CHAPTER - II

FINE TOPOLOGICAL MODIFICATIONS OF DENSITY TOPOLOGIES
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2.0. INTRODUCTION

In many aspects of density topological treatments fine modifications of density topologies are seen to be as useful as indispensible tools. Moreover as we shall come across these modifications frequently to take advantage of their roles during our exposition, we feel it worthwhile to present some of them for introduction which may not readily be available for readers in a compact form.

So, this chapter is devoted to identify and compile some of such modification topologies from dispersed literature. We hint their implications in aspects of density topologies and put our critical views without going deep or details of anyone. However for working satisfaction proofs of some results are presented in a comprehensive manner and also some results are extended. For completion a short study on \( \mathbb{R} \) with ordinary metrical density (2.1) and its category analogue (2.3) is introduced at the beginning.

2.1. DENSITY TOPOLOGY ON \( \mathbb{R} \)

2.1.0. In measure case, we recall that open sets in the d.T. \( D \) on \( \mathbb{R} \) are those Lebesgue measurable sets \( U \) from the complete \( \sigma \)-finite measure space \( (\mathbb{R}, \Sigma, \mu) \) such that for each \( x \in U \),

\[
\lim_{h \to 0} \frac{\lambda(U \cap (x-h, x+h))}{2h} = 1 \quad \text{(i)}
\]

\( \lambda \) being the Lebesgue measure.
In terms of §1.2 this means that
\[ D = \{ \Lambda \in \Sigma : \Lambda \subseteq \phi_{I,\lambda}(\Lambda) \}, \]
where \( \Sigma = \sigma \)-algebra of all Lebesgue measurable sets,
\[ \phi_{I,\lambda}(A) = \{ x \in \mathbb{R} : d(A, x) = 1 \}, A \in \Sigma \]
and \( \phi_{I,\lambda} : \Sigma \to \Sigma \) is a lower density with the properties that \( \lambda(\phi_{I,\lambda}(A) \Delta A) = 0 \), and \( I_{\lambda} \) denotes the \( \sigma \)-ideal of Lebesgue null-sets.

We note that such a mapping \( \phi_{I,\lambda} \) with the above property exists naturally by the most significant Lebesgue density theorem (cf. e.g. [Ox], [Saks]) a special version of which is proved below.

2.1.1 Theorem-1: Let \( E \in \Sigma \). If \( \lambda(E) > 0 \), then
\[ \lim_{h \to 0} \frac{\lambda(E \cap (x-h,x+h))}{2h} = \chi_{E}(x) \text{ a.e.} \]

Proof: Let \( F(x) = \int_{-\infty}^{x} \chi_{E} \, dt \). Since \( \chi_{E} \) is continuous on \( E \) and \( F \) is differentiable, \( F \) is finite and
\[ F'(t) = \chi_{E}(t) \text{ a.e.} \]

On the other hand, the given limit
\[ = \lim_{h \to 0} \frac{1}{2h} \{ F(x+h) - F(x-h) \} \]
But if \( h > 0 \), \[ \frac{1}{h} \{ F(x+h) - F(x) \} \to F'(x) \]
and \[ \frac{1}{h} \{ F(x-h) - F(x) \} \to -F'(x); \text{ a.e.} \]

On substraction, the result follows.

Note:1: As a consequence of this theorem, a set \( E \) is measurable iff, almost every point of \( E \) is a density point of \( E \). Hence measurability of \( E \) implies that of \( \phi_{I,\lambda}(E) \); and also \( \lambda(E \Delta \phi_{I,\lambda}(E)) = 0 \)
We shall write \( \phi(E) \) instead of \( \phi_{I,\lambda}(E) \) if there is no confusion.

Note:2: (i) If \( b \) is the density b.o.given by
b \Lambda = \{ x \in \mathbb{R} : d_\Lambda (E, x) > 0 \}, \text{ Then } D = T_b = T_\phi.

ii) If \Lambda \in \Sigma, \text{ then } \phi(\Lambda) \in D,

iii) If \Lambda \in \Sigma, \text{ then } (\phi(\Lambda))_D = \phi(\Lambda) \cup N \text{ where } N \in \mathcal{I}_\lambda \text{ and }

\phi(\Lambda) \text{ is the interior of } (\phi(\Lambda))_D

**Proof of (iii):** \(\phi(\Lambda) = \phi(\Lambda) \cup N\), where \(\phi(\Lambda)\) is open and \(N \in \mathcal{I}_\lambda\).

Let \(\phi(\Lambda_i) - N_i \in D\) s.t. \(\phi(\Lambda_i) - N_i \subseteq \phi(\Lambda)\).

Then \(\phi(\Lambda_i) - N_i \subseteq \phi(\Lambda_i) = \phi(\phi(\Lambda_i)) - N_i\)

\(\subseteq \phi(\phi(\Lambda) \cup N) = \phi(\Lambda)\).

\(\Rightarrow \phi(\Lambda) \text{ contains any such } \phi(\Lambda_i) - N_i\).

**Theorem-2:** (i) \(\phi(\Lambda) \subseteq (\Lambda)_D\), (ii) \((\bar{\Lambda})_D = \phi(\Lambda)_D\).

**Proof:** (i) We have \(\phi(\Lambda) \cap \phi(A^c) = \phi\).

Let \(x \notin (\bar{\Lambda})_D\). Then \(x \in (A^c)_D \subseteq \phi(A^c)\).

\(\therefore x \notin \phi(\Lambda), \text{ since } \phi(\Lambda) \cap \phi(A^c) = \phi\)

(ii) Easy and omitted.

**Theorem-3:** Every Borel subset of \(\mathbb{R}\) is a \(\mathcal{D}-\mathcal{G}_\delta\) set.

**Proof:** Let \(\Lambda\) be a \(D\)-Borel subset of \(\mathbb{R}\). Then \(\Lambda\) is measurable.

Hence given \(\varepsilon = \frac{1}{n}, (n=1,2,\ldots)\), there exists an usual open set \(G_n \supset \Lambda\) s.t.

\(\lambda(G_n \setminus \Lambda) < \frac{1}{n}\).

