CHAPTER V

CHARACTERISATION OF REAL FUNCTIONS ON
$
\mathbb{R}$ - TOPOLOGICAL SPACES
CHAPTER V
CHARACTERISATION OF REAL FUNCTIONS ON d-TOPOLOGICAL SPACES*

5.0 INTRODUCTION

Till now we were dwelling upon the aspects of density topological methods, characterisations for different aspects of behaviour and properties of d-topological spaces. Although inexhaustively, all of these aspects were treated not only merely as an essential causes for the investigation of d.T. itself, but also in the behavioural aspects of the d-topologies for which our objectives were laid to make the d-topological methods applicable to the theory of real functions.

In this chapter our chief aim is to characterise some real functions on the density topologies and thereby implicate the role of density topologies in real function theory.

All functions mentioned here are real-valued. The symbols and notations used here are already enlisted in table "List of symbols and notation" given at the beginning".

5.1. GENERAL CHARACTERISATION OF FUNCTIONS ON d-SPACE

In this section we study the relation between continuity, measurability, and functions having the property of Baire with respect to a density topology.

An earlier version of this chapter was lectured by me at the "Theory of real functions semester "at the Stefan Banach International Mathematical Centre" Warsaw, Poland, during October 1989, as an invited speaker.
Definition: A real function $f$ on a t.s. is said to have the property of Baire if it is measurable with respect to the $\sigma$-algebra of all sets with the Baire property.

It is well-known that the relation between continuity and measurability is given by the Lusin's theorem [Saks, p72]. In an abstract density topology $T$ Lusin's theorem is true iff $T$ is quasi-regular ([LM]), i.e. iff $\mu$ is regular (cf. 3.6.2). Below we give a category analogue of Lusin's theorem to hold for general $d$-topological space.

Theorem-1: Let $f$ be a real-valued function on the d.T. $(X,D)$. Then the following are equivalent:

(i) $f$ is continuous a.e.

(ii) $f$ has the Baire property

(iii) $f$ is measurable.

Proof: (i) $\Rightarrow$ (ii). Let $C$ be the set of all points of discontinuity of $f$, and $\mu C = 0$. Then $C$ is of 1-category. Let $g = f|_{X-C}$ be continuous. So for every open set $G \subseteq R$, $g^{-1}(G) = U \sim C$ for some open set $U$.

Since $g^{-1}(G) \subseteq f^{-1}(G) \subseteq g^{-1}(G)\sim C$, so

$U \sim C \subseteq f^{-1}(G) \subseteq g^{-1}(G)\sim C = (U \sim P)\sim C = U\sim C$ for some $P \subseteq C$.

$\Rightarrow f^{-1}(G) = U \Delta P_1$ for some $P_1 \subseteq C$.

$\Rightarrow f$ has the property of Baire.

(ii) $\Rightarrow$ (i). Let $G_i = (a_i, b_i) \subseteq R$ where $a_i, b_i \subseteq Q$, the set of rationals.

Let $f^{-1}(G_i) = U_i \Delta C_i$ where $U_i$ is open and $C_i$ is of 1-category in $X$. Also let $g = f|_{X-(U_i \cup C_i)}$.

Then $g^{-1}(G_i) = f^{-1}(G_i) \sim C_i$, where $C = \cup_i C_i$.

$= U_i \Delta C_i \sim C = U_i \sim C$ which is open relative to $X \sim C$, for each $i$. 

for every open set $G$, $g^{-1}(G)$ is open.

$\therefore$ $f$ is continuous except a null set.

$(ii) \iff (iii)$ is obvious from the fact that $\mu$ is a category measure (cf. 1.2.1, Th-3 and 3.6.2)

**Cor:** $f$ is measurable iff for every open set $U \subseteq \mathbb{R}$, the set $f^{-1}(U) = V \cup P$, where $V$ is density open and $P$ is a 1-category (null) set.

Let $(X,T)$ be a t.s., and

- $B_a = \text{the family of all functions with Baire property}$
- $B_0 = \text{the family of all } T\text{-Borel functions}$
- $\zeta = \text{the family of all } T\text{-continuous a.e. functions}$
- $\mathcal{M} = \text{the family of all measurable functions on } X$.

