CHAPTER IV

CHARACTERISATION OF $d$ - SPACE AS FINE TOPOLOGY
CHAPTER-IV

CHARACTERISATION OF d-SPACE AS FINE TOPOLOGY

4.0 INTRODUCTION

In the last chapter-III we studied some general properties of density topologies without considering the existence of a pre-topology on the set. However a lot of topological aspects of abstract density topologies can be studied by making them dependent on their pre-topological structure. They include, for example, the lifted properties, the Lusin-Menchoff property, regularity and normality conditions, insertion property, connectedness and various others relative to the modifications of density topologies. We shall study in brief some of these.

In section 4.1 we prove some assertions concerning the refinements of abs. d.T. induced by a lifting and their behaviour. In section 4.2 we study the aspects of Lusin-Menchoff (L.M.) property with reference to d.T. considering a general approach. In the line of M.Brelot (1971) and Lukes et. al (1989) we show that d-topology having the L.M. property can be expressed as a weak topology generated by a family of semi-continuous functions and thereby implying the complete regularity.

Common examples of d-topologies are not normal and it is not even pseudo-normal. However they are connected. In 4.3 and in 4.4. we put forth some sufficient conditions for these properties and analyse them*.

Some portions of this chapter has appeared in our paper [Gos 3].
4.1. FINE ABSTRACT DENSITY TOPOLOGY

Let $D$ be an abs.d.T. on $(X, \Sigma, \mu)$, and $T_\Theta$ be a topology on $X$ such that $D$ is finer than $T_\Theta$. In this case $D$ is called a fine topology on $X$ and $T_\Theta$ will be called a pre-topology of $D$ on $X$.

The following proposition gives general construction of finer topology on an abstract density topology.

Let $T$ be an abs. d.T. on $(X, \Sigma, \mu)$. If $\Theta$ be a topology on $X$ such that $\Theta$ is finer than $T$ and for every $G \in \Theta$ ($G \neq \emptyset$), there exists a $B \in \Sigma$ such that $B \subset G$ and $\mu B > 0$. Then it is obvious that for every $A \in I_\mu$, $A$ is $\Theta$-closed and for every $A \in \Sigma$, $A \setminus (A)^\Theta \in I_\mu$. Therefore $\Theta$ is also an abs. d.T. (By Th-4 1.3.4). This gives

**Proposition:** Any topology on $X$ finer than an abs.d.T. on $X$ and satisfying the condition (C) of Theorem-4 of 1.2.1. is an abs. d.T.

We shall see below that the lifted topologies are the finest topologies on abstract density spaces and that every abstract density space can be refined to one such space. This indicates a possible topological approach to many assertions of the theory of lifting.

4.2. LIFTED ABSTRACT DENSITY TOPOLOGY

Topologies associated with a lifting are first seen to study by Oxtoby [cf.[Tul-1] as the ones which do not admit further refinements.

**Definition-1:** If the lower density $1$ determining an abs.d.T. on
Proposition: Let $T$ be an abs. d.T. induced by a lifting $l$. Then for any $A \in \Sigma$,

$$\overline{\alpha_T} = A \cup (1 \setminus (A^c))$$

Proof: $X = l(X) = l(\alpha \cup A^c) = 1(A) \cup 1(A^c)$

But $1(A) \cap 1(A^c) = \emptyset$. \therefore $1(A^c)^c = 1(A)$.

Hence $\overline{\alpha_T} = A \cup (1 \setminus (A^c))^c = A \cup 1(A)$. (cf. Note, 1.3.5).

Theorem 1: Let $J$ be an abs. d.T. on $(X, \Sigma, \mu)$ induced by the lower density $l$. Then $J$ is lifted if there exists a perfect b.o. $b$ such that for every $A \in \Sigma$,

(i) $bA = (l \setminus A^c)^c$ and (ii) $bA^c = (bA)^c$.

If in addition $J$ is such that for all $U \in J$, $\overline{U} \in J$, then the converse holds.

Proof: Let $b$ be the given base operator. We have

$$[l(\alpha \cup B)]^c = (l(A \cap B^c))^c$$

$= b(\alpha \cap B^c)$ by (i)

$= b(\alpha \cup B)^c = [b (\alpha \cup B)]^c$

$= (b \alpha \cup B)^c = (b \alpha)^c \cap (b B)^c$ \quad (\because (b \alpha)^c = 1(A^c))

$= bA^c \cap bB^c$

$= (lA)^c \cap (lB)^c$

\therefore $l(\alpha \cup B) = l \cup B => l$ is lifting.