Define \(G = \bigcap_{i=1}^{\infty} G_n\). Then \(G\) is an usual \(\mathcal{G}_\delta\) set with \(G \supset \Lambda\) and \(\lambda(G \setminus \Lambda) \leq \lambda(G_n \setminus \Lambda) < \frac{1}{n}\) (\(\forall n \in \mathbb{N}\)). Let \(\lim_{n \to \infty} \lambda(G_n \setminus \Lambda) \geq 0\). If we let \(G_n \setminus (G \setminus \Lambda) = H_n\), then \(H_n \in D\) and \(\bigcap H_n = \Lambda\). Thus \(\Lambda\) is \(\mathcal{D}-\mathcal{G}_\delta\).

**Remark(1):** There exists a \(d\)-open set in \((\mathbb{R}, D)\) which is not \(\mathcal{G}_\delta\).

**Example:** Let \(\Delta = \{ x_n : n \in \mathbb{N} \}\) be any countable dense subset of the Cantor set \(C\). Let \(I = [0,1]\) and \(\Lambda_n = (I \setminus C) \cup \{x_n\}\). Then \(\Lambda_n\) is both \(\mathcal{P}_\sigma\) and \(\mathcal{G}_\delta\) in \(I\). But \(\Lambda = \bigcup_{n \in \mathbb{N}} \Lambda_n \in D\) and \(\Lambda = (I \setminus C) \cup \Delta\).
Hence \( I - C \) is \( D \)-open. Since \( \triangle \) is not \( G_\delta \), \( A \) cannot be \( G_\delta \) at least.

2) There exists a \( D \)-open set which is not \( F_\sigma \).

For example, the set of all irrationals in \( R \).

3) There exists a \( D \)-open set which is not Borel, since

Cardinality of the family of \( D \)-open sets = \( 2^c \) and cardinality of the family of \( D \)-Borel sets = \( c \).

4) Every \( d \)-Borel subset of \( R \) is in fact the intersection (or union) of an open \( F_\sigma \) and a closed \( G_\delta \). Also every regular open set is an Euclidean \( F_\sigma \) (cf[T]) (not true in general abs. d.T. space).

**Theorem-4:** A set \( A \subseteq R \) is regular \( D \)-open (i.e. \( A = (\overline{A})^o \)) iff \( A = \phi(E) \) for some \( E \in \Sigma \).

**Proof:** If \( A = (\overline{E})^o \) for some \( E \in \Sigma \), then \( \phi(E) = (\overline{\phi(E)})^o \) follows from Th-1 (Note-2 (iii)).

Conversely, let \( A = (\overline{E})^o \). Then \( A = \phi(E) \sim N \), for some \( E \in \Sigma \) and \( N \in I\lambda \).

Now \( A \Delta \phi(E) = [(\phi(E) \sim N) \cup \phi(E)] \sim [(\phi(E) \sim N) \cap \phi(E)]. \)

\( \subseteq N \) a nowhere dense set.

(\( \therefore \)) \( A \sim \phi(E) \) where both \( A \) and \( \phi(E) \) are regular open. \( \therefore \) \( A = \phi(E) \).

**Note:** The regular open algebra \( A \) of \( R \) can be characterised as open sets modulo first category sets ([Hal-1]). Therefore it is seen that \( A \) is the reduced measure algebra of the measurable sets modulo null-sets ([T]).

### 2.2. Properties of \((R,D)\):

We have seen many properties for \((R,D)\) in general d.T. space. Some of its specific and remarkable properties are [see also 3.1.1.].
(i) $T_u \subset D$ (a proper inclusion). So $D$ is $T_2$.

(ii) $(R,D)$ has a basis of cardinality $2^{\aleph_0}$ ([T]).

(iii) $(R,D)$ is neither separable nor Lindel"{o}f ([GW]).

(iv) It is C.R., non-normal. ([GN]) where every compact sets are finite ([Sc]).

(v) D-Borel sets are precisely the Lebesgue measurable sets and are $G_\delta$ ([Sc]).

(vi) Every subspace of $R$ is the union of a closed discrete subspace and a subspace satisfying the CCC ([T]).

(vii) Disjoint collection of open sets are countable.

(viii) $(R,D)$ is the coarsest topology relative to which every approximately continuous function is continuous and which is of Baire class-1, and also has the Darboux property ([GW]).

(ix) Every subinterval of $R$ is connected in $(R,D)$.

2.3. I-DENSITY TOPOLOGY

INTRODUCTION

The definition of a categorial density point was first given by Wilczynski in 1982 ([Wil]). The corresponding d-topology on $R$ called the I-d.T. was introduced by W. Poreda et al. ([PWW]) as a topology determined by a special lower density in the category sense. This topology were studied for its properties in some subsequent papers (cf. e.g. [Wil-2], [WA], [Wil-1], [PWW], [Laz], [WP] etc). Extension of these notions to plane sets and density were later investigated in [CW], [BLW] etc.). In [Kuz], [Wil-3] some topological analogues between measure and category of some well-known theorems such as Vitali, Lebesgue etc. with their applications were established.
I-density topology is anyway a generalisation of the ordinary d.T. to the setting of category in lieu of measure.

Original definition of I-density topology uses the algebraic structure of $\mathbb{R}$. By defining 'porosity', Zajicek in [Zj] and [Zj-1] showed that it is possible to generalise Wilczynski's definition of I-d.T. on $\mathbb{R}$ to an arbitrary metric space by alternative use of the notions of topology and porosity.

In this section and the next, we shall discuss on some fundamental notions of $I$ and deep-$I$ density topologies.

2.3.0. If $E$ is measurable, the following are equivalent ([Wil-1])

(i) $d(E,0) = 1$

(ii) $\lim_{h \to 0} \frac{\lambda(E \cap (-h,h))}{\lambda(-h,h)} = 1$

(iii) $\lim_{h \to 0} \frac{\lambda(E \cap (-h,h))}{2h} = 1$

(iv) $\lim_{n \to \infty} 2^{-l} n \lambda [E \cap (-\frac{1}{n}, \frac{1}{n})] = 1$ (putting $h = \frac{1}{n}$)

(v) $\lim_{n \to \infty} \mu(nE) \cap (-1,1) = 2$, where $nE = \{nx : x \in E\}$

Using these equivalences and a well-known theorem of Riesz, the following reformulation of the definition of an ordinary density point is obtained in [PWW] so as to derive the corresponding topology.