**Theorem-2:** If $(X,T)$ is an abs. d.T. then $B_0 = B_a = \zeta = \mathcal{M}$

**Proof:** Every nowhere dense set in $X$ is $T$-closed. Thus Borel sets in $X$ are the sets having the Baire property. Now in $X$, if $f \in \mathcal{M}$, then $f \in B_a$ and conversely. Hence by above $f \in B_a$ iff $f \in B_0$.

Also if $f \in B_a$, then there exists a 1-category set $K$ in $X$ such that $f|_{X \setminus K}$ is continuous in $T$ on $K^C$. Since converse is also true (cf. [Kur], Th 8.1), the other parts of the theorem easily follow.

### 5.2. CONTINUITY-RELATED FUNCTIONS

#### 5.2.1. In this section we study the behaviour of semi-continuous, approximately continuous functions, measurable functions and the functions of Baire classes on a d-topological space.
It is known that the set of all points of discontinuity of a real-valued semi-continuous functions on a space $X$ is a 1-category in $X$. Also if a space $Y$ is of the 1-category in itself, then there exists a bounded upper semi-continuous (U.S.C.) or lower semi-continuous (L.S.C.) function which is not continuous at any point ($\{\text{Fr}-1\}$). On the other hand, on a space $X$ of 2-category, there must be some point of $X$ for which every semi-continuous function $f$ is continuous. Since every open subset of a d-topological space is of 2-category, this gives us that:

Theorem-1: If $f$ is a bounded semi-continuous function on a d-topological space $X$, then the set of points of continuity of $f$ is dense in $X$.

Another characterisation of semi-continuous functions on d-space is given below.

Theorem-2: Let $(X,d)$ be an abstract d-space on $(X,\Sigma,\mu), \mathcal{F}$ be a family of L.S.C. functions on $X$ such that for each $x \in X$, the set $\{f(x) | f \in \mathcal{F}\}$ is bounded above. Then for every non-empty set $'A'$ of positive measure, there exists a non-empty d-open set $V \subseteq A$ and a $K \in \mathbb{Z}^+$ such that if $v \in V$ and $f \in \mathcal{F}$, then $f(v) \leq K$.

Proof: For every $t=1,2,3,\ldots$, let

$F_t = \{x : x \in A, f \in \mathcal{F} \Rightarrow f(x) \leq t\}.$

Since $f$ is LSC, $F_t$ are closed and measurable. Now $(X,d)$ being a density space, $A$ contains an open set $U$ $(1,2; A \sim (A)_d^\circ)$ which is of 2-category. Hence there exists a positive integer $n$ such that $F_n^\circ \neq \emptyset$. Let $V=(A)^\circ_d$, then the result follows.

5.2.2. The functions which are semi-continuous with respect to
the d.T. d have some important characterisations. They coincide with the class of approximately semi-continuous functions. These functions belong to that of Baire class I, and also have the Darboux property ([G.W]). Ridder extended these results to real functions of n-variables.

**Definition-1:** A function $f$ from a t.s. $X$ to a space $Y$ is said to be approximately continuous (abbr. app.c. or A.C.) at a point $p \in X$, if for every open set $V$ containing $f(p)$, the set $f^{-1}(V)$ has density 1 at $p$.

Equivalently, the set $X^{-f^{-1}(V)}$ has density 0 at $p$.

The approximately continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ are also the continuous functions in the density topology ([GN]). In general, it follows from the above equivalent definition that $f$ is app.c. at $p$ if for every $\varepsilon > 0$ the set $\{x : |f(x) - f(p)| > \varepsilon\}$ has outer density zero at $p$; i.e. iff for every neighbourhood $U$ of $f(p)$, $p$ does not belong to the set $[X^{-f^{-1}(U)}]_D$ in the d.T. $D$ on $X$. Since continuity of $f$ implies measurability and having the property of Baire (Theorem-2.5.1, $B_a = \mathcal{C} = \mathcal{M}$), from above we see that app.c. functions are measurable and measurable functions are app.c. almost everywhere.

