Chapter 5

FUZZY SEMI INNER PRODUCT OF FUZZY POINTS*

5.0 INTRODUCTION

In this chapter we extend the idea of fuzzy semi inner product space of crisp points to that of fuzzy points. The notion of orthogonality on the fuzzy semi inner product of fuzzy points is introduced. Some of its properties are studied. Also the concepts like fuzzy numerical range of 'fuzzy linear maps' on the set of fuzzy points is introduced and some results are obtained.

5.1 FUZZY SEMI INNER PRODUCT OF FUZZY POINTS

Definition 5.1.1 [WO3]

Let X be a set, then a fuzzy subset \( x_\lambda \), where \( \lambda \in (0,1] \) is

* Some results contained in this chapter have been included in a paper communicated for publication in the International Journal for Fuzzy Sets and Systems.
called a fuzzy point on $X$

if

$$x_{\lambda}(y) = \lambda, \text{ if } y = x$$

$$= 0, \text{ otherwise}$$

**Note 5.1.2**

Let $X$ be a $C'(I)$ module. If $\alpha \in C'(I)$ and $x_{\lambda} \& y_{\mu}$ be two fuzzy points on $X$ then

(a) $\alpha x_{\lambda}$ is defined as the fuzzy point $(\alpha x)_{\lambda}$.

(b) $x_{\lambda} \& y_{\mu}$ is the fuzzy point which takes the value $
\lambda \land \mu$ at $x+y$

(c) The set of all fuzzy points on $X$ is denoted by $\hat{X}$

**Definition 5.1.3**

Let $X$ be a $C'(I)$ module. A fuzzy semi inner product of fuzzy points on $X$ is a function

$*: \hat{X} \times \hat{X} \rightarrow C'(I)$ satisfying the following conditions

(i) $*$ is a linear in first argument

$$i.e \ (x_{\lambda} \& y_{\mu}) \ast z_{\psi} = x_{\lambda} \ast z_{\psi} + y_{\mu} \ast z_{\psi} \text{ and}$$
(\alpha x_\lambda) \ast z_\psi = \alpha (x_\lambda \ast z_\psi)

where \ x_\lambda, y_\mu and z_\psi \in \hat{X} & \alpha \in C'(I).

(ii) \ x_\lambda \ast x_\lambda > 0, \ \forall x_\lambda \neq 0 \beta ,\ \text{where} \ \beta \in (0,1]

(iii) \ [x_\lambda \ast y_\mu]^2 \leq [x_\lambda \ast x_\lambda] [y_\mu \ast y_\mu], \ \text{where} \ x_\lambda, y_\mu and z_\psi \in \hat{X},

\text{then} \ <X,*> \ \text{is called a fuzzy semi inner product space of fuzzy points.}

Note 5.1.4

As \hat{X} is not a fuzzy linear space the term 'fuzzy semi inner product' used above is not in the usual sense.

Theorem 5.1.5

Let <X,*> be a fuzzy semi inner product space of fuzzy points. Then treating X as a real vector space, the function \|\|:\hat{X} \rightarrow R^*(I) defined by \|x_\lambda\| = [x_\lambda \ast x_\lambda]^{1/2} is a fuzzy norm on \hat{X}. 
Proof:

Similar to the proof of 4.1.3.

Note 5.1.6

Let \( \langle X, \ast \rangle \) be a fuzzy semi inner product space of fuzzy points. If \( \parallel \parallel \) is the fuzzy norm generated from the fuzzy semi inner product \( \ast \) of fuzzy points. Then the fuzzy semi inner product space of fuzzy points is denoted by \( \langle X, \ast, \parallel \parallel \rangle \).

Definition 5.1.7

Let \( \langle X, \ast, \parallel \parallel \rangle \) be a fuzzy semi inner product space of fuzzy points. Let \( x_\alpha, y_\beta \in X \), then \( x_\alpha \) is said to be orthogonal to \( y_\beta \) (denoted by \( x_\alpha \perp y_\beta \)) or \( y_\beta \) is transversal to \( x_\alpha \) if \( y_\beta \ast x_\alpha = 0 \).

Proposition 5.1.8

Let \( x_\alpha, y_\beta \) and \( z_\gamma \) be three fuzzy points in \( \langle X, \ast, \parallel \parallel \rangle \) such
that \( x_\alpha \perp y_\beta \) and \( x_\alpha \perp z_\gamma \) then \( x_\alpha \perp (a y_\beta \oplus b z_\gamma) \) for every \( a, b \in C'(I) \).