Theorem-1: For any Lebesgue measurable set $E$, the equation (i) in 2.1.0 has the following equivalences:

(i) $\chi_{nE} \cap (-1,1)$ converges to $\chi_{(-1,1)}$ in measure,

(ii) For any increasing sequence $\{n_m\}$ of natural numbers
there exists a subsequence \( \{n^p_n\}_n \subset \mathbb{N} \) such that
\[
\lim_{p \to \infty} \chi_{n^p_n \cdot E \cap (-1,1)} = \chi_{(-1,1)} \quad \text{a.e.}
\]

(iii) For any sequence \( \{t_n\}_n \subset \mathbb{N} \) of positive real numbers diverging to infinity, there exists a subsequence \( \{t^p_n\}_n \subset \mathbb{N} \) such that
\[
\lim_{p \to \infty} \chi_{t^p_n \cdot E \cap (-1,1)} = \chi_{(-1,1)} \quad \text{a.e. (cf. [CLO])}
\]

Wilczynski's observation in defining the categorial density follows from the fact that it was not essentially the measure function but the \( \sigma \)-algebra \( \Sigma \) of measurable sets and the \( \sigma \)-ideal \( I \) of \( \lambda \) null-sets that really matters for the definition of density. Hence on the same setting of the measurable space \( (X, \Sigma, J) \) (1.4.1.) with arbitrary \( \sigma \)-ideal \( J \) and considering \( \Sigma = \mathfrak{F} = \sigma \)-algebra of all the sets having the property of Baire and \( J = I = \sigma \)-ideal of the 1st category sets in \( \mathbb{R} \), the categorial analogue of I-d.T. is obtained. As Lebesgue measure is translation invariant the equivalences in Th-1 also holds good for any density point \( x_0 \in \mathbb{R} \).

**Definition-1:** \( x_0 \in \mathbb{R} \) is an I-density point of \( A \in \mathcal{B} \) iff
\[
\chi_{n \Delta \cap (-1,1)} \xrightarrow{n \to \infty} 1. \text{In this case we shall write } d_I(A, 0) = 1
\]

**Definition-2:** \( x_0 \in \mathbb{R} \) is an I-density point of \( A \in \mathcal{B} \) iff
\( \circ \) is an I-density point of \( A \sim x_0 = \{x-x_0 : x \in A\} \).

We say \( x_0 \in \mathbb{R} \) is an I-dispersion point of \( A \in \mathcal{B} \)
iff \( \chi_{n(A-x_0) \cap (-1,1)} \xrightarrow{n \to \infty} 0. \)

Equivalently \( x_0 \) is an I-density point of \( A^c \), i.e. \( d_I(A^c, x_0) = 1 \)

**Note:** (i) \( d_I(A, 0) = 1 \) iff \( \lim \inf_{n \to \infty} (n.A \cap (-1,1)) \) is residual in \( (-1,1) \).
(ii) \( d_I(A^c, o) = 1 \) iff \( \limsup_{n \to \infty} nA \cap (-1, 1) \) is of first category.

(iii) Right hand and left hand density points are defined as usual.

(iv) Condition (iii) of Th-1 above gives a n.a.s.c. for a point \( x_0 \in \mathbb{R} \) to be an I-density point of a set \( A \) ([PWW]). In [CLO] this point \( x_0 \in \mathbb{R} \) has been termed as strong I-density point of \( A \).

(v) Above definitions of I-density become equivalent to those of metrical density when \( S = \sum \mu \) of all measurable sets and substituting \( I^\mu \) for \( I \).

(vi) The justification of the definition of I-d.T. to pre-requisite the restriction of \( \sigma \) - algebra conveniently to the sets with Baire property and the \( \sigma \) - ideal to those of \( 1 \)-category sets can be seen in [Wil-1], [CL-5]. This set up makes I-d.T. an abs. categorial d.T.

(vii) A result in (Th-2 in [Wil-1]) shows that the notion of an I-density point really differs from that of a residual point.

On the other hand an interesting results in (Th-1, [WA]) makes it clear that there is no connection between density and I-density points or dispersion or I-dispersion points.

2.3.1. Theorem-1: Let \( \phi_I(A) = \{ x \in \mathbb{R} : d_I(A, x) = 1 \} \) for \( A \in \mathcal{B} \), and let \( A \sim B \) means that \( A \Delta B \in I \).

Then (i) \( \phi(\phi) = \phi ; \phi(\mathbb{R}) = \mathbb{R} \)

(ii) \( A \sim B \Rightarrow \phi(A) = \phi(B) \)

(iii) \( A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B) \).

(iv) \( A^c \subseteq \phi(A) \subseteq \overline{A} \) (interior, closures are w.r.t. natural topology \( T_u \))
(v) \( A \sim \phi(A) \) (analogous to \textit{Lebesgue density theorem})

(vi) \( \phi(A \cap B) = \phi(A) \cap \phi(B) \).

\textbf{Proof}: (i), (ii), (iii), (iv) are implied by definition and the theorem-1 (2.3.0).

(v) For every \( A \in \mathcal{B} \) consider \( R_A = \text{a regular open set relative to} \) \( A \) \text{ w.r.t.} \( T_u \), i.e. \( R_A = \overline{R}_A^o \).

If \( A \in \mathcal{B} \), then \( A = R_A \Delta P \) where \( P \in I \).

\( \Rightarrow A \sim R_A \) \text{... (1)} [The representation \( R_A \Delta P \) is unique in the Baire space \( R \)].

Now \( R_A \subseteq \phi(R_A) \subseteq \overline{R}_A = R_A \sim \phi(R_A) \) \text{... (2)}

(\therefore R_A \Delta \overline{R}_A \subseteq I, R_A \text{ is } T_u \text{-open}).

Also \( \phi_m(R_A) \sim \phi(A) \) \text{... (3)}

From (1), (2) and (3), \( A \sim \phi(A) \).

(vi) \( A \cap B \subseteq A \Rightarrow \phi(A \cap B) \subseteq \phi(A) \).

\( A \cap B \subseteq B \Rightarrow \phi(A \cap B) \subseteq \phi(B) \).

\( \phi(A \cap B) \subseteq \phi(A) \cap \phi(B) \).