It is known that the ordinary d.T. on $\mathbb{R}^n$ is the coarsest topology relative to which every approximately continuous function is continuous. However it is not true for the strongly app.c. functions since strong d.T. in $\mathbb{R}^n$ is not C.R.

Every A.C.function has the Darboux property and belongs to the 1st class of Baire. Any bounded A.C.function is a derivative. It is known as Denjoy's theorem. A real valued
function is A.C. at 'a' iff f is continuous at 'a' in the density topology (we call it as \((D_\circ - \cup\) continuous). In particular f is A.C. on \(R\) iff the sets,

\[\{x \in R : f(x) > \alpha\}\] and \(\{x \in R : f(x) < \alpha\}\) are density open for any \(\alpha \in \mathbb{R}\). Hence every app.c. function is measurable.

Two other important facts that is true for the class \(A\) of all app.c. functions possesses are:

a) \(A\) is closed under pointwise addition and multiplication and also under uniform convergence.

b) The set of all bounded app.c. functions forms a Banach space.

In an important theorem Zahorski \([Z]\), proved (he termed as \(F_\sigma\)- and density open set as \(M_5\) in his paper) the following theorem providing the existence of app.c. functions on d-spaces.

**Theorem-2:** Given a \(G_\delta\) set \(A \subset R\) which is density closed, there exists an app.c. and upper-semi-continuous function \(h\) on \(R\) such that

\[0 \leq h \leq I\] and \(A = \{x \in R | h(x) = 0\}\).

5.2.3. The following results (Th-1) extends a result in \([Saks], 10.6\) to an abs. d.T. on \(X\) without using Lusin's theorem. This also extends a similar result of Sion \([Si]\) where the space \(Y\) was assumed to have a countable base.

First we define

**Definition-1:** A collection \(P\) of non-empty sets in a t.s. \((X,T)\) is called a \(\mathcal{Q}\)-base if for every point in every non-empty open
subset of $X$, there exists at least one member $U$ of $P$, such that $x \in U \in P$.

**Definition-2:** Let $X$ be a t.s. and $A \subseteq X$. A family $F$ of non-empty subsets of $X$ is said to be dense in $A$ if for every $x \in A$, every neighbourhood of $x$ contains a member of $F$.

**Theorem-1:** Let $(X, T)$ be an abs. $d$-topological space on $(X, \Sigma, \mu)$ with $\mu$ regular and $(Y, T')$ be a t.s. having a countable $q$-base. If $f : X \to Y$, then $f$ is measurable iff $f$ is app. continuous a.e.

**Proof:** By definition of app. continuity, $\forall V \in T'$, $X - f^{-1}(V)$ has density 0 a.e. on $f^{-1}(V)$. By density theorem $X - f^{-1}(V)$ is measurable and so $f^{-1}(V)$ is measurable. Thus $f$ is measurable.

Conversely, let $Y$ have the countable $q$-base $P = \{V_n : n \in \mathbb{N}\}$.

For any $V_n$, $\mu(f^{-1}(V_n) \Delta U_n) = 0$ where $U_n = (f^{-1}(V_n))^c_T$. Let $C_n = f^{-1}(V_n) - U_n$. $C_n$ is of 1-category. Hence $C = U \cup C_n$ is of 1-category and $\mu C = 0$. Let $x \in X - C$.

$\therefore d(U_n, x) = 1$ since $x \notin C$.

Now let $B$ be the $T'$-neighbourhood of $f(x)$. Then by hypothesis, there exists $V_n \in P$ such that $f(x) \in V_n \subseteq B$. Then $x \in f^{-1}(V_n)$.

Since $x \notin C$, $x \notin C_n$ for any $n$, $x \in (f^{-1}(V_n))^c$.

$\therefore d(f^{-1}(V_n), x) = 1$.

But $f^{-1}(V_n) \subseteq f^{-1}(B)$ which is measurable.