Proof:

Given \( y_\beta \ast x_\alpha = \tilde{0} \) & \( z_\gamma \ast x_\alpha = \tilde{0} \)

then \( (a y_\beta \oplus b z_\gamma) \ast x_\alpha = (a y_\beta) \ast x_\alpha + (b z_\gamma) \ast x_\alpha \)

\[ = a(y_\beta \ast x_\alpha) + b(z_\gamma \ast x_\alpha) \]

\[ = a \tilde{0} + b \tilde{0} = \tilde{0} \]

ie \( x_\alpha \perp (a y_\beta \oplus b z_\gamma). \)

Proposition 5.1.9

Let \( \langle X, \ast, \| \| \rangle \) be a fuzzy semi inner product space of fuzzy points. If \( x_\alpha \perp y_\beta \) then

\[ \| x_\alpha \ast a y_\beta \| \geq \| x_\alpha \| \text{ for every } a \in C'(I). \]

Proof:

Given \( x_\alpha \perp y_\beta \) \( \text{ ie } y_\beta \ast x_\alpha = \tilde{0}. \)

Consider
\[(x_\alpha \otimes ay_\beta) ^* x_\alpha \leq [(x_\alpha \otimes ay_\beta) ^* (x_\alpha \otimes ay_\beta)]^{1/2} x [x_\alpha ^* x_\alpha]^{1/2}\]

\[x_\alpha ^* x_\alpha \otimes ay_\beta ^* x_\alpha \leq \|x_\alpha \otimes ay_\beta\| \|x_\alpha\|\]

\[\|x_\alpha\|^2 \leq \|x_\alpha \otimes ay_\beta\| \|x_\alpha\|\]

\[\|x_\alpha\| \leq \|x_\alpha \otimes ay_\beta\| \text{ for every } a \in C'(I).\]

**Definition 5.1.10**

Let \(\langle X, *, _1, \|_1 \rangle\) and \(\langle Y, *, _2, \|_2 \rangle\) be two fuzzy semi inner product spaces of fuzzy points. Then the function \(f: \hat{X} \rightarrow \hat{Y}\) is called a `fuzzy linear map' if \(f(x_\lambda \otimes z_\gamma) = f(x_\lambda) \otimes f(z_\gamma)\) and \(f(\alpha x_\lambda) = \alpha f(x_\lambda)\) for every \(x_\lambda, z_\gamma \in \hat{X}\) and \(\alpha \in C'(I).\)

**Definition 5.1.11**

Let \(\langle X, *, \| \rangle\) be a fuzzy semi inner product space of fuzzy points. The function \(f: \hat{X} \rightarrow C'(I)\) is called a `fuzzy linear functional' if \(f(x_\lambda \otimes y_\mu) = f(x_\lambda) + f(y_\mu)\) and \(f(\alpha x_\lambda) = \alpha f(x_\lambda)\) for every \(x_\lambda, y_\mu \in \hat{X}\) and \(\alpha \in C'(I).\)
Definition 5.1.12

Let $T$ be a 'fuzzy linear map' on $<X,*,\|\|>$. Then $T$ is said to be bounded if there exists a $k \in \mathbb{R}^*(I)$ such that $\|Tx_\lambda\| \leq k\|x_\lambda\|$ for every $x_\lambda \in \hat{X}$ in this case we define

$$\|T\| = \inf \{k \in \mathbb{R}^*(I) | \|Tx_\lambda\| \leq k\|x_\lambda\|\}.$$ 

5.2 FUZZY NUMERICAL RANGE, WEAK LIMITS AND 'FUZZY LINEAR FUNCTIONALS'

Definition 5.2.1

Let $T$ be a 'fuzzy linear map' on $<X,*,\|\|>$ then by the fuzzy numerical range of $T$, denoted by $w(T)$ we mean the set

$$w(T) = \left\{Tx_\lambda * x_\lambda | \|x_\lambda\| = \bar{1}\right\}$$

and

$$[w(T)] = \sup_{x_\lambda \in \hat{X}} \left\{[Tx_\lambda * x_\lambda] | \|x_\lambda\| = \bar{1}\right\}.$$
Proposition 5.2.2

Let \( <x,*,||\cdot||> \) be a fuzzy semi inner product space of fuzzy points. Let \( T \) & \( T' \) be two fuzzy bounded linear maps on \( \hat{X} \) then

(i) \( [w(T)] \leq \|T\| \)

(ii) \( [w(T+T')] \leq [w(T)] \odot [w(T')] \)

Proof:

(i) \( w(T) = \left\{Tx_\lambda \ast x_\lambda : \|x_\lambda\| = 1 \right\} \)

\[
[w(T)] = \sup_{x_\lambda \in X} \left\{ [Tx_\lambda \ast x_\lambda] : \|x_\lambda\| = 1 \right\}
\]

\[
= \sup_{\|x_\lambda\| = 1} \left\{ \|Tx_\lambda\| \ast \|x_\lambda\| \right\}
\]

\[
= \sup_{\|x_\lambda\| = 1} \|Tx_\lambda\| \leq \|T\|
\]

i.e. \( [w(T)] \leq \|T\| \)
\[(ii) \quad \|w(T+T')\| = \sup_{x_\lambda \in X} \left\{ \|(T+T')x_\lambda \ast x_\lambda\| \mid \|x_\lambda\| = 1 \right\} \]

\[= \sup_{x_\lambda \in X} \left\{ \|(Tx_\lambda + T'x_\lambda) \ast x_\lambda\| \mid \|x_\lambda\| = 1 \right\} \]

\[\leq \sup_{x_\lambda \in X} \left\{ \|(Tx_\lambda \ast x_\lambda)\| \mid \|x_\lambda\| = 1 \right\} \ast \sup_{x_\lambda \in X} \left\{ \|(T'x_\lambda \ast x_\lambda)\| \mid \|x_\lambda\| = 1 \right\} \]

\[\text{i.e.} \quad \|w(T+T')\| \leq \|w(T)\| \ast \|w(T')\| \]