Converse: Let $l$ be a lifting for the lifted abs. d.T. $J$ on $X$. So by Th.1, 1.3.5, there exists a perfect b.o. $b$. It remains to show that

(i) for all $U \in J$, $\overline{U} \in J$, and (ii) for all $A \in \Sigma$, $bA^c = (bA)^c$. 

Part(i) \( U \cup \mathcal{J} \Rightarrow U \cup \mathcal{L}(U) \)

\[
\therefore \quad \overline{U \cup \mathcal{J}} = U \cup \mathcal{L}(U) = \mathcal{L}(U) = \overline{\mathcal{L}(U)}
\]

\( \Rightarrow \quad \mathcal{L}(U) \subset \mathcal{L}(\overline{U \cup \mathcal{J}}) \)

\( \Rightarrow \quad \overline{U \cup \mathcal{J}} \in \mathcal{J} \).

Part(ii'): We have, \( A \sim bA \) and \( bA = A \leq bAc \Rightarrow bAc = (bA)c \), otherwise \( bAc \not\subseteq (bA)c \).

\[
\begin{align*}
\text{Cor(i): If 1 is a lifting then} & \\
1(A) \cup (B) &= (bAc) \cup (bBc) \\
\therefore & 1A \cup 1B = bA \cup bB
\end{align*}
\]

Theorem-2: Let \( \mathcal{J} \) be the abs. d.T. determined by the lifting 1. Then, following are equivalent (i) For any \( A \in \Sigma \), there exists a \( \mathcal{J} \)-closed as well as a \( \mathcal{J} \)-open set \( V \) such that \( V \sim A \).

(ii) \( \mathcal{J} \)-closure of any \( \mathcal{J} \)-open set is \( \mathcal{J} \)-open.

Proof: (i) \( \Rightarrow \) (ii): Easily follows by contrapositive argument.

(ii) \( \Rightarrow \) (i): Let for all \( U \in \mathcal{J}, \overline{U} \in \mathcal{J} \). Since \( \mathcal{J} \) is lifted, for every \( A \in \Sigma \), there exists a perfect base operator \( b \) such that \( bA = (1A \cap \overline{C})C \) and \( bA \cap (bA)^C \), i.e. \( 1A \cap (1A^C) \), where 1 is the lifting. Hence considering \( V = A \cup [1(A^C)]^C \) it can be verified easily that \( V \) is both \( \mathcal{J} \)-closed (cf. Note 1.3.4.) and \( \mathcal{J} \)-open (we note that \( 1A^C = A \)). Thus for every \( A \) we have a \( \mathcal{J} \)-clopen set \( V \) such that \( A \sim V \).
Theorem-3: Let $\mathcal{J}$ be an abs. d.T. on $(X, \Sigma, J)$ (cf. 1.4.1.) determined by the lower density $\lambda$. $\mathcal{J}$ is a lifted topology iff for any $A \in \Sigma$, there is a $\mathcal{J}$-clopen set $V$ such that $\lambda(A) = V$.

Proof: Let $\mathcal{J}$ be a lifted topology. By theorem-1 for every $U \in \mathcal{J}$, $\overline{U} \in \mathcal{J}$. By Theorem-2 we get the result.

Converse is obvious.

Remark: We can observe that corresponding to any lower density or lifting $\lambda'$ on $(X, \Sigma, \mu)$, there exist naturally two abstract density topologies on $X$. First is the one (1.2) as $T_{\lambda} = \{ \lambda(A) \mid A \in \Sigma, N \in \mathcal{I}_A \}$ and the second is $G_{\lambda}$ which is induced by the base $\{ \lambda(A) : A \in \Sigma \}$. Then $G_{\lambda} \subseteq T_{\lambda}$. A few basic properties regarding $G_{\lambda}$ and $T_{\lambda}$ can be seen in [Tul]. The above theorems both hold equally for each of $T_{\lambda}$ and $G_{\lambda}$, and as well as for an abs. d.T. or the lifted topology on $(X, \Sigma, J)$ of 1.4.1.

Theorem-4: Let the abs. density topology $T$ on $(X, \Sigma, \mu)$ be determined by the lifting $\lambda$. Then any abs. d.T. $D$ finer than $T$ on $(X, \Sigma, \mu)$ implies $D = T$.

Conversely, if $T$ does not admit any refinement, then $T$ is a lifted topology.

Proof: Let $D \supset T$ such that $D$ is induced by a lower density $\rho$.