$\therefore d((f^{-1}(B))^C, x) = 0$, $\forall x \notin C$.

Thus $f$ is app. c. except on a set of measure zero.

5.2.4. Goffman and Waterman [GW] have shown that if $X$ is a metric space and $f : \mathbb{R}^n \to X$ is app. c., then

(i) $f(\mathbb{R}^n)$ is a separable subspace of $X$,

(ii) $f$ is of Baire class 1,

(iii) $f$ takes $d$-connected sets into $T_\mu$-connected sets,
(T_u = standard topology).

An identical but extended result of (i) above has been proved below for an abs. d.T.

Theorem-1: Let T be an abs. d.T. on \((X, \Sigma, \mu)\) and \((Y, \Sigma', \mu')\) be a separable \(\sigma\)-finite measure space with a topology \(T' \subseteq \Sigma'\). Let \(f: X \to Y\) be an app.c. function a.e. on X. Then there exists a countable family \(P \subseteq \Sigma\) such that \(\{f(A) : A \in P\}\) is dense in \(f(X)\).

Proof: Since \(Y\) is separable measure space, \(Y\) possesses a regular net structure which is monotone and has the strong Vitali property ([Br], p-35). It also admits a natural contraction by inclusion.

Let \(I = \{I_x\}_{x \in X}\) be the net differentiation basis for \(X\). We denote \(I_x \to x\) if \(x \in I_x \subseteq I_x\) where each \(I_x\) is a net at \(x\).

Let \(V = \{V_n\}_{n=1}^\infty\) be a sequence of contracting nets in \(Y\) and \(C\) be the set of points of \(X\) at which \(f\) is not app.c. Then \(\mathcal{M}C = 0\).

Let \(x \in E = (f^{-1}(F_x))^c\) where \(F_x \subseteq V_n\) for some \(n\). We note that \((f^{-1}(F_x) - E) \subseteq E\).

Since \(E \in \Sigma\), \(d(E, x) = 1\). So there exists an \(I_n \in I_x\) such that \(\mu(E \cap A) > (1 - \varepsilon), \mu A\) for every \(A \in I_n\). Now \(E \cap A \in \Sigma\) and since \(T\) is an abs. d.T., so there exists a measurable set \(A' \subseteq E \cap A\) (e.g. \(A' \subseteq (E \cap A)^c\)) where \(A' \sim E \cap A\) such that \(\mu A' > (1 - \varepsilon), \mu A\) for all \(\varepsilon \ldots \ldots (i)\).

But \(A' \in I_{n+1}\) (\(\subseteq I_n \subseteq I_x\)) \((I_L)\) is monotone) for every such \(A'\) obtained as in \((i)\).

Keeping \(n\) fixed and varying \(x\) over \(X\) we get a family
\[ \bigcup_n \text{ of all } A'_i \in \Sigma \text{ with above property (i), and also a family } C_n = \bigcup \left( f^{-1}(F_n) \sim (f^{-1}(F_n))^\circ \right). \text{ Since } V_n \text{ is a Vitali cover so we see that } U_n \text{ is also a Vitali cover for } X - C. \text{ Hence there exists a countable subcover } W_n \subset U_n \text{ such that } \mu((X - C) - \bigcup W_n) = 0. \text{ Now consider the countable family } P = \bigcup \{ W_n \} \text{ of measurable sets of } X \text{ obtained as before.}

If } F \in P, \text{ then there exists a cell } E \text{ in the covering net } I \text{ in } X, \text{ such that } F \subset E \text{ and } F \sim E \text{ such that } \mu(F) > (1 - \varepsilon) \mu E. \text{ We show that } P \text{ is our required countable family.}

Consider any } x_0 \in X, \text{ and any open neighbourhood } V_o \text{ of } f(x_0) \in Y.

