CHAPTER V I

IMPLICATIONS OF DENSITY TOPOLOGY IN REAL ANALYSIS
CHAPTER - VI
IMPLICATIONS OF DENSITY TOPOLOGY IN REAL ANALYSIS

6.0 INTRODUCTION

Although density topology has been introduced during fifties its methods similar to those of the identical fine topologies of the forties could be traced back in applications many decades ago. Some of such examples are Zahorski's theorem, approximation theorems, Lusin-Menchoff properties, Urysohn's Lemma, construction of functions of Pompeiu or Kopcke type including their extensions to unbounded derivatives and so on. On the other hand we have already noticed that this topology is serving as a nice example of a Tychonoff co-compact pseudo complete Baire space which does not have the Blumberg's property or which is also not a generated space.

Our aim in this chapter is to mention a few examples of such implications including the applicability of d.T. in real function theory. More such can also be noted in chapter-III, V. Appendix and in our paper [Gos-2].

6.1. SOLUTIONS OF LIFTING PROBLEMS

Existence of a lifting for a particular space has significant demand in a wide domain of applications such as in the problems of disintegration of measures, in integral operators, in ergodic theories or in Stochastic processes and so on.

However it remained as an unsolved problem as to whether or not there exists a lifting for an arbitrary algebra of bounded functions on a measure space.
We see below how the density topology is used to prove the existence theorem in lifting theory.

The problem whether there exists a lifting of the Banach Algebra $B$ of all bounded Lebesgue measurable functions corresponding to $\mathbb{R}$ was solved first by J. Von Neumann in 1931. In 1958 D. Maharam established that for an arbitrary space of $\sigma$-finite measure the lifting of the algebra $B$ of all bounded measurable functions always exists. Her proof [Ma] was followed by another independent proof by A and C Tulcea in [Tul-1]. Its variation by constructing a lifting from a lower density using the abstract density topology for the first time was given by both Tulceas in their elegant book 'Topics in the theory of lifting'. (Ch.V, p-57, [Tul]).

Another useful and simple solution for the above lifting problem so-called as 'Von Neumann theorem' was given by Scheinberg [Sc] by constructing a topology $U$ on $\mathbb{R}$ finer than $D_0$ (cf.2.5.6). With this topology Scheinberg showed that

**Theorem:** If $f$ is a bounded measurable function then there exists a unique $(U-T_u)$ continuous function $\hat{f}$ such that $f = \hat{f}$ a.e. where $T_u$ is the natural topology on $\mathbb{R}$.

This result plays a pivotal role in his proof. In brief it runs as follows:

Let $B$ be as described in the beginning with norm $||f|| = \sup_{x} |f(x)|. N$ is the space of all bounded functions a.e. equal to zero as a closed ideal in $B$. $L^\infty$ is the quotient algebra $B/N$ with norm $||f + N|| = \inf_{\varphi \in N} ||f + \varphi||$. The lifting problem is to find, if one exists, a lifting from $L^\infty$ to
B. Scheinberg proved it by using the above theorem by which he could select a map for every function in $L^\infty$ such that the algebraic operations and limits are consistent with this selection (cf. [Sc] for details).

6.2. IMPLICATIONS IN DIFFERENTIATION THEORY

Use of the density topology methods such as the well-known Lusin-Menchoff ([L.M]) property of the density topology (Ch-IV) can be seen in many of the constructions and characterisations of functions in real function theory such as approximately continuous and differentiable functions of peculiar types or the boundary behaviour of functions.

6.2.1. In providing an existence theorem of app.c. functions on $\mathbb{R}^n$ Zahorski in his monumental work ([Z]) proved the following important theorem (cf. also 5.2.2.)

Theorem-1: If $H_1', H_2 \subset [0,1]$ are $G_\delta$ and density closed sets and $H_1 \cap H_2 = \emptyset$, then there exists a function $f \in \mathcal{A}$ such that $f(x) = 0$ if $x \in H_1$; $f(x) = 1$, if $x \in H_2$, and $0 < f(x) < 1$ if $x \notin H_1 \cup H_2$.

Based on this result Petruska and Laczkovitch [PL-1] under the density topological consideration established that the necessary and sufficient condition for any Baire-1 function restricted to a subset $H$ of $[0,1]$ to have its extension to an app.c. function is that $\lambda(H) = 0$. In fact they have proved:

Theorem-2: The following conditions are equivalent:

a) $\lambda(H) = 0$  
(b) $b_{\mathcal{A}_H} = b_{\mathcal{B}_H}$  
(c) $\mathcal{A}_H = \mathcal{B}_H$.