If $S \in \Sigma$, then $S \cap \rho(S) = S_D \subseteq \text{sup}(S) \subseteq \overline{S}_D$ ... (i)

$\Rightarrow \mu(S \triangle \overline{S}_D) = 0$ or $\rho(S) = \rho(\overline{S}_D)$ ........(ii)

Since $T \subseteq D$, for any $S$, $\overline{S}_D \subseteq \overline{S}_T$ ........(iii)

Suppose $G \in D$. $G \in \Sigma$. To show $T = D$, it is enough to prove that $G \cap (G^c)_T = \emptyset$.

Now $G \subseteq \rho(G) = \rho(\overline{G})_T$, since $G \sim \overline{G}_T$ and $G \in \Sigma$.

$\subseteq (\overline{G}_T)^c_D$ by (i) ( \because $\rho(S) \subseteq \overline{S}_D$)

$\subseteq (\overline{G}_T)^c_T = \overline{G}_T$, (by (iii))
Then, \( G\cap(G^c)^T = G \cap (G\cap(G^c))^T \)
\[ = G \cap (G^c)^T \]
(We note that \( T \) is a lifted topology and \( \overline{A}_T = A \cup (A) \).)

Again \( G\cap(G^c)^T = (G\cap(G^c))^T \) \( G \subseteq 1(G^c) \) .... (v)

From (iv) and (v),
\[ G\cap(G^c)^T \subseteq 1(G\cap(G^c))^T \]
\[ = 1(G\cap(G^c))^T \subseteq 1(G^c) = 1(\emptyset) = \emptyset, \]
Thus every \( D \)-open set \( G \) is \( T \)-open.

**Conversely:** Let there be not any abs. \( d.T. \) on \( X \) strictly finer than \( T \). By Th-1 above we simply require to prove that \( \forall A \subseteq T, \overline{A} \subseteq T \). Suppose this is not true and there exists \( U \subseteq T \) such that \( \overline{U} \not\subseteq T \). Define a family \( t \) on \( X \) such that \( G \subseteq t \) if \( G = \cup(A_1 \cup A_2) \) where \( A_1, A_2 \subseteq T \). It can be verified that (i) \( t \) is a topology (ii) \( t \) is strictly finer than \( T \) and (iii) if \( G \subseteq t, G \not\subseteq \emptyset \) then \( \mathcal{M}G > 0 \).

Then by our Theorem-4 (1.3.4), \( t \) is an abs. density topology on \( X \) contradicting the assumption.

Thus \( t = T \). By Theorem-1 \( T \) is a lifted topology.

Next Theorem shows that every abs. \( d.T. \) can be refined to a lifted topology.

**Theorem-5:** There exists a finer lifted topology \( T \) for every abs. \( d.T. \) \( D \) on \( X \).

**Proof:** By Theorem-4 a lifted topology is a maximal one. If \( F \) be the partially ordered family of all abs. \( d. \) topologies \( T_i (i \in \Lambda) \) on \( X \) which are finer than \( D \), then by Zorn's Lemma there exists a maximal topology \( T_m \) which must be the lifted one by Th-4 Hence \( T_m = T \).
To construct $T_m$, consider the topology $T_m = \operatorname{Sup}\{\bigcup_{i \in \Lambda_i}\}$, the topology generated by the sub-basis $\bigcup_{i \in \Lambda_i}$. Then $T_m \supset D$. Let $G = \bigcup_{i \in \Lambda_i} G_i \in T_m$, where $G_i \in T_i$. To complete the proof, we need to show,

(i) $T_m$ is an abs. d.T. and (ii) $T_m$ is lifted. Proceeding as in Th-2,1.2.1, it can be shown that $G$ is measurable and if $G \neq \emptyset$ and $\mu G > 0$, then there exists a measurable set $B$ such that $B \subset G$ and $\mu B > 0$. Also since $G$ is measurable $\mu[G \Delta \operatorname{cl}(G)]_{T_m} = 0$ or $G \sim \operatorname{cl}(G)_{T_m}$. By Th-4 (1.3.4), $T_m$ is an abs. d.T. on $X$. Also by Th-4 (converse) above $T_m$ is a lifted topology.

Theorem-6: There exists a lifting on every categorial density topology $T$ on $(X,\Sigma, J)$, where $(X, T)$ is a Baire space and $J$ is the $\sigma$-ideal of all $T$-first category sets.

Proof: Hashimoto *-modification $T^*$ of $T$ is an abs. d.T. by Theorem-1 (2.5.3.) By its Cor-3 and Th-5 above the result follows.