Then } \mu((f^{-1}(V_o))^\circ) \neq 0 \text{ except a set } A_o \text{ of 1-category. This implies there exists a } u_o \in f^{-1}(V_o) - A_o \text{ and a } k \text{ such that } f(u_o) \in V \subset V_k \text{ and } V \subset V_o. \text{ Because } \{ V_n \} \text{ is a contraction } u_o \text{ can be chosen so that it does not belong to any } A_n \text{ (n=0,1,2,...,n-1). So } u_o \in F \text{ for some } F \in W_k. \text{ Then there exists a } V \subset V_k \text{ such that } F \subset f^{-1}(V). \text{ So } f(u_o) \in V \subset V_k \text{ and } V \subset V_o. \text{ Therefore } f(F) \subset V \subset V_o. \text{ Thus } \{ f(F) : F \in P \} \text{ is dense in } f(X).

5.3. BAIRE FUNCTIONS
5.3.1. In this section we discuss the relation between measurable functions and the Baire functions associated with the density topology.

The following theorem-1 is due to Zink[Zi].

Lemma: Let } (R^n, D) \text{ be the d.T. and } f \text{ be a bounded Lebesgue measurable function on } R^n. \text{ Let } A \text{ be the class of all app.c.functions that are pointwise bounded above by } f, \text{ i.e.,}

\[ A = \{ g : g \in C(R^n, D), g \leq f \} \text{ where } C(R^n, D) \text{ denotes the class of all continuous real valued function defined on } (R^n, D). \]
If for every \( x \in \mathbb{R}^n \),
\[
h(x) = \sup \{ g(x) : g \in A \}
\]
then \( f = h \) a.e.

**Theorem-1:** Every bounded Lebesgue measurable function defined on \((\mathbb{R}^n, \mathcal{D})\) is almost everywhere equal to the limit of a non-decreasing sequence of approximately continuous functions.

**Proof:** Let \( A \) be the class of approx. continuous functions that are pointwise bounded above by \( f \). Let \( \forall x \in \mathbb{R}^n, h(x) = \sup \{ g(x) : g \in A \} \). Then \( f = h \) a.e. Let \( E(g_n) = \{ x : h(x) - g_n(x) > \frac{1}{n} \} \) where \( g_n \in A \). In [Zi], it is shown that \( \forall n \in \mathbb{N}, \) there exists a \( g_n \in A \) for which \( \lambda(E(g_n)) < \frac{1}{n} \).

\[
\text{If } f(n) = \sum_{k=1}^{n} g_k, \text{ then for some } \varepsilon > \frac{1}{n}, \text{ we get }
\lambda(\{ x : h(x) - f_n(x) > \varepsilon \}) < \lambda(E(g_n)) < \frac{1}{n}
\]
\[
\Rightarrow \lambda(\{ x : h(x) - \lim_{n} f_n(x) > \varepsilon \}) = 0.
\]

Making \( \varepsilon \to 0 \), we get,

\( h(x) \) is a.e. equal to \( \lim_{n} f_n(x) \),

or \( f \) is a.e. equal to \( \lim_{n} f_n(x) \). (''' \( h = f \) a.e.)

Since the approximately continuous functions are of Baire type 1, the above theorem gives us that,

**Cor:** Every bounded Lebesgue measurable function defined on \((\mathbb{R}^n, \mathcal{D})\) is a.e. equal to a function of the second Baire class.

It is known that the family of app.c. functions coincides with the family of all semi-continuous functions in the d.T. \((\mathbb{R}^n, \mathcal{D})\). Also every semi-continuous function \( f: (\mathbb{R}, \mathcal{D}) \to (\mathbb{R}, \mathcal{T}_u) \) is of Baire class one which is also continuous a.e. on a perfect set. Anyway on general abs.d.T. we cannot expect such a result since it holds on an arbitrary topological measure space \( Y \) iff \( Y \) is perfectly normal ([N1]).
However the following results due to Zink [Zi] concerning the semi-continuous functions that are almost everywhere equal to the functions belonging to first Baire class are significant in this context.

**Theorem-2:** Let $f$ be a bounded measurable function on a topological measure space $(X, \Sigma, \mu)$. The following are equivalent:

(i) $f$ is equivalent to an l.s.c (u.s.c.) function,

(ii) $A \in \Sigma \Rightarrow A$ is equivalent to an open set.