Let \( x \in \phi(A) \cap \phi(B) \). So \( x \in \phi(A) \). Let \( \{n_m\}_{m \in \mathbb{N}} \) be an increasing sequence of natural numbers. Hence by Th-1 there exists a subsequence \( \{n_{m_p}\}_{p \in \mathbb{N}} \) such that

\( \chi(n_{m_p}) \) \( (A-x) \) \( \cap (-1,1) \overset{I}{\rightarrow} 1 \), I-a.e. Similarly as \( x \in \phi(B) \), there exists a subsequence \( \{n_{m'_q}\}_{q \in \mathbb{N}} \) of \( \{n_{m_p}\} \) such that

\( \chi(n_{m'_q}) \) \( (B-x) \) \( \cap (-1,1) \overset{I}{\rightarrow} 1 \), I-a.e. Both imply

\( \chi(n_{m_{pq}}) \) \( (A \cap B - x) \) \( \cap (-1,1) \overset{I}{\rightarrow} 1 \), I-a.e.

Thus \( x \in \phi(A \cap B) \).
The conditions (i), (ii), (v) and (vi) above in Th-1 shows that \( \phi_I: \mathcal{B} \rightarrow \mathcal{B} \) is a lower density and so as in 1.2.1., we let
\[
T_I = \{ A \in \mathcal{B} : A \subseteq \phi_I(A) \}.
\]

\( T_I \) as determined by \( \phi_I \) gives a topology (cf. [WP], Th-1) called as \( I \)-density topology (or Wilczynski Topology).

The only essential parts in proving \( T_I \) for topology are the following requirements. We note that finite intersection of Baire sets is Baire.

**Theorem-2:** (a) \( \phi_I \) is a lower density,

b) Arbitrary union of \( T_I \)-Baire sets is \( T_I \)-Baire.

**Proof:** We need to prove only (b).

Let \( \{ A_\alpha : \alpha \in \Lambda \} \subseteq T_I \). So \( A_\alpha \subseteq \mathcal{B} \), \( \forall \alpha \).

Let \( G = \bigcup A_\alpha \). Then \( \Lambda \subseteq \phi_I(A_\alpha) \subseteq \phi_I(G) \), \( \forall \alpha \)

\[ G \subseteq \phi_I(G) \]. We show \( G \in \mathcal{B} \).

Corresponding to every \( A_\alpha \), let \( R_{A_\alpha} \) denotes the regular open set in \((X, T_u)\). Consider

\[ C = \{ R_{A_\alpha} : \alpha \in \Lambda \} \], Let \( C_u = \bigcup R_{A_\alpha} \).

\( C \) is a cover of \( C_u \).

Let \( S = \{ R_{A_\alpha} : \alpha_n \in \Lambda, n \in \mathbb{N} \} \) be a countable subcover of \( C_u \).

Also let

\[ S_u = \bigcup_{n \in \mathbb{N}} R_{A_\alpha_n} \] \( \quad \) and \( B = \bigcup_{n \in \mathbb{N}} A_{\alpha_n} \) \( \in T_u \)

\( S_u \) and \( B \) are Baire and \( S_u \sim B \).

Now proceeding as in Th-2 (1.2.1) we have

\[ B = \bigcup_{\alpha \in \Lambda} A_\alpha \subseteq \bigcup_{\alpha \in \Lambda} (=G) \subseteq \bigcup_{\alpha \in \Lambda} \phi_I(A_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} \phi_I(R_{A_\alpha}) \]

\[ = \bigcup_{\alpha \in \Lambda} \phi_I(R_{A_\alpha}) \subseteq \bigcup_{\alpha \in \Lambda} \phi_I(C_u) = \phi_I(B). \]

Since \( B \in \mathcal{B} \) and \( B \Delta \phi_I(B) \in I \), therefore \( G \Delta B \in I \Rightarrow G \in \mathcal{B} \).
Definition of \( T_I \) is compatible with the following definitions.

**Definitions**

1. \( T_I = \{ \phi_I(A) \cap N : A \in \mathcal{B}, N \in I \} \).

2. A \( \subset R \) is a \( T_I \)-neighbourhood of a point \( x \in R \) iff \( x \in A \) and there exists a set \( B \in \mathcal{B} \) such that \( B \subset A \) and \( x \) is an \( I \)-density point of \( B \).

3. A set \( F \subset R \) is \( T_I \)-closed iff every point which does not belong to \( F \) is an \( I \)-dispersion point of \( F \).

2.3.2. Below we list some basic properties of \( T_I \),

[cf. eg. [Wil-1], [WP], etc.]

(i) \( T_I \) is stronger than \( T_U \); so is Haudorff.

(ii) Every \( T_I \)-Borel set has the Baire property.

(iii) Closed intervals are connected but not compact.

(iv) \( T_I \) is not a regular topology, since in \( Q \) (rationals) \( 0 \) and \( Q - \{0\} \) cannot be separated by sets from \( T_I \).

(v) It is not Lindelof and not separable.

(vi) A set \( A \subset R \) is closed and discrete w.r.t. \( T_I \) iff \( A \in I \).

(vii) A is \( T_I \)-compact iff \( A \) is finite.

(viii) Let \( A \in \mathcal{B} \). Then

1. \( A^\circ = A \cap \phi(A) \)
2. \( A = A \cup (\phi(A^C))^C \)
3. \( \phi(A) \subset A \).

(ix) The derivative of a set in \( T_I \)-topology has analogous properties (cf [WP]) as its derivative in the d.T. on \( R \) [See 3.2].

**Theorem-1**: Let \( A \subset R \), \( \phi(A) = \{ x \in R : d_I(A,x) = 1 \} \). Let \( M \) and \( K \) be the measurable cover and the measurable kernel of \( A \).

Then (i) \( \phi(A^\circ) = \phi(K^\circ) \) (ii) \( \phi(A) = \phi(M) \).
(Interiors, closures are w.r.t. $T_T$)

**Proof:** If $\lambda \in B$, results follow easily.

Let $A \in B$, then $K^o = K \cap \phi(K)$.

Also $K^o \subseteq A^o$. Let $x \in A^o$. Then there exists a set $U \in B$ such that $x \in U$, $U \subseteq A$ and $d(A, x) = 1$.

Then $U \cap K \in I$. Therefore $d_I(U \cap K, x) = 1$ and so $x \in \phi(K)$.

Then $A^o - K^o \subseteq \phi(K) - K \subseteq I \Rightarrow A^o \sim K^o$.