4.3. LUSIN-MENCHOFF PROPERTY

Common examples of d-spaces lack the property of normality. However Lusin-Menchoff property plays a role intermediate between normality and complete regularity. It plays also a significant role in study of fine topological properties. It's proof for different topologies can be seen in various papers (e.g.[Ch.], [GN], [TZ], [Om], [Lu] etc.)

Definition-1: Let $(X, T_0)$ be a topology and $T$ be a topology on $X$ finer than $T_0$, $T$ has the Lusin-Menchoff (abbr.LM) property w.r.t. $T_0$ if for any pair of disjoint closed subsets $F$ w.r.t. $T_0$ and $F^T$ w.r.t. $T$ in $X$, there exists a $T_0$-open set $G$ and a
T-open set $G^T$ such that $F^T \subseteq G$, $F \subseteq G^T$ and $G \cap G^T = \emptyset$.

Equivalently:

**Definition:** \(T\) has the L.M. property w.r.t. \(T_o\) if for every pair \(F \subseteq U^T\) (\(F\) is \(T_o\)-closed, \(U^T\) is \(T\)-open), there is a \(T\)-open set \(G^T\) such that $F \subseteq G^T \subseteq \overline{G^T} \subseteq U^T$.

The following results are well known -

1. The ordinary d.T. on \(R^n\) has the L.M. property,
2. \(R^n\) with strong d.T. does not have the L.M. property,
3. \(R\) with I.d.T. and deep-I-d.T. has the L.M. property.

**Theorem 1:** The necessary and sufficient condition that a space \((X,T)\) have the L.M. property with respect to a pre-topology \((X,T_o)\) is that:

Given a pair of disjoint closed sets \(F\) w.r.t. \(T_o\) and \(F^T\) w.r.t. \(T\), there exists a \(T\)-continuous and upper-semi-continuous (U.S.C.) function \(f\) on \(X\) such that

(i) \(0 \leq f(x) \leq 1\)

(ii) \(f(x) = 0\) if \(x \in F^T\)

= 1 if \(x \in F\).

**Proof:** Let for the given pair of closed sets \(F,F^T\), the open sets \(H\) and \(H^T\) (resp. w.r.t. \(T_o\) and \(T\)) exists such that $F^T \subseteq H$ and $F \subseteq H^T$ where $H \cap H^T = \emptyset$. \(\therefore F^T \subseteq H \subseteq (H^T)^C \subseteq F^C\).

Thus that \((X,T)\) has the LM property is equivalent to saying that for any pair \(F^T\) (T-closed) and an open set \(U(=F^C, T_o\)-open), there exists an open set \(G_{1/2}\) such that $F^T \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq U \ldots (i)$

Now \(G_{1/2}\) is \(T_o\)-open set containing T-closed set \(F^T\) and \(U\) is \(T_o\)-open set containing T-closed set \(G_{1/2}\). So by LM property (i) we have

$$
F^T \subseteq G_{1/4} \subseteq \overline{G_{1/4}} \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq G_{3/4} \subseteq \overline{G_{3/4}} \subseteq U.
$$
We continue this process exactly the same way as in the proof of Urysohn's Lemma and obtain for each \( t \in D \) (the set of dyadic fractions in \((0,1)\) and which is dense in \([0,1]\)) and open set \( G_t \) with the property that if \( t_1, t_2 \in D, \ t_1 < t_2 \) then \( G^T_{t_1} \subseteq G_{t_2} \).

Define \( f \) on \( X \) as

\[
 f(x) = \begin{cases} 
 \inf \{ t : x \in G_t \} & \text{if } x \notin U^C \\
 1 & \text{if } x \in U^C 
\end{cases}
\]

Then \( 0 \leq h(x) \leq 1 \ \forall x \in X \), and \( f \) maps \( X \) into \([0,1]\). It is apparent that \( \forall t \in D, F^T \subseteq G_t \), i.e. \( f[F^T] = \{0\}, f[U^C] = \{1\} \). The following facts can be proved as usual (\( r, s \in D \)).

a) If \( x \notin G^T_r \), then \( f(x) \geq r \)

b) If \( x \in G_r \), then \( f(x) \leq r \)

c) If \( x \in G_r \setminus G^T_s \), where \( s < r \) then \( s \leq f(x) \leq r \).