**Theorem-3:** Let $(X,T)$ be a completely regular t.s. on $(X,\Sigma,\mu)$. Then every bounded semi-continuous function on $X$ is a.e. equal to a function belonging to Baire class 1, and every bounded function of first Baire class is a.e. equal to a l.s.c. (or u.s.c.) function.

**Definition:** A measure space $X$ is called a o-space if every bounded measurable function $f$ is equivalent to a continuous function; $X$ is a l-space if every $f$ is equivalent to a l.s.c. (or u.s.c.) function and $X$ is not a o-space.

$X$ is called a $B_1$-space if every $f$ is equivalent to a function of Baire type 1.

From these it is obvious that

**Theorem-4:** A completely regular abs.d.T. space $(X,\Sigma,\mu)$ is a l-space which is $B_1$.

5.3.2 Petruska and Laczkovich under the density topological consideration have proved that [PL-1].

**Theorem-1:** For any function $f(x)$ in Baire class-2 there exists a sequence $\{f_n(x)\}$ of bounded and app.c. functions such that

$$\lim_{n \to \infty} f_n(x) = f(x).$$
This result is stronger than the similar ones proved earlier by A. Bruckner (who took \( \{ f_n(x) \} \) as the sequence of Darboux functions in \( B_1 \)) and then by D. Preiss (who proved that every \( f(x) \in B_2 \) is the pointwise limit of derivatives; i.e. \( f(x) = \lim_{n \to \infty} f_n(x) \) where \( f_n(x) = F'(x), \ n = 1, 2, 3, \ldots \).)

Interesting corollaries of the above theorem is:

**Corollary:** Let \( \mathcal{B}_* = \mathcal{A} \) and \( \mathcal{B}^*_\alpha = (\bigcup_{\beta < \alpha} \mathcal{B}_\beta) \lambda \).

Then (i) \( \mathcal{B}_0 \subset \mathcal{B}_* \subset \mathcal{B}_1 \),

(ii) \( \mathcal{B}_n^* = \mathcal{B}_{n+1}^* \) (\( 1 \leq n < \omega \)) (iii) \( \mathcal{B}_\alpha^* = \mathcal{B}_\alpha \) (\( \alpha > \omega \)).

Laczkovitch in [L] defined closure operations for subclasses of Baire-1 functions. Thus he considered the topology \( T \) generated by a subclass \( \mathcal{J} \) with the subbasis \( \{ x : f(x) > 0 \}, \ f \in \mathcal{J} \). Then he characterised some such subclasses of Baire-1 functions in terms of three separation properties \( S_1, S_2, S_3 \) (for definitions and details see [L]) where \( S_3 \Rightarrow S_2 \Rightarrow S_1 \). It was shown that the classes \( \mathcal{B}_1 \) and \( C[0,1] \) have the property \( S_3 \), whereas \( \mathcal{A} \) and \( DB_1 \) have the property \( S_2 \). But none of the classes \( \mathcal{A}, DB_1 \) and \( \mathcal{A} \cap D_\infty \) has the property \( S_3 \) (see also [L-1]).

O'Malley in [Om-1] has shown that (i) \( \mathcal{A} \cap D_Z \) has \( S_2 \) (ii) \( \mathcal{A} \cap D_f \) has \( S_1 \) but not \( S_2 \) and (iii) \( \mathcal{A} \cap D_2 \) does not have even \( S_1 \).

**Remark:** Topologies associated with the subclasses of a family of functions are seen to be investigated in many works. The modification topologies described in chapter-\( \mu \) are some of these. In fact many of these topologies are generated by classes of functions of interest to differentiation theory.

A topology associated with such subclasses of Baire-1 functions are also studied by Nishiura [Ni]. Some
interesting and useful characterisations of subclasses $A, A \cap C_e$, $B_1(A \cap C_e)$ and $B_1(A) \cap B_1(C_e)$ are produced. He has shown that the subfamily,

$\mathcal{A} = \{ E : \exists f \in A \cap C_e, \text{s.t. } E(f(x) > 0) = E \}$ of $A$ which is the co-zero subset of $A \cap C_e$ form a basis of a topology $(A_e)$ for which the class $A \cap C_e$ is the class of continuous functions in this topology. Z.Grande (cf.[Ni] for reference) also considered the relationships between those above classes for $R$ and showed that $B_1(C) = B_1(A \cap C_e)$.