The analogous result for the extension of $f|_H$ to a derivative on $[0,1]$ also has the same n.a.s.c., i.e., $\lambda(H) = 0$, ...
Theorem-3: For $H=[0,1]$, $\mathcal{D}_H=\mathcal{P}_H'$ iff $\lambda(H)=0$

Among many other important auxiliary results the following is an application of these extension theorems which can be considered as an analogue of the Tietze extension theorem.

Theorem-4: If $H \subseteq [0,1]$ is perfect, then $\mathcal{D}(H)=\mathcal{P}_H$.

The above result in Theorem-2 has been obtained by Pasquale Vetro [Pas] in $\mathbb{R}^n$ by using an insertion function $F$ from an in-between theorem (P.417, Th-1, [Pas]). He has defined:

Definition: Let $\phi: \mathbb{R}^n \to I(\mathbb{R})$ (the class of real intervals) and $f: \mathbb{R}^n \to \mathbb{R}$. Then $f$ is called a selection for $\phi$ if $f(x) \in \phi(x)$, $\forall x \in \mathbb{R}^n$.

Theorem-5: The function $\phi: \mathbb{R}^n \to I(\mathbb{R})$ admits an ordinarily app.c. selection iff it has the property of ordinarily app. continuity (5.2.2.) on $\mathbb{R}^n$.

6.2.2. Preservation of points of density under homeomorphism was studied by A. Bruckner [Br-2].

Bruckner proved that if $g$ is a homeomorphism of $[0,1]$ onto itself, then a necessary and sufficient condition for $f \circ g$ to be app.c. for every app.c. real function $f$ defined on $[0,1]$ is that $h=g^{-1}$ preserves density points. In [Wi] U. Wileżyńska has extended this result to the case of functions of several variables. In fact this property of a function on the d-topological spaces has been taken as a preamble in the study of density continuity (i.e. D-D continuity) by Ostaszewski in [Ost] and [Ost-1].
Niewiarowski [Nie] gives an example of a homeomorphism $h$ such that $h$ is $d$-continuous but $h'$ is not, (for definitions of $d$-continuity etc. cf. Appendix). In [Ost], it is shown that the set $Y$ of all $d$-continuous $1$-$1$ functions $f:[0,1] \to [0,1]$ is a first category proper subset of the set $\mathcal{H}$ of all homeomorphisms $h:[0,1] \to [0,1]$ which is a complete metric space ([Ox]), p.50). Every function $h \in \mathcal{H}$ is also not absolutely continuous.

Let $H$ be the class of all homeomorphisms $h: \mathbb{R} \to \mathbb{R}$ (with the $d$-$T.$) whose inverses are density continuous. Considering $H$ as a semigroup with composition as operation it has been shown in [Ost-2] that the semigroup and three of its subgroups have the inner automorphism property. $\mathcal{C}_D$ is not a vector space [CL-6], but it is a Lattice [CL-4]. If $\mathcal{C}^\infty$ is the set of all functions $f: \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable at every point, then there exist functions in $\mathcal{C}^\infty$ which are not $d$-continuous [CL-6]. However unlike the app.c. functions the $d$-continuous functions are in the class Baire*-1 ([CLO-1]).

6.2.3. In the theory of integrals, the importance of the following condition(N) was recognised by N.Lusin, ([Saks]).

Let $I \subset \mathbb{R}$ be an interval. A function $f:I \to \mathbb{R}^n$ is said to satisfy the Lusin condition(N) if $f$ maps every set of Lebesgue measure zero onto the set $f(E)$ of measure zero.

We note that by virtue of Th-1 (1.1.3) the above definition remains valid if we let the domain set$(E)$ or the range set $(f(E))$ to belong to any member of the four classes, viz, the null sets, nowhere dense sets, $1$-category sets or the
closed and discrete sets. Any monotone d-continuous function is continuous and satisfies the Lusin (N) condition.

In response to a query (No.1 in Real Anal. Exch. 1(1), (1976) 63) J. Maly ([Mal]) produced a counter-example to show that the first co-ordinate of the classical Peano curve is a density continuous function which does not satisfy the Lusin condition (N). However for a 1-1 d-continuous function $f$, both $f$ and $f^{-1}$ have the Lusin condition (N). A homeomorphism $h$ in standard topology is continuous a.e. iff $h^{-1}$ is absolutely continuous. It is shown in [Ost-1] that:

**Proposition:** Let $f:[a,b] \rightarrow \mathbb{R}$ and let there exists an onto function $g: f([a,b]) \rightarrow [a,b]$ such that

(i) $g$ is d-continuous a.e.