Then the continuity w.r.t. \( T \) and the upper semi-continuity follows from the results

\[
 f^{-1}((0,\infty)) = U(G_t: t<\infty), \text{ a } T_o \text{-open set and } f^{-1}(\beta,1) = U((G^T_t)^C: t>\beta), \text{ a } T \text{-open set.}
\]

**Sufficiency:** Let \( F \) and \( F^T \) are disjoint; \( F \) is \( T_o \)-closed, and \( F^T \) is a \( T \)-closed set.

Let \( f:X \to [0,1] \) be a \( T \)-continuous and U.S.C. function on \( X \) s.t. \( 0 \leq f(x) \leq 1 \) and \( f[F^T] = 0, f[F] = 1 \).

Then apparently

\[
 G = f^{-1}[0,\frac{1}{2}], \text{ and } G^T = f^{-1}(\frac{1}{2},1) \text{ will be two disjoint open sets containing } F^T \text{ and } F \text{ respectively i.e. } F^T \subseteq G, F \subseteq G^T.
\]

So \((X,T)\) has the L.M. property w.r.t. \((X,T_o)\)

Following propositions proved in [LMZ] (2.B and 3.B) are useful for us.
Proposition-1: Let $T$ be the weak topology on $X$ induced by a family $\mathcal{F}$ of functions. If $x \in G \subseteq X$, $G \in T$, then there exists $f, g \in \mathcal{F}$ and $f \leq g$ on $X$ such that $x \in \{x : f(x) \leq g(x)\} \subseteq G$. In particular, $T$ is completely regular.

Proposition-2: Let $(X, T_0)$ be a topology and $T$ is finer than $T_0$. Given sets $F, F^T$, closed in $T_0$, $T$ respectively, and $A, A^T \subseteq X$ such that $A$ is $T_0$-closed and $A^T$ is $T$-closed, also $F^T \subseteq A$, $F \subseteq A^T$. Then $(X, T)$ has the L.M. property w.r.t. $T_0$ iff there are $T$-continuous and upper semi-continuous functions $f$ and $g$ on $X$ such that $f \not\leq 0$, $g \not\leq 0$, and $F^T \subseteq Z(f) \subseteq A$, $F \subseteq Z(g) \subseteq A^T$, (where $Z(h) =$ zero-set of $h$).

Proposition-3: A lifted topology $T$ on $(X, \mathcal{X}, \mu)$ is completely regular if any of its coarser pre-abstract density topology is completely regular.

Remark: 1) Every closed set in a semi-metric space $(X, T_0)$ is a zero set. Hence it can be asserted from the above proposition-2 that a topology $T$ on a semi-metric space has the L.M. property for any $T$-closed set $F^T$ and for each $G_0$-set $A \supseteq F^T$, there exists a $T$-continuous and U.S.C. function $f_\alpha (\not\leq 0)$ such that $F^T \subseteq Z(f_\alpha) \subseteq A$.

2) The collection $\mathcal{F} = \{f_\alpha : \alpha \text{ runs over all pairs } (F^T, A)\}$ of all $T$-continuous and U.S.C. functions on $X$ is a family of functions (in fact a convex cone) separating points from closed sets. Hence the sets $f_\alpha^{-1}(V)$ ($V$ is open in $R$) of co-zero sets form a basis for a topology $T'$. An application of prop-1 will show that the weak topology $T'$ generated by $\mathcal{F}$, is exactly $T$. Thus $T$ is completely regular.
Above establishes.

**Theorem-2**: Let \((X,T_0)\) be a \(T_1\)-space and \(T\) is a topology on \(X\) finer than \(T_0\) having the L.M. property. Then \((X,T)\) is completely regular.

**Cor-1**: \(\mathbb{R}^n\) with ordinary d.T. is C.R.

**Cor-2**: \(\mathbb{R}\) with I.d.T. or with deep-I-density topology is C.R.

4.3.1. **Definition**: Let the topology \(T\) be finer than \(T_0\). \((X,T)\) is said to have the **complete Lusin Menchoff property** (abbr. CLM) w.r.t. \((X,T_0)\) if every subspace of \(X\) has the L.M. property. [Equivalently, if \(T\) has the L.M. property on every \(T\)-open subset of \(X\)].