$I$. \begin{align*}
\phi(A^o) &= \bigcap \phi(K) \\
\phi(K^o) &= \bigcap \phi(K)
\end{align*}

(ii) is proved similarly by proceeding through complementation of $M$ and $A$.

**Note-(1):** It is not true however that if $A \in B$ then $\bar{A} - A^o \in I$.

For example there exists a non-Baire, non-measurable set $B$ (e.g. any Bernstein set, (cf.[Ox]) such that $B^o = \emptyset$.

(2) If $A \subseteq R$, then $(\bar{A})_{T_I} \in B$.

2.4. **DEEP-I-DENSITY TOPOLOGY:** Due to non-regularity of $(R, T_I)$, the topology $T_I$ is not the coarsest one w.r.t. which every approximately continuous function is continuous, - a result unlike to $(R, D)$. But it has been proved (cf.[PWW]) that if $f$ is an approximately continuous function, then for every open interval $(a, b) \subseteq R$, the set $f^{-1}((a, b))$ can be expressed as $GUZ$ where $G \in T_u$ and $Z \in I$. Hence the topology (cf.[WP]),

$T'_I = \{U \in T_I : U = GUZ\}$ is a natural trial for looking for such a coarsest topology on $R$. (compare $a_e$-topology in 2.6).

Here $T_u < T'_I < T_I$, and $T'_I$ is proved to be a connected separable Baire space where also compact sets are finite.
However $T'_I$ is not the coarsest topology. The coarsest such topology corresponding to the I-approximately continuous function was found in independent works of [Laz] and [WP]. It is called the deep-I-density topology, and is obtained by introducing some variations in the definition of I-density point.

**Definition-1:** A point $a \in R$ is said to be a deep-I-density point of $A \subseteq R$ iff there exists a closed set $F \subseteq A \cup \{a\}$ such that $a$ is an I-density point of $F$.

**Definition-2:** Let $T_d = \{A \in T_I : \text{every point } x \in A \text{ is a deep-I-density point of } A\}$. Then $T_d$ is a topology and this topology is called the deep-I-density topology on $R$.

In fact, it can be shown that $\phi_d(A) = \{x \in R : x$ is a deep-I-density pt. of $A\}$ induces this topology $T_d$, and that $T_d = \{A \in \mathcal{B} : A \subseteq \phi_d(A)\} \subseteq T_I$. Also for every $x \in R$, the set $U(x) = \{U \cup \{x\} : U \text{ is an open interval at } x \text{ and } x \in \phi_d(U)\}$ forms a base for $T_d$ at $x$.

Also $T_U < T_d < T'_I < T_I$.

$T_d$ is a non-normal, C.R, $T_2$-space. It is the required coarsest topology enquired for at the beginning.

### 2.5. FINE TOPOLOGICAL MODIFICATIONS*

By a fine topological modification in a t.s. $(X,D)$ we

*A portion of this section is included in our paper [Gos 5].
mean a t.s. \((X,T)\) such that \(T\) is comparable to \(D\) and \(T\) is obtained by modifying either \(D\) itself or a pretopology (initial topology) \(T_0\) on \(X\).

In some of the particular examples of the modification topologies we discuss below the standard topology \(T_u\) on \(\mathbb{R}^n\) will play the role for \(T_0\) above.

2.5.1. HASHIMOTO *-MODIFICATION ([Has]): Let \((X,T_0)\) be a \(T_1\)-space. A set \(A \subseteq X\) is said to have a property \(P\) (we write 'A has \(P\)') at \(p \in X\), if there exists a neighbourhood \(V\) of \(p\) such that \(V \cap A\) has \(P\). Let \(Z = \{A \subseteq X : A\) has \(P\}\).

Let \(A^* = \{x \in X : V \cap A \notin Z, \forall \text{nbd } V \text{ of } p\}\).

= Set of points at which \(A\) is not locally \(P\).

Assume that (i) \(Z\) is an ideal.

(ii) \(A \in Z \Leftrightarrow A \cap A^* = \emptyset \Leftrightarrow A^* = \emptyset\)

and (iii) \(\forall p \in X, \{p\} \in Z\).

Define *-closure of \(A\) as \(\overline{A}_* = A \cup A^*\). Then this closure operator \(\gamma : A \rightarrow \overline{A}_*\) gives a new topology called the *-topology or simply \(T^*\) w.r.t. the property \(P\), or w.r.t. the ideal \(Z\).

Theorem-1: \(A \subseteq X\) is *-open iff \(\overline{A^*} \cap N = \emptyset\), where \(G\) is \(T_0\)-open and \(N \in Z\).

Proof: Let \(A\) be *-open. So \(A^c\) is *-closed.

\[
A^c = \overline{A^c} = A^c \cup (A^c)^* = (A^c)^*
\]

\[
\Rightarrow A^c = (A^c)^* \cup (A^c^c)^*
\]
= (a closed set) U (a set in Z)

(Since \( A^C - (A^C)^* = \phi \Rightarrow A^C - (A^C)^* \in Z \), by (i))

\[ \therefore \ A \text{ is the difference of a } T_0 \text{-open set and a set } N \text{ belonging to } Z. \]

Conversely: Let \( A = G \setminus N \Rightarrow A^C = (G \cap N^C)^C = G^C \cup N \).

\[ \therefore (A^C)^* = (G \cup N)^* = (G^C)^* \subseteq G^C \subseteq A^C \]

\[ \Rightarrow A^C \text{ is } ^*-\text{closed} \Rightarrow A \text{ is } ^*-\text{open}. \]

We shall, in particular, be interested with defining the \(^*-\)Topology of a t.s. \((X, T_0)\) where \(Z\) is assumed to be the family having the property of 1st category.

**Definition-1:** Let \((X, T_0)\) be a t.s., \(Z\) be the \(\sigma\)-ideal of the \(1\)-category sets in \(X\). We define

\[ T^* = \{ G \setminus N : G \in T_0 \text{ and } N \text{ is a } T_0\text{-first category set} \} \]

Then \(T^*\) is called a \(^*-\)modification of \(T_0\) w.r.t. \(Z\).

Obviously, \(T^*\) is finer than \(T_0\). It can be seen that a subset \(F \subseteq R\) is \(^*-\)closed iff \(F = V \cup P\) where \(V\) is \(T_0\)-closed and \(P \in Z\).