5.4. GENERATED SPACE: The concept of 'generated space' seems to be originated from the study of semigroups of continuous functions by some authors (see e.g.[Mag]. In 1962 Shuepermann proved that the class of all generated spaces is S-admissible (cf.[Shu] for definition and proofs.) The property of a space being 'generated' is interesting due to the fact that the semigroups of its continuous self maps has the inner automorphism property ([CLO]). However, it is interesting to find that the d-topology does not belong to this class of generated spaces.

**Definition-1:** A t.s.X. is said to be generated provided the family.

$\mathcal{F} = \{ (f^{-1}(x)) : x \in X \text{ and } f : X \to X \text{ is continuous} \}$ forms a subbase for $X$.

That the density and I-density topology on $R$ are not generated is shown by following theorem in [CL-2].

**Theorem-1:** (a) $R$ with ordinary density topology $D_0$ is not generated.

(b)The I-density topology $(R,D_1)$ is not generated.
In fact let \( \mathcal{F} \) be the class of all continuous functions from \((\mathbb{R}, D_I)\) to \((\mathbb{R}, D)\).

Let \( S = \{ (f^{-1}({x}))^C : x \in \mathbb{R} \text{ and } f \text{ is } (D - D_I) \text{ continuous } \} \), and

\[
S_1 = \{ (f^{-1}({x}))^C : x \in \mathbb{R} \text{ and } f \in \mathcal{F} \}.
\]

Then obviously \( S \subseteq S_1 \).

Thus obviously \( S \) cannot be a subbase for \( D_I \) because \( D_I \) is not regular.

\( \therefore \) \( S \) cannot be a subbase for \( D_I \).

Hence \( (\mathbb{R}, D_I) \) cannot be generated.

Remark: However the deep-I-density \( D_d I \) can be used as an example of a generated space as Ciesielski, Larson and Ostaszewski ([CLO]) have shown that this topology \( D_d \) is generated.

5.5. BLUMBERG'S FUNCTIONS ON d.T.SPACES

In real function a natural study is to characterise functions in terms of Blumberg's property. This property may arise while studying special types of functions such as semi-continuous functions, c-continuous functions, almost continuous functions, Borel measurable functions etc. This leads to investigate topological spaces which permit Blumberg's functions.

Henry Blumberg in 1922 ([Bl-l]) proved that for any real valued function \( f \) defined on \( \mathbb{R} \) there is a dense subset \( D \) of \( \mathbb{R} \) such that \( f|_D \) is continuous.

In 3.7. we have seen that \((\mathbb{R}^n, T_u)\) is such a space. However many common refinements of \((\mathbb{R}^n, T_u)\) such as ordinary density topology, their category analogues or many other abstract
measure density spaces are quite pathological in respect of this property. In this section we intend to identify some d-topologies which admit Blumberg's functions.

**Theorem-1:** Let \((X,T)\) be a Baire space and \(D\) be a topology on \(X\) such that

1. \(D\) is finer than \(T\)
2. \(D \not\sim T\).

Then if \((X,T)\) is Blumberg, so is \((X,D)\).

**Proof:**

(i) First we show that every \(T\)-dense subset of \(X\) is \(D\)-dense.

Let \(Y\) be a \(T\)-dense set in \(X\), and \(U\) be any \(D\)-open set \((U \neq \emptyset)\).

\[ U^\circ \neq \emptyset \quad (\because \text{ } T \underset{\subset}{\sim} D) \]

But \(U^\circ \cap Y \neq \emptyset\). This implies \(U \cap Y \neq \emptyset\).

Hence \(Y\) is \(D\)-dense \((\because U^\circ \subset U)\)

(ii) Next, suppose \(f: X \to \mathbb{R}\). For any \(r \in \mathbb{Q}\), let \(M_r = \{x \in X : f(x) > r\}\) and \(N_r = \{x \in X : f(x) < r\}\).

