Appendices and bibliography

Appendix A:

According to the First Law of Thermodynamics, if a quantity of heat \(dQ\) is added to a volume during an elementary time \(dt\) and this increases its internal energy by an amount \(dE_I\) and performs work \(dW\) then [Schlichting (1968)]

\[
dQ/dt = dE_I/dt + dW/dt \tag{A-1}
\]

The amount of heat added by conduction in the x-direction during time \(dt\) to a volume \(\Delta V\) can be written as:

\[
dQ = dt \Delta V \left\{ \left( \partial / \partial x \right) (k \partial T / \partial x) + \left( \partial / \partial y \right) (k \partial T / \partial y) + \left( \partial / \partial z \right) (k \partial T / \partial z) \right\} \tag{A-2}
\]

The surface force \(\vec{P}\) can be written in terms of normal stresses \((\sigma_x, \sigma_y, \sigma_z)\) and shearing stresses \((\tau_{xy}, \tau_{yz}, \tau_{zx})\) in the form

\[
\vec{P} = \partial \vec{P}_x / \partial x + \partial \vec{P}_y / \partial y + \partial \vec{P}_z / \partial z
\]

\[
= (\partial \sigma_x / \partial x + \partial \tau_{xy} / \partial y + \partial \tau_{xz} / \partial z) \hat{i} +
\]

\[
+ (\partial \tau_{xy} / \partial x + \partial \sigma_y / \partial y + \partial \tau_{yz} / \partial z) \hat{j} +
\]

\[
+ (\partial \tau_{xz} / \partial x + \partial \sigma_z / \partial z + \partial \tau_{zx} / \partial y) \hat{k} \tag{A-3}
\]
where the stress tensor couple stress fluid is

\[ T_{ij} = -p \delta_{ij} + \mu (v_{ij} + v_{ji}) + \eta \nabla^2 (v_{ij} - v_{ji}) \]

(A-4)

and the equation of motion is

\[ \rho \left( \frac{d \mathbf{w}}{dt} \right) = \mathbf{F}, \quad \rho \text{ is density} \]

(A-5)

The change in total energy is therefore

\[ \frac{dE}{dt} = \rho \Delta V \left\{ \frac{d}{dt} \left( \frac{1}{2} (u^2 + v^2 + w^2) \right) \right\} \]

(A-6)

Using equations (A-3) and (A-5) in equation (A-6), we obtain

\[
\begin{align*}
\frac{dE}{dt} &= \Delta V \left\{ \rho \frac{d}{dt} \left( \frac{1}{2} \sigma_x \frac{\partial}{\partial x} + \frac{1}{2} \sigma_y \frac{\partial}{\partial y} + \frac{1}{2} \sigma_z \frac{\partial}{\partial z} \right) \\
&\quad + v \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \\
&\quad + w \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \right\} \\
&= \Delta V \left\{ \frac{1}{2} \left( \frac{\partial}{\partial x} \left( \rho \frac{du}{dt} \right) + \frac{\partial}{\partial y} \left( \rho \frac{dv}{dt} \right) + \frac{\partial}{\partial z} \left( \rho \frac{dw}{dt} \right) \right) \\
&\quad + v \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \\
&\quad + w \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \right\} \\
&= \Delta V \left\{ \frac{1}{2} \left( \frac{\partial}{\partial x} (\rho u \frac{du}{dt}) + \frac{\partial}{\partial y} (\rho u \frac{dv}{dt}) + \frac{\partial}{\partial z} (\rho u \frac{dw}{dt}) \right) \\
&\quad + v \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \\
&\quad + w \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \right\}
\end{align*}
\]

(A-7)

The total work performed by the normal and shearing stresses per unit time can be written as
\[ \frac{dW}{dt} = - \nabla \left\{ \left( \frac{\partial}{\partial x} (u \sigma_x + v \tau_{xy} + w \tau_{zx}) + \left( \frac{\partial}{\partial y} (u \tau_{zx} + v \tau_{xy} + w \sigma_x) \right) \right\} \]  

(A-8)

Using equation (A-4) and substituting equations (A-2), (A-6) and (A-8) in equation (A-1), we obtain the energy equation for couple stress fluid model in tensor notation as follows:

\[ \rho c_p \left( \frac{\partial T}{\partial t} + u^i T_{,i} \right) - (k T_{,i})_{,i} = \mu \left( u^i_{,j} + u^j_{,i} \right) + \eta \left( u^i_{,j} - u^j_{,i} \right) \nabla^2 \left( u^i_{,j} - u^j_{,i} \right) \]  

(A-9)

where \( T \) and \( k \) are the temperature and thermal conductivity, \( \mu \) and \( \eta \) the viscosity-coefficient and couple stress parameter, \( u_i \) the fluid velocity components, \( c_p \) the specific heat of constant pressure, \( \rho \) the fluid density.

We consider the dimensionless scheme

\[ \varepsilon_1 = \frac{h_o}{l_o} \ll 1 \text{ and } \varepsilon_2 = \frac{h_o}{d_o} \ll 1 \]

(lubrication approximation), where \( l_o, h_o \) and \( d_o \) are scales of length, height and breath (along x, y and z direction) respectively and introduce other dimensionless coordinates \( X, Y \) and \( Z \) by

\[ x = l_o X, \quad y = h_o Y = \varepsilon_1 l_o Y \quad \text{and} \]
\[ z = d_o Z = \left( \frac{1}{\varepsilon_2} \right) h_o Z = \left( \frac{\varepsilon_1}{\varepsilon_2} \right) l_o Z = \left( \frac{1}{\lambda} \right) l_o Z \]
and fluid velocities

\[ u = V_0 U, \quad v = V_0 V = V_0 \varepsilon_1 U, \quad w = V_0 W = (1/\lambda)V_0 U, \]

where \( V_0 \) is the velocity scale in equation (A-9), we obtain three dimensional Cartesian form of energy equation with constant \( k \) and prominent terms of order \( \mu (\varepsilon_1^{-2}) \) and \( \eta(\varepsilon_1^{-3}) \) as follows:

\[
k(\partial^2 T/\partial y^2) = -\mu \left[ (\partial u/\partial y)^2 + (\partial w/\partial y)^2 \right] + \eta \left[ (\partial u/\partial y)(\partial^3 u/\partial y^3) + (\partial w/\partial y)(\partial^3 w/\partial y^3) \right] \quad (A-10)
\]

In two-dimensional form, \((1/\lambda) \rightarrow 0\) then the equation (A-10) reduces to

\[
k(\partial^2 T/\partial y^2) = -\mu (\partial u/\partial y)^2 + \eta(\partial u/\partial y)(\partial^3 u/\partial y^3) \quad (A-11)
\]