**Example-1**: The a.e-modification (\(a_e\)-topology, 2.5.8) of the d.T. on \((\mathbb{R},T_u)\) has the CLM property.

**Example-2**: The lifted topology \(U\) defined by Scheinberg (2.5.6) has the CLM w.r.t. the d.T. \((\mathbb{R},D)\).

**Example-3**: The \(r\)-modification (2.5.7) of the d.T. on \(\mathbb{R}\) has the CLM.

4.4. **L.M. PROPERTY FOR BASE OPERATOR SPACE**: In consideration with many useful properties of density topologies, a treatment with b.o. spaces is more workable rather than the topology itself. It has been shown in [LMZ] that not only these properties (including L.M.) are preserved under different operations (such as binary or set theoretic) on b.o. spaces, but also these find their practical uses to many other comparable topologies. It is noticed that sometimes the density b.o. has the L.M. property while the t.s.
induced by itself does not have it. The superdensity topology described in Appendix is a good example for this. In Theorem-1 below we see how the L.M. property of b.o. serves a better way of investing this property for a modification of density topology.

**Definition:** Let \((X,T_0)\) be a t.s., \((X,b)\) be a b.o. space such that \(b\)-topology is finer than \(T_0\). Then \(b\) is called a **fine base operator**.

The b.o. \(b\) is said to have the L.M. property with respect to \(T_0\) if for every \(T_0\)-open set \(G \subseteq X\) and for every set \(A \subseteq X\) such that \(A \cup bG \subseteq G\), there exists a \(T_0\)-open set \(U \subseteq X\) where \(A \subseteq U \subseteq U \cup bU \subseteq G\).

**Note:**

1) It can be seen that \(b\) has the L.M. property iff the closure operator \(\gamma\) w.r.t. \(b\) where \(\gamma A = A \cup bA\) has the L.M. property.

2) A strong b.o. \(b\) has the L.M. property iff the corresponding \(b\)-topology has the L.M. property. \(b\) is said to have the C.L.M. property, if \(\forall Y \subseteq X\), the operator \(\gamma A = \gamma \cap bA \forall A \subseteq X\), has the L.M. property w.r.t. the topology \(T_0|_Y\).

Following facts are related and fundamental to the study of L.M. property (cf. [LMZ], p.103) through base operators.

1) The topology induced by a b.o. \(b\)' has the CLM iff \(b\) is strong and \(b\) has the CLM.

2) A b.o. \(b\) has the CLM iff the following condition [C] holds:

[C] For every pair of subsets \(A\) and \(B\) of \(X\) for which \(\overline{A} \cap B = \emptyset = A \cap bB\), there exists an open set \(G\) such that \(B \subseteq G \subseteq A^c\) and \(A \cap bG = \emptyset\).
3) If \( b,\gamma \) be two fine base operators on a t.s. \((X,T_0)\) then \( b\gamma \) has the L.M. (or CLM) property if \( b \) and \( \gamma \) both have the L.M. (or CLM).

4) In (3) if \( \gamma \) is a fine closure operator having the CLM, then \( b\gamma \) has the CLM or LM according as \( b \) has the CLM or L.M. property respectively. [Here \( (b\gamma)_A = b(\gamma A) \)].

5) Let \((X,T_0)\) be a t.s. and \( K \subseteq X \). If \( b \) be a fine b.o. on \( X \) having the CLM, then the operator \( b_A : A \rightarrow bA \setminus K \) also has the CLM. We have

**Theorem-1:** The K-modification of the d.T. on \( R \) with respect to \( T_u \) is completely regular.

**Proof:** Let the K-modification of the d.T. \( D \) on \( R \) be \( D_k \) [cf.Ch-2]. Then \( T_u < D_k < D \) and that \( D \) has the CLM property w.r.t. \( T_u \). It is enough to show that \( D_k \) has the L.M. property w.r.t. \( T_u \).

By Th-1, 2.5.4*, we have, for any \( X \subseteq R \),

\[
(\overline{X})_{D_k} = \left[ (\overline{X} - K)UX \right]_D \quad \text{... (i) where } \overline{X} = (\overline{X})_{T_u}.
\]

Consider the operators \( \alpha, \beta, \gamma \) such that \( \alpha(X) = (\overline{X})_D \), \( \beta(X) = (\overline{X} - K)UX \) and \( \gamma(X) = (\overline{X})_D \).

From (i), \( \alpha(X) = (\beta \gamma)(X) = \gamma(\alpha(X)) \).

Now \( \gamma \) has the CLM w.r.t. \( T_u \). By Note(4) above, \( \alpha \) has the CLM if \( \beta \) has the same.

But \( \beta(X) = (\overline{X} - K)UX \)

\[
= \beta(X)U_iX \quad \text{where } \beta(X) = \overline{X} - K, \text{ and } i \text{ is the identity b.o.}
\]

which has the CLM w.r.t. \( T_u \). Also \( \beta \) has obviously the CLM. since \( (K, T_u) \) is completely normal. Therefore \( \beta \), i.e., the \( D_k \) topology on \( R \) has the CLM.