Consider the set

\[ F_r = [(M_r)^\circ \setminus (M_r)^\circ] \cup [(N_r)^\circ \setminus (N_r)^\circ] \]

By prop-1 (b) above, \(F_r\) is a 1-category set in \(T\).

If \(f\) is \(D\)-continuous at a point \(x\) where \(f\) is not \(T\)-continuous, then there exists a \(r \in \mathbb{Q}\) such that \(x \in F_r\).

So the points of discontinuity of \(f\) is an \(F_{\sigma}\) set \(P\) such that

\[ P = \bigcup_{r \in \mathbb{Q}} F_r. \]

Hence \(P\) is a \(T\)-1st category set and since \(X\) is Baire the residual set \(P^c\) is dense (see 7.4, Oxtoby, 1971). Thus every \(D\)-continuous function is \(T\)-continuous almost everywhere on \(X\) and the converse is also true.
Above (i) and (ii) show that if \((X,T)\) is Blumberg, so is \((X,D)\). The proof is completed.

It is an immediate consequence of the above theorem that

**Cor-1:** There exists a Blumberg function on \((\mathbb{R}^n,D)\) if \(D\) is finer than \(T_u\) and \(D \not\sim T_u\).

**Cor-2:** Since \(D\) and \(I\) both are not \(S\)-related to \(T_u\), so \((\mathbb{R}^n,D)\) and \((\mathbb{R},I)\) are not Blumberg.

**Cor-3:** \((\mathbb{R}^n, a_e)\) and \((\mathbb{R}^n, r)\) are both Blumberg.

**Theorem-2:** The deep-\(I\)-density topology \(T_{d,\mathbb{R}}\) is \(S\)-related to \(T_o\) and independent of \(CH\), it admits a Blumberg function.

**Proof:** First we show \(T_{d,\mathbb{R}} \not\sim T_o\).

Let \(B \subseteq \mathbb{R}\) be \(T_u\)-dense. \((\mathbb{R},T_u)\) is a Baire space and so \(B^c\) is of first category.

From 2.4 (Definition and properties) we have that if \(x\) belongs to any \(T_{I,\mathbb{R}}\)-open set \(V\), then \(x\) is an \(I\)-density point of \(V\) and \(\forall \delta > 0, (x-\delta, x+\delta) \cap V\) cannot belong to \(I\), i.e. \(V\) is of 2-category. Let \(U \in T_{I,\mathbb{R}}\) and \(U \not\supseteq B\). Then \(U\) has Baire property and (by Th. 4.4 of [Ox]) \(U\) is uniquely expressible as a disjoint union of a \(G^\delta\) set and a set of 1-category. By above arguments we see that \(U\) is the complement of a 1-category set i.e. \(U\) is residual.

Let \(x \in B^c\). Then any \(T_{I,\mathbb{R}}\)-open set \(A\) containing \(x\) is of 2-category and hence \(A\) must intersect \(U\). Otherwise \(A \subseteq U^c\) is of 1-category. Therefore \(U\) cannot be separated from \(x \in B^c\) in \(T_{I,\mathbb{R}}\) and so in \(T_{d,\mathbb{R}}\) (i.e. \(T_o \subset T_{d,\mathbb{R}} \subset T_{I,\mathbb{R}}\)). In the regular space \(T_{d,\mathbb{R}}\) this is possible only if \(B\) is dense w.r.t. \(T_{d,\mathbb{R}}\). Hence \(T_{d,\mathbb{R}} \not\sim T_u\).

Hence by Theorem 4.1, \(T_{d,\mathbb{R}}\) is Blumberg since (under \(CH\)) \(T_u\) is Blumberg.
Remark: The results on Blumberg property on D-topological space
on $\mathbb{R}$ (3.7) and that of the property of $(\mathbb{R},D)$ for generated
space clearly shows that the density topology acts as a counter-
example in many aspects of topology (cf. [Gos-2]).