximately and I-app. differentiable. Also the well-known fact that every bounded app.c. function is a derivative is not true for the bounded I-app.c. function. A counterexample for this is given in [CL-2].

The notion of deep-I-app. differentiability of a function $f$ can be introduced through the notion of a deep-I-density point of a set. However it can be proved that for the I-app.c. functions the notions of I-app. and deep-I-app differentiability of a point coincide. If a function $f: \mathbb{R} \to \mathbb{R}$ is I-app.d. at $x$ then $f$ is I-app.c. at $x$. I-app. differentiable I-almost everywhere functions (for definition see [CLO-2]) are Baire. Other important properties and the interrelations among various classes of I-app. differentiable functions can be seen in [CLO-2]. In this paper the classes of I-density continuous and deep-I-density continuous functions are considered as semigroup with composition as operation. Using the fact that the notions of I-app. derivatives, app. derivatives and ordinary derivatives coincide on the class of all homeomorphisms, Ciesielski, Larson and Ostaszewski have proved that the groups of automorphisms of these semigroups and several of their subsemigroups have their inner automorphism property.

If $f$ has an I-app. derivative at all $x \in [0,1]$, then there exists a selection $S$ such that $f$ has a selective derivative (cf.[Om-3]) at all $x \in [0,1]$. Using the notion of balanced selective derivatives ([Om-4]) Mrs. Lazarow proved that every finite I-app. derivative is of Baire Class-1. Some of the I-app. derivatives are of Baire Class-2. Important characterisations
of I-app. differentiable functions and I-approximate derivatives are contained in the following theorem.

**Theorem:** (P.21, [LW]): Let \( F: [0,1] \to \mathbb{R} \) have a finite I-app. der \( F'_{I-\text{ap}}(x) \), for all \( x \in [0,1] \). Then

a) there is a sequence of closed sets \( E_n \), whose union is \([0,1]\) such that \( F|_{E_n} \) is continuous for each \( n \).

b) \( F \) has the Darboux property.

c) there is a dense open set \( U \) on which \( F \) is continuous.

d) the I-app. derivative has the Darboux property and is of Baire class-2.

e) the set of points of continuity of \( F'_{I-\text{ap}}(x) \) is dense in \([0,1]\) and \( F \) is differentiable at any point of continuity of the I-approximate derivative.

f) \( F \) is differentiable for I-almost all \( x \in [0,1] \).

g) \( F \) has an app. der. \( F'_{\text{ap}}(x) \) for almost all \( x \in [0,1] \).

Many other interesting results analogous to those of ordinary derivatives found in [Br] such as the M.V. theorem can be found in [LW]. Some of these describe the relations between the I-approximate differentiation and monotonicity of the function \( F \). If \( F: [0,1] \to \mathbb{R} \) is Baire-1 and Darboux, and \( F'_{I-\text{ap}}(x) \) exists except on a denumerable set such that \( F'_{I-\text{ap}}(x) > 0 \) a.e., then \( F \) is non-decreasing and continuous on \([0,1]\), (Th-10,[LW]).

The notion of the symmetric I-app. derivative of a function \( f: \mathbb{R} \to \mathbb{R} \) is being introduced only recently in [LL]. Let us define
Definition: Let \( f: [a,b] \rightarrow \mathbb{R} \) have the Baire property. The upper symmetric I-app. derivative \( f_{I-ap}^S \) of \( f \) at a point \( c \in [a,b] \) is defined as the g.l.b. of the numbers \( \lambda \in \mathbb{R} \cup \{+\infty\} \) for which the set \( \{ t: \frac{f(c+t) - f(c-t)}{2t} < \lambda \} \) has \( 0 \) as a point of \( I \)-density.

Similarly the lower symmetric I-app. derivative is defined. When they are equal, their common value is denoted by \( f_{I-ap}^S \) at \( c \), is termed as symmetric I-app. derivative of \( f \) at \( c \)
(at the end points \( a \) and \( b \) only ordinary derivatives need to be considered).

For an I-app.c. function \( F \), the derivatives \( f_{I-ap}^S \) and \( f_{I-ap}^S \) have the property of Baire. They belong to the third class of Baire if \( f \) is a continuous function, \( f \) is equivalent to its respective ordinary I-app. derivative if \( f \) is monotone on \( (a,b) \). Thus if \( f \) is monotone and symmetrically I-app.d. function on \( [a,b] \), then \( f \) is a symmetrically differentiable function.