The above theorem underlines a general approach to prove the LM property for many modifications d-topologies. For example

* If \( \{b_\alpha\} \) be a family of base operators we define \( \bigvee b_\alpha = \text{Sup } b_\alpha \) is the base operator \( b : A \rightarrow \bigcup b_\alpha A \).
Theorem-2: Let $T^*_k$ be the extended K-modification (2.5.5) of the d.T. on $(R, T_u)$ and let $D_o$ be another topology on $R$ such that $T_u < D_o < D$ and $D_o$ has the CLM w.r.t. $T_u$. Then $D_k$ has the LM or CLM according as $D_o$ has respectively the LM or CLM.

Proof: We note that $T_u < D_o < T_k < D$ and the closure property in $T_k$ is given by

$$(\overline{A})_{T_k} = \{(\overline{A})_{D_o_{K}} U A\}^D.$$

Then proceeding similarly as in Theorem-1 above, the proof follows then by using the results (5) after Note above.

Below we state a few more results. Their proofs are not at much variance with above. However different proofs are also available in literature based on their positional circumstances.

Theorem-3: Let $b$ be a density b.o. on $(R, T_u)$ and $\alpha$ be the a.e-modification (2.5.10) of $b$ which has the C.LM w.r.t. $T_u$. Then $\alpha$-topology also has the CLM.

Proof: Omitted.

Theorem-4: The a.e-modification (a.e-topology (2.5.10) of the d.T. on $(R, T_u)$ has the CLM.

Proof: Omitted.

It can be observed that a modification (e.g. the a.e-topology) may have the CLM property even though some pre-topology finer than the initial topology does not have the CLM.

Theorem-5: Consider the r-modification of the d.T. on $R$. Let $U$ be r-open and $F$ be a closed subset of $U$ such that there exists an r-basis set $B$ with property $F \subset B \subset U$. 

97
Then the r-topology has the CLM.

**Proof:** (cf. [Om-1]).

### 4.5. NORMALITY CONDITIONS IN d.T.

Generally a d.T. even with a L.M. or a CLM property or even with a completely regular pre-topology (e.g. $T_u$ in case of $(R,D)$) need not be normal. F.D. Tall in [T-1] however has characterised normal subspaces on $R$ with d.T. considering various set-theoretic approaches. It will be relevant here to mention a few important assertions proved in [T-1].

**Proposition-1:** (Th-4, [T-1]) Assume $2^\aleph_0 < 2^\aleph_1$ (respectively Martin's Axiom). Then $Y \subseteq R$ is normal iff $Y$ is the union of a Sierpinski (resp. generalised Sierpinski) set $S$ and a null-set $T$ such that $S \cap T = \emptyset$.

**Proposition-2:** (Th-12, [T-1]) Assume M.A. and also $2^{\aleph_0} > \aleph_1$. Let $X$ have the following properties:

(i) C.C.C.  (ii) Co-metrizable  
(iii) no isolated points  
(iv) Martin's Axiom works  
(v) uncountable closed discrete subspace exists.

Then $X$ has a dense hereditarily normal non-collectionwise Hausdorff subspace.

**Cor:** $R$ with density topology has the above property.

A sufficient condition for non-normality of d.T. is obtained by Tall [T] as follows:

**Proposition-1:** Let $Y$ be a normal t.s. such that the cardinality of the regular open algebra of $Y$ is less than $2^k$. Then every closed discrete subset of $Y$ has cardinality $< k$ (i.e. $Y$ is $k$-compact).
It is seen that the regular open algebra of $\mathbb{R}$ with density topology has the cardinality $2^{\aleph_0}$, and $\mathbb{R}$ is not $2^{\aleph_0}$ compact. Hence $(\mathbb{R}, D)$ is not normal.

**Proposition-2:** Let $(X, D)$ be a d.T. on a separable Baire space $(X, \mathcal{T}_0)$. If every $D$-continuous function is of Baire class one (i.e. $\in B_1$), then $(X, D)$ is not normal.

**Proof:** Let $(X, D)$ be normal and $P, Q$ be two disjoint countable $\mathcal{T}_0$-dense subset of $X$. So, $P, Q$ are $D$-closed first category sets. Hence by Urysohn's Lemma, there exists a $D$-continuous function $f$ on $X$ such that $f|_P(x) = 0$ and $f|_Q(x) = 1$. Since $f \in B_1$ the sets $f^{-1}[0, \alpha]$ and $f^{-1}[\beta, 1]$ (where $0 < \alpha < \beta < 1$) are closed ($G_\delta$).