2.5.2. In search of a weaker topology on \(R\) so that the Borel sets generated by it are exactly the Lebesgue measurable sets, Scheinberg ([Sc]) used the above definition - 1 (2.5.1.) of \(T^*\) taking \(T_u\) as base topology \((T_0)\) and \(I_\lambda\) for \(Z\). So an open set \(A\) for this topology (let \(T'\)) is of the form \(U \setminus Z\) where \(U\) is standard open and \(\lambda Z = 0\). This modification gives the weaker topology \(T'\) such that \(T_u \preceq T' \preceq D\).

\(T'\) has many properties common to those of the d.T.D. such as \(T'\)-compact sets are finite or the \(T'\) is translation
invariant with regard to continuity. Also \((\bar{A}^o)^T = A \iff A = (\bar{A}^o)^T_u\).

However, \(T'\) is not regular since there exists a countable set \(C = \{ \frac{1}{n}: n = 1, 2, 3, \ldots \}\) which is \(T'\)-closed discrete such that it cannot be separated from its limit point \(o\) where \(o \notin C\).

2.5.3. Following Hashimoto, Hayashi [Hay-1,2] and Samuels [Sam] established properties of \(*\)-modification of an initial topology \((T_o)\) under varied restriction on the conditions (e.g., (a) and (b) below) of the ideal set \(Z\). Hayashi abstracted the following condition (b) from the well-known property of \(\mathcal{U}\)-ideal of first category sets.

\[(a) \forall x \in X, \{x\} \in Z\]
\[(b) \text{If } \forall x \in E \subset X, \text{ there is a } T_o-\text{nbd. } N(x) \text{ of } x \text{ such that } E \cap N(x) \in Z, \text{ then } E \in Z.\]

However, that under general conditions (b) is not satisfied by many ideals in an arbitrary topological measure space was shown in [Sam]. Hence, as a consequence, the theorem-1 (2.5.1) which shows that the family of sets of the form \(G - N\) where \(G \in T_o\) and \(N \in Z\) need not form a topology in general. This negates a comment of Schienberg ([Sc], P.236) that his construction of \(T'\)-topology applies quite generally to measures on topological spaces. That the condition (b) is too restrictive to make all \(*\)-open sets of the form \(G - N\) can be seen in an example due to [Sam], constructed upon his topological (global) property.

Samuels [Sam] defined the \(*\)-topology in a more usual way without those conditions.
Definition: Let \((X, T_0)\) be a t.s. and \(Z\) be any ideal of \(X\). Then 
\[ T^* = T(Z, T_0) = \left\{ \bigcup \{ G \setminus Z \} : G \in T_0, Z \in Z \right\}, \]
is a topology on \(X\) and \(T^* \supset T_0\).

If \(P \in Z\) then \(P\) is \(T^*\)-closed-discrete, and condition (a) gives that \(T^*\) is \(T_1\).

He showed that 
\[ (E^d)_{T^*} = \{ x \in X : \forall x \in G \in T_0, G \setminus E \cap \{ x \} \notin Z \}. \]
which shows the equivalence of \(T^*\) with Hayashi and Martin [M-1]. Also \((E^d)_{T^*}\) is \(T_0\)-closed.

Properties of \(T^*\):

a) If \((X, T^*)\) is \(T_1\), every \(T^*\)-perfect set is \(T_0\)-closed.

b) If \(G \in Z \cap T_0\), every subset of \(G\) is \(T^*\)-open/closed.

c) \(T^* = T_0\), iff every member of \(Z\) is \(T_0\)-closed.

d) The class of all \(T_0\)-closed-discrete sets is the largest ideal such that \(T^* = T_0\).

e) \(A \subseteq X\) has the Baire property iff \(A = F^*_1 \cup F^*_2\), where \(F^*_1, F^*_2\) are closed sets and \(F^* \supseteq F^*_1 \cup F^*_2\).

f) \(A\) is measurable iff \(A\) is both \(F^*_0\) and \(G^*_0\) w.r.t. Lebesgue measure zero.

Theorem-1: If \((X, T)\) is a Baire space, then the Hashimoto *-modification, \(T^*\)- is an abstract d.T. on \(X\), if \(Z\) is an ideal of 1-category sets in \(X\).

Proof: By definition

\[ X^* = \{ x \in X : \forall x \notin Z, \forall \text{ nbd. } V \text{ of } x \}. \]

As \(X\) is Baire, every open subset is of second category. So \(X^* = X\).

From property (c) above, it is evident that in \((X, T^*)\)
every * 1-category set is *-n.d. and if \( A \subseteq Z \), then \( A = \emptyset \cup A = (a \text{ T-closed set}) \cup (a \text{ set belonging to } Z) \). So \( A \) is *-closed.

Moreover from property (e) and (f) above we have \( A \) is measurable iff \( A \) has the Baire property.

Hence \((X,T^*)\) is an abs.d.T.

**Cor - 1:** \( T^* \) is a category d.T. on \( X \).

**Cor - 2:** The Hashimoto *-modification \( T^* \) on \( \mathbb{R}^n \) is an abs. category d.T. space finer than the d.T. on \( \mathbb{R}^n \).

**Cor - 3:** Let \( T \) be an abs. d.T. on \( X \). \( T^* \) is a topology on \( X \) which is finer than \( T \) such that every non-empty \( T^* \)-open set contains a measurable set of positive \( \mu \)-measure. Then \( T^* \) is an abs.d.T. on \( X \).

**Theorem-2:** Let \((X,T)\) be d.T. and \( f \) be a real function on \( X \). Then \( f \) is \( T^* \)-continuous iff it is \( T \)-continuous ([Zj]).

2.5.4. K-modification of d.T.

Let \((R,D)\) be the d.T. on \( R \), \( T_u \) the standard topology on \( R \). So \( T_u \) < \( D \). Let \( K \subseteq R \) and \( D_K \) be the family of all subsets \( U \) of \( R \) such that

(i) \( U \) is a \( D \)-nbd of every point \( x \in K \cap U \).

and

(ii) \( U \) is a \( T_u \)-nbd of every point \( x \in U \setminus K \).

Then \( D_K \) is a topology on \( R \) and is called the K-modification of \( D \) w.r.t. \( T_u \).

It can be checked that \( T_u \) < \( D_K \) < \( D \).