**Definition 5.2.3**

A fuzzy semi inner product space of fuzzy points \(\langle X, \ast, \| \| \rangle\) is said to be strictly convex if \(\|x_\lambda \ast y_\mu\| = \|x_\lambda\| \ast \|y_\mu\|\) then \(\gamma_\mu = [\alpha]x_\lambda\), where \(x_\lambda \neq 0 \neq y_\mu\) and \(\alpha \in C'(I)\).

**Definition 5.2.4**

A sequence \(y_n\) in \(\langle X, \ast, \| \| \rangle\) is said to converge weakly in \(\mu_n\).
the second component to \( y \psi \) if \( x \lambda \cdot y_n \mu_n \) converges to \( x \lambda \cdot y \psi \mu_n \) for all \( x \lambda \in \hat{X} \).

**Proposition 5.2.5**

Let \( <X, *, \| \|> \) be a strictly convex fuzzy semi inner product of fuzzy points. Then the weak limit in the case of weak convergence with respect to the second component of the fuzzy semi inner product in \( \hat{X} \) is unique.

**Proof:**

Let \( y_\mu \) and \( y_\psi' \) be two weak limits of the sequence \( y_n \mid n \in \mu_n \) \( <X, *, \| \|> \). Then

\[
x \lambda \cdot y_\mu = x \lambda \cdot y_\psi' \quad \forall x \lambda \in \hat{X}.
\]

Let \( x \lambda = y_\mu \) then \( y_\mu \cdot y_\mu = y_\mu \cdot y_\psi' \)

\[
\text{ie } [y_\mu \cdot y_\mu] = [y_\mu \cdot y_\psi']
\]

\[
\text{ie } \|y_\mu\|^2 \leq \|y_\mu\| \cdot \|y_\psi'\|
\]
ie $\|y_\mu\| \leq \|y'_\psi\|

similarly taking $x_\psi = y'_\psi$ we get

$\|y'_\psi\| \leq \|y_\mu\|$

ie $\|y_\mu\| = \|y'_\psi\| \quad (1)$

also $\|y_\mu\|^2 = \|y_\mu\| \cdot \|y'_\psi\|

ie $[y_\mu * y'_\psi] = \|y_\mu\| \cdot \|y'_\psi\|

$\rightarrow y_\mu = [\alpha] y'_\psi$

ie $y_\mu = y'_\psi$ by (1).

Proposition 5.2.6

Let $<X,*,\|\|>$ be a fuzzy semi inner product space of fuzzy points. Consider the map $f_{X_\lambda} : \hat{X} \rightarrow C'(I)$ defined by

$f_{X_\lambda}(y_\mu) = y_\mu * x_\lambda$, then $f_{X_\lambda}$ is a 'fuzzy linear functional' on $\hat{X}$. 
Proof:

\[ f_{x_\lambda} (y_\mu \oplus z_\psi) = (y_\mu \oplus z_\psi) * x_\lambda = y_\mu * x_\lambda + z_\psi * x_\lambda \]

\[ = f_{x_\lambda} (y_\mu) + f_{x_\lambda} (z_\psi) \]

also \[ f_{x_\lambda} (\alpha y_\mu) = (\alpha y_\mu) * x_\lambda = \alpha (y_\mu * x_\lambda) \]

\[ = \alpha f_{x_\lambda} (y_\mu). \]

Notation 5.2.7

\( \hat{X}_L \) denotes the set of all 'fuzzy linear functionals' on \( \hat{X} \) of the form \( f_{x_\lambda} \)

Proposition 5.2.8

Let \( \langle \hat{X}, *, \| \| \rangle \) be a fuzzy semi inner product space of fuzzy points. Then the map \( *' \) defined on \( \hat{X}_L \times \hat{X}_L \) by

\[ f_{x_\lambda} *' f_{y_\mu} = y_\mu * x_\lambda \]

satisfies the following conditions

(i) \[ f_{x_\lambda} *' f_{x_\lambda} > \bar{0}, \text{ when } x_\lambda \neq 0 \]
Proof:

(i) \( \lf x \r_x^* \lf y \r_y > 0 \) when \( x \neq 0 \)

(ii) \( \lf x \r_x^* \lf x \r_x \leq \lf y \r_y^* \lf y \r_y \)

Remark 5.2.9

If \( y \perp x \) in \( \hat{X} \) then \( x \perp y \) in \( X \cap y \).