(ii) $g \circ f$ is an identity function,

Then $f$ satisfies the Lusin condition (N).

6.2.4. **APPROXIMATE DERIVATIVES**

A measurable function $f$ is said to have a finite approximate derivative $k$ at a point $x_0 \in \mathbb{R}$, if $\forall \varepsilon > 0$, the set $\{ x : \left| \frac{f(x) - f(x_0)}{x-x_0} - k \right| < \varepsilon \}$ has density 1 at $x_0$.

Every approximately differentiable (app.d) function is app.c., and is of Baire* 1. The differentiable functions with respect to density topology are precisely the app.d. functions. Unfortunately unlike approximate continuity, d.T. is not the coarsest one making the app.d. functions continuous. However it is the r-topology which is the coarsest and also the coarsest one for which the two subclasses of Baire-1 functions
i.e. the Baire* 1 app.c. functions and the ambivalent app.c. functions are both continuous (A function f is ambivalent if ∀ C, the sets \( \{ x: f(x) > C \} \) and \( \{ x: f(x) < C \} \) are both \( F_\sigma \) and \( G_\delta \), ([O'm-1])). O'Malley has also shown that another class \( \mathcal{F} \) of functions namely the 'app.c. everywhere and continuous a.e.' is contained in the class of r-continuous functions. However in this case the topology which is the coarsest one making every \( f \in \mathcal{F} \) continuous is not the r but the a.e.topology (\( C_{\text{h-11}} \)) that O'Malley introduced. It is proved that

**Theorem:** Let \( U \) be an ambivalent d-open set with \([0,1]) - U \neq \emptyset \). \( X_0 \) is any closed subset of \( U \). Then there exists a function \( g \) such that

1. \( g(x) \) is USC and \( g(x) = 1, \forall x \in X_0 \).
2. \( g(x) \) is approximately differentiable, \( \forall x \in U \)
3. \( g \) is differentiable with derivative zero, \( \forall x \in [0,1] - U \).

Thus \( g \) is app.d. ([O'm-1]). By virtue of Lusin-Menchoff property of the r-topology, it can be easily seen that any r-continuous function can be uniformly approximated by functions from the class of all app.d. functions.

**6.3. SOLUTION OF UNIFORM APPROXIMATION THEOREMS OF CONTINUITY**

It is known that a real function \( f \) on \( R \) is a Baire one iff \( f \) is a uniform limit of differences of lower-semi-continuous (l.s.c.) functions. A similar result ([LMZ]) for an app.c. function is given as:

**Theorem-1:** Let \( f \) be a bounded app.c. function on \( R \). Then \( f \) can be uniformly approximated by differences of non-negative app.c. and l.s.c. functions.
Let $\mathcal{L}$ be the class of all differences of l.s.c. functions on $\mathbb{R}$ and $\mathcal{A}_c$ be the class of all sums of absolutely convergent series of continuous functions. Sierpinsky in 1921 showed that $\mathcal{L} = \mathcal{A}_c$. In the same year, using transfinite induction S. Mazurkiewicz proved that any bounded Baire 1 function can be uniformly approximated by functions of $\mathcal{L}$. Later in another paper S. Kempisty [Kem] proved along with some other interesting results that a function belonging to $\mathcal{L}$ can be always inserted between a pair of unequal Baire one functions. This result built up the most significant theorem called the uniform approximation theorem of functions of Baire class one ($B_1$) by functions from $\mathcal{L}$. Sierpinski (1924) used this theorem to prove that $\mathcal{L} = B_1$. Later on, F. Hausdorff (1919), Tucker (1968), Mauldin (1974) etc. presented independently some other abstract and more generalised forms of this uniform approximation theorem.

Any of these above results however could not remove the natural conjecture as whether any app.c. function $f$ can be uniformly approximated by the differences of l.s.c. and app.c. functions. It was answered in 1985 by S. Vaněček [V] for the case of bounded functions that the approximation is true. His method of proof consisted of the use of L-M property of the density topology and the application of the generalised form of the Stone-Weirstrass property.