**Theorem-1:** Let \( D_K \) be the K-modification of \( D \) on \((R,T_u)\). Then \( \overline{\overline{A}}_{D_K} = (\overline{\overline{A}} \setminus K) \cup A \) where \( \overline{\overline{A}} = (\overline{A})_{T_u} \) and \( A \subseteq R \).
Proof: Let \([\tilde{A} - K)UA\]_D = Q which is D-closed. To show that Q is the smallest \(D_K\)-closed set containing Q.

(i) It is required to show that for every \(x \in Q \cap K^C\) there exists a \(D_K\)-nbd contained in \(Q \cap K^C\).

\[x \notin Q \Rightarrow x \notin \tilde{A} - K \Rightarrow x \notin \tilde{A} \cap K^C \]

\[\Rightarrow x \in (\tilde{A})^C\] which is \(T_u\)-open.

Since \((\tilde{A})^C \subseteq Q^C\), so Q is \(D_K\) closed,

(ii) \[Q = (\tilde{A} - K)_D \cup \tilde{A} \subset (\tilde{A})_D D_K \cup (\tilde{A})_D D_K = (\tilde{A})_D D_K\]

\[\Rightarrow Q\ is\ the\ D_K\-closure.\]

2.5.5. Extended K-modification:

Let D be a fine topology on \((R, T_u)\) and \(D_o\) be another topology on R such that \(T_u < D_o < D\). For \(K \subset R\), let \(T_K\) denote the K-modification of D relative to \(D_o\). Then \(T_K\) is a fine topology on \((R, D_o)\)

\[T_u < D_o < T_K < D\ .\]

The closure property is defined similarly as Th-1 (2.5.4.) above where \(\tilde{A} = (\tilde{A})_D D_o\).

2.5.6. The U-topology

As an example of a maximal enlargement of \(d.T\) on R, this topology is produced by Scheinberg (cf. [Sc] for details).

The topology \(U\) is finer than the density topology and so Hausdorff. In fact \(U\) is a lifted topology (Chapter - IV).

The topology \(U\) has many special properties that any bounded measurable function is equal a.e. to a uniquely
associated U-continuous function and an application to an easy solution of the problem of existence of a lifting.

2.5.7. $r$-Modification: (O'Malley topology)

In his study of approximately continuous (app.c) function and to find the coarsest topology relative to which some subclass of app.c. functions may become continuous, R.O'Malley presented two topologies called as '$r$' and '$a_\mathcal{E}'' topologies. They are modifications of density topology on $R$.

**Definition:** [Om-1] Let $(R, D)$ be the d.T., and

$$B = \{ U \subseteq R : U \text{ is } D\text{-open and } U \text{ is both } F_\sigma \text{ and } G_\delta \}.$$  

The topology generated by $B$ as its basis is called the $r$-modification (or simply the $r$-topology) of the ordinary density topology.

It is denoted by '$r$'. Every $r$-open set is $D$-open.

**Definition-2:** A set $A \subseteq R$ is called ambivalent if it is both $F_\sigma$ and $G_\delta$.

All $r$-open sets are not ambivalent. (Prop-3.2 [Om]) and (Unlike d.T.) any set of measure zero need not be $r$-closed (e.g. '$Q$'). However if $X$ is closed, $\lambda(X) = 0$, then $X$ has no $r$-limit point. Every subset of $X$ is closed. Also if $G$ is $r$-open and if $d(G, x_0) = 1$, then $G \cup \{x_0\}$ is $r$-open.

2.5.8. Almost everywhere modification: Let $(R, D)$ be the d.T. on $R$. 

Definition: A set U is said to be almost open if U is $D$-open and $\lambda(U) = \lambda(U^\circ)_{Tu}$.

The collection of almost open sets in R forms a topology, called as a.e. topology on R. We denote it by a.e. topology.

In fact the a.e. open subsets of R are those which are of the form $EUZ$, where E is $T_u$-open and every point of Z is a density point of E. We show by the following theorem a method of construction of the a.e.-modification of density topology, which can be extended to $\mathbb{R}^n$.

Theorem: Let D be the d.T. on $(\mathbb{R}, \Sigma, \lambda)$ and

$$D_1 = \{U \subseteq \mathbb{R} : U \in D, U = G \cup N \text{ where } G \in T_u, \lambda(N) = 0\}.$$ 

Then $D_1 = a_e$ and a.e. is a topology on R.

Proof: $a_e = \{U \subseteq \mathbb{R} : U \in D, \lambda(U) = \lambda(U^\circ)_{Tu}\}.$

That $D_1 = a_e$ is evident.

Let $U_1, U_2 \in a_e$. Then $U_1, U_2 \in D$. So, $U_1 \cap U_2 \in D$.

Now $U_1 = U_{Tu} U_{TN_1} = G_1 U Z_1$ (Say)

and $U_2 = U_{Tu} U_{TN_2} = G_2 U Z_2$ (Say)

$$U_1 \cap U_2 = (G_1 U Z_1) \cap (G_2 U Z_2)$$

$$= (G_1 \cap (G_2 U Z_2)) \cup (Z_1 \cap (G_2 U Z_2))$$

$$\subseteq (G_1 \cap G_2) \cup (Z_1 U Z_2).$$

$$\Rightarrow U_1 \cap U_2 \subseteq G_1 \cap G_2 \subseteq Z_1 U Z_2.$$

$$\Rightarrow \lambda(U_1 \cap U_2) \leq \lambda(Z_1 U Z_2) = 0$$

$$\Rightarrow \lambda(U_1 \cap U_2)^\circ = \lambda(U_1 \cap U_2)^\circ_{Tu}.$$ 

$$\Rightarrow G_1 \cap G_2 \subseteq T_u \text{ and } G_1 \cap G_2 \subseteq (U_1 \cap U_2)^\circ.$$
Next let $\forall i \in \mathcal{A}$, $U_i \in \mathcal{A}$ such that $\lambda(U_i) = \lambda(U_i^0)$

$\Rightarrow U_i = G_i \cup N_i$ for some $G_i, N_i$.

Let $U = \bigcup_i U_i$ and $G = \bigcup_i G_i$. Then $U \in \mathcal{D}$, $G \in \mathcal{T}_U$.

Let $N = \bigcup_i N_i \sim G \ (\in \mathcal{F})$ ...(i)

If possible let $\lambda(N) > 0$ $\Rightarrow N_D^0 \neq \emptyset$.