But there are $\alpha, \beta$ such that the sets $U = f^{-1}[0, \alpha], V = f^{-1}[\beta, 1]$ are disjoint and $U, V$ are both complements of first category sets. This shows that $(X, D)$ is not a Baire space - a contradiction.

**Definition:** $P$-insertion property: Let $(X, T)$ be a t.s., $P \subset \mathcal{P}(X)$. The topology $T$ is said to have the $P$-insertion property if for every $T$-closed set $F$ and for every $T$-open set $G \subset F$, there exists a set $A \in P$ such that $G \subset A \subset F$.

(Equivalently, if $\forall S \subset X$, $\exists$ a set $A \in P$ s.t. $A^o_T \subset S \subset A_T$).

Above is a very useful notion introduced by Lukes and Zajicek[LMZ]. Using this the following Theorem and the proposition (showing another condition of non-normality of fine topology was obtained in[LMZ] (6.A.10). We again omit their proofs.

**Theorem:** $\mathbb{R}^n$ with ordinary density topology has the $G_\delta$-insertion property.
Proposition-3: Let \((X, T_0)\) be a t.s. and \(T\) be a topology on \(X\) finer than \(T_0\). Let \(P\) be a family of subsets of \(X\) such that \(|P| = \infty\).

If (i) \(T\) has the \(P\)-insertion property and (ii) there exists a set \(A \subseteq X\) s.t. \(A^{\text{d}_{T_0}} = \emptyset\).

Then \(T\) is not normal.

4.6. CONNECTEDNESS OF abs.d.T.

It is known that most examples of density topologies on \(R^n\) are connected ([GN], [GW], [Wil.-1], [Ea-2], [LZ], [TZ] etc.) In [TZ] this property was extended to \(\phi\)-topology (cf.1.1.2) and proved a sufficient condition for connectivity as follows.

Proposition-1: If the parameter of regularity \(\phi\) is positive for every point \(x\) in any open connected subset of \(R^n\) then \(S\) is \(\phi\)-connected.

For \(R_1\) proposition-1 is also a necessary condition.

In [Ea-2] the above result is modified to density topology on a metric space. In [LZ] a more general approach for proving connectivity of d.T. and other fine topologies is being made. The following result in [LMZ] is in fact more general that particularises also the theorem of [TZ] above.

Proposition-2: Let \(T\) be a fine topology on a strong Baire, connected space \((X, \mathcal{P})\). Assume that any subset of \(X\) which is simultaneous both \(T\)-open and closed is of type \(G^\delta\), and the \(T\)-closure of any open set \(G\) is dense in the boundary of \(G\). Then \(X\) is \(T\)-connected.

Corollary: Let \((X, T_0)\) be a locally connected strong Baire t.s. and let \(T\) be a fine topology on \(X\) having the \(G^\delta\)-insertion
property. Then any open connected set is $T$-connected iff the $T$-closure of any open set $G$ is dense in the boundary of $G$.

Let us now consider that $(X,T)$ is an abs.$d$.$T$. finer than $(X,T_0)$. Let every $T$-open set $U$ which is a $T_0$-connected be also a $T$-connected. Then it is not difficult to show that for every pair of $T$-open and $T_0$-connected sets $U$ and $V$ such that $U \cap V = \emptyset$. We can have $U \cap (\overline{V})_{T_0} = \emptyset$. By proposition-5.4 [LMZ] the converse of this holds if both $T$ and $T_0$ are locally connected.

It is observed that all the above assertions includes that a $d$-space is connected iff it is connected w.r.t. a pre-topology. However any set in ordinary $d$.$T$. on $\mathbb{R}^n$ is $d$-connected if it can be represented as $\mathbb{R}$ (a first category set).

If the abstract density topology $(X,T)$ is also a lifted one then closure of every open set in $(X,T)$ is open. Hence the lifted topology is extremally disconnected. Thus both the topologies $G_\uparrow$ and $T_\uparrow$ discussed in 4.2. (Remark) are extremally disconnected. In fact the topology $T$ above is zero-dimensional if $T$ is regular. On the other hand there exists abs. $d$.$T$. which is not even $T_\uparrow$. Thus its lifted refinement may also be non-zero dimensional. Moreover, if $T$ is regular then the density space $(X,T)$ cannot admit a regular dispersion point ([KR]) (for definitions see [Kur] or [KR]).