A more general version of the approximation theorem for continuous Baire-1 functions by avoiding boundness and some other restrictions is proved independently by Lukes, Maly and Zajicek [LMZ] in the following theorem by using density
topology methods - (cf. also Th-1).

Theorem-2: Let $T$ be a fine topology (including $d$-topology) on a metric space $(X, \mathcal{T})$ having the L-M property. Then any real Baire one and $T$-continuous function on $X$ can be uniformly approximated by the differences of two positive $T$-continuous l.s.c. functions.

6.4. PROBLEM OF CONSTRUCTION OF FUNCTIONS

In a Baire space (for example, completely metrizable space, Locally compact Hausdorff space or density topological spaces etc) the Baire's category theorem helps to identify the existence of functions of peculiar type. However construction of such functions with desirable properties for proving the existence is not so easy and even some are the problems of century old. Construction of a function which is continuous and nowhere differentiable appeared in 1875 by Weierstrass (Example-1) and in 1916 by Hardy (Example-2 below).

1) \[ f(x) = \sum_{k=0}^{\infty} b^k \cos (a^2 \pi x); \text{ a is odd and } 0 < b < 1. \]

2) \[ g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin (n^2 \pi x). \]

However construction of a nowhere monotone differentiable function appeared for the first time in 1887 by Kopcke and then in 1915 by Denjoy. Later a number of constructive proofs of the main problem are seen in literatures (cf.[KS], [BM]) and all of them are complicated in nature and also at the same time not free of errors. However, based on the
Pompeiu* construction of a strictly increasing differentiable function whose derivative vanishes on a dense set, Weil [Weil] had forwarded a lucid proof using Baire's category theorem on a Banach space $B$ of bounded derivatives on $[0,1]$ with sup. norm. He observed that for each interval $I \subset [0,1]$ the subset $B_2 = \{f \in B_1: f \geq 0 \text{ on } I\}$ of the complete space.

$B_1 = \{f \in B: f = 0 \text{ on a dense set}\}$ is nowhere dense in $B^1$. He constructed a pair of differentiable functions $f_1$ and $f_2$ belonging to $B_1$ such that the sets 

\[ \{x : f_1(x) > 0 \text{ and } f_2(x) = 0\} \text{ and } \{x : f_2(x) > 0 \text{ and } f_1(x) = 0\} \]

are dense. Then the function

\[
 f(x) = \int_0^x (f_1(t) - f_2(t)) \, dt.
\]

is the function which is nowhere monotonic differentiable.

Still however, looking for such a pair $(f_1, f_2)$ as above is highly cumbersome. Recent use of density topology by Goffman has supplied an easy proof of this problem (see also [Br-3]). He used the L-M property of the density topology to show that every $(D-T^*)$ continuous function on $\mathbb{R}$ is app.c. [G-3] he defined two sequences $(f_n)$ and $(g_n)$ of app.c. functions on two disjoint countable dense subsets $A$ of $\langle a_1 \rangle$ and $B$ of $\langle b_1 \rangle$ such that

(i) $0 \leq f_n \leq 1$: $f_n(a_n) = 1$: $f_n(b_n) = 0$ and

(ii) $0 \leq g_n \leq 1$: $g_n(b_n) = 1$: $g_n(a_n) = 0$

---

*Definition: Functions of Pompeiu type: Let $f$ be a real valued function on an open interval $(a,b)$ which has a bounded derivative. If $\{x: f'(x) = 0\}$ and $\{|x: f'(x) \neq 0\}$ are both dense in $(a,b)$, then $f$ is said to be of Pompeiu type.

If $\{x: f''(x) > 0\}$ or $\{|x: f''(x) < 0\}$ is dense in $(a,b)$ then the Pompeiu function $f$ defined above is said to be of Koppcke type.*
Then the bounded app.c. function given by

\[ f = \sum_{n=1}^{\infty} \frac{1}{2^n} (f_n - g_n) \]

is nowhere monotonic differentiable.

**Remarks:** It appears that d-topology is the natural tool with which we can verify the approximate behaviour of functions.

The Denjoy's Theorem mentioned earlier gives us that certain behaviour of differentiable functions can be obtained by those of app.c. functions with specified behaviour. The Zahorski property of density topology also offers one of the possible ways of constructing such functions. Use of such method of topological construction has already been seen in the works of Petruska and Laczkovich [PL-1].

A history of nowhere differentiable monotone functions is described by Bruckner and Leonard in "Derivatives" (Amer. M. Monthly, 73 (1966)).