We have by (i)

$$N_D^0 \cap G_i = \emptyset \ \forall i \Rightarrow \lambda(N_D^0 \cap G_i) = 0 \ \forall i$$

But for some $i = j$, $U_j \cap N_D = \emptyset$

$\Rightarrow (G_j \cup N_j) \cap N_D = \emptyset$

$\Rightarrow (G_j \cap N_D^0) \cup (N_j \cap N_D^0) \neq \emptyset$

$\Rightarrow G_j \cap N_D^0 \neq \emptyset$, contradicting (ii)

$\therefore \lambda(N) = 0 \Rightarrow \bigcup_i U_i \in \mathcal{A}$.

Proposition 1 [LMZ] Let $T$ be the a.e. modification of the d.T. $(\mathbb{R}^n, D)$. Then

a) $T$ has the Lusin-Menchoff property (cf. 4.2.0) on every subspace of $\mathbb{R}^n$.

b) A function $f$ is $T$-continuous iff $f$ is approximately continuous (A.C.) everywhere and almost continuous everywhere. Also $T$ is the weak topology induced by the family of all A.C. functions which are continuous a.e.

c) $(\mathbb{R}^n, T)$ is a strong Baire space.

d) Any $T$-Baire one function is continuous at all points of a residual set).

2.5.9. The following relation defined in [Tod] is very useful to derive many pre-topological properties.
Definition: Let $D$ be a topology finer than $T$ on $X$. $D$ and $T$ are said to be $S$-related if for every $A \subseteq X$,

$$A^o_D \not\subset \emptyset \iff A^o_T \not\subset \emptyset.$$ 

We denote it by $D \mathcal{S} T$.

$a_e$ and $r$-topologies are both $S$-related to $T_u$ on $R$ [Om].

Following are some consequences of the definition.

Result: Let $D \mathcal{S} T$ on $X$, and $D \succ T$. Then

(a) Following are equivalent:

(i) $D \mathcal{S} T$, (ii) $A \subseteq X$ is $D$-dense (or $D$-nowhere dense) iff $A \subseteq X$ is $T$-dense ($\pi \in \mathfrak{P}$, $T$-nowhere dense),

(b) $A^o_D \not\subset A^o_T$ is both $D$ and $T$ nowhere dense,

(c) $(X,D)$ is a Baire space iff $(X,T)$ is a Baire space.

d) $A \subseteq X$ has the $D$-Baire property iff $A$ has the $T$-Baire property.

Theorem: Let $(X,T)$ be a Baire space and $D$ be a topology on $X$, such that

1) $D$ is finer than $T$ (ii) $D \mathcal{S} T$.

Then $D$ is an abs. $d$-$T$ on $X$ iff $T$ is an abs. $d$-$T$ on $X$.

Proof: This follows from above definition and the theorem-3 (1.2.1.).

Proposition: Let the a.e-modification of the d.$T$ on $R$ be $a_e$. Then the family $a_e^* = \{G - Z : G \in a_e, Z \text{ is a 1-category set} \}$ is an abstract d.$T$ on $R$.

Proof: From Th-1 it easily follows that $T^*$ is an abs. d.$T$. iff $T^*$ is a Baire space. Density topology on $R$ is Baire. This implies that $a_e$ is also Baire. Then proceeding similarly to the
Theorem-1 (2.5.3) the result follows.

Remark: Proposition-1 is valid if $a_e$ is replaced by $r$.

A $*$-topology helps to construct a general category $d.T.$ in the following way:

**Proposition 2:** Let $T \sim D$, on $X$. $(X,T)$ is a Baire space. Then $(X,D^*)$ is a category $d.T.$ on $X$ where

$$D^* = \{G \sim Z : G \in D, Z \text{ is a } T\text{-first category set} \}$$

**Proof:** Easily follows from the definition and above Th-1.

2.5.10 Nishiura in [N1] used the equivalent form of O'Malley's modification on $R^N$ while investigating for a topology in his study of a.e. continuous and approximately continuous functions.

The $a_e$-modification has been generalised in [LMZ] in the following form.

Let $(X,D)$ be a fine topological space on a pre-topology $(X,T_o)$. Consider

$$B=\{U: U \in D, U=GU\{x\}, \text{ where } G \in T_o \text{ and } x \in X\}$$

Then the topology formed by $B$ together with $\emptyset$ as the basis for a new topology is also termed as the a.e.modification of $D$ or in short a.e. topology (denoted by $a_e$). As before $T_o < a_e < D$.

2.5.11. A generalisation of a.e. topology is being studied by Grande, E. ([Gr], 1984) on a t.s. $(X,T)$ with a complete finite measure space $(X,M \mu)$ where $T \subset M$. If $\mathcal{F}$ be a lifting of the algebra of all the real bounded measurable functions on $X$, let

$$T_1 = \{\mathcal{F}(A) - N : A \in M, \mu N = 0\}$$

be the Oxtoby topology
(cf. section 2.1.) where $\chi_\rho(\Lambda) = \rho(\chi_\Lambda)$ for the corresponding characteristic function. With this he proved that

1) $T_2 = \{A \in T_1 : \mu A = \mu(A^c)\}$ is a topology.

2) A function $f : \mathbb{X} \to \mathbb{R}$ is $T_2$-continuous iff it is $T_1$-continuous, and the set of all the points where $f$ is not $T$-continuous has measure zero.

2.5.12 Porosity and $\sigma$-Porosity: The notions of porosity seems to be widely used for applications in differentiation theory.

The $\sigma$-porous topology on a metric space $(X, \rho)$ was introduced by Zajicek in $[Z_j]$ and $[Z_j-1]$ as an alternative definition of I-d.T. It was shown that $T_1$ is the $\sigma$-modification of a topology called the $\rho$-topology and is found useful in developing intuition about the I-d.T.

The details (cf. [Gos-5]) we omit but simply quote the following results.

**Theorem-1:** ([Z_j-1]) The $\rho^*$-topology on $\mathbb{R}$ is exactly the I-density topology on $\mathbb{R}$.

**Proposition-1:** The porosity topology $P$ on $X$ is $S$-related to the initial metric topology $\rho$ on $(X, \rho)$.

**Proposition-2:** If $(X, \rho)$ is a Baire space then a real function $f$ on $X$ is $X^*$-continuous iff it is $\rho$-continuous on $X$. 