Chapter 4

FUZZY SEMI INNER PRODUCT SPACES*

4.0 INTRODUCTION

Lumer, G [LUM] introduced the idea of semi inner product space with a more general axiom system than that of inner product space. The importance of semi inner product is that whether the norm satisfies the parallelogram law or not, every normed space can be represented as a semi inner product space, so that the theory of operators can be extended further by Hilbert space type arguments. Parallel to this on a C'(I) module we are able to introduce the notion of fuzzy semi inner product. We prove that a fuzzy semi inner product generates a fuzzy norm and further that every fuzzy normed space can be made into a fuzzy semi inner product space.

*Some results contained in this chapter have been included in a paper accepted for publication in The Journal of Fuzzy Mathematics.
Also the notion of fuzzy orthogonal set is introduced. Existence of a complete fuzzy orthogonal set is established. The concept of generalized fuzzy semi inner product is introduced.

4.1 FUZZY SEMI INNER PRODUCT

Definition 4.1.1

A fuzzy semi inner product on a $C'(I)$ module $X$ is a function $\star : X \times X \to C'(I)$ which satisfies the following conditions

(i) $(x+y) \star z = x \star z + y \star z$

$$\lambda \star x \star y = \lambda (x \star y)$$

ie, $\star$ is linear in the first argument where $x, y, z \in X$ and $\lambda \in C'(I)$

(ii) $x \star x > 0$ for every nonzero $x \in X$

(iii) $[x \star y]^2 \leq [x \star x][y \star y]$

then $(X, \star)$ is called a fuzzy semi inner product space.

Note 4.1.2

(i) If $\star$ is linear in first and conjugate linear in the
second arguments also satisfies the above conditions
(i) & (ii) then \( <x,*> \) will be called a fuzzy inner
product space. Clearly a fuzzy inner product space is
a fuzzy semi inner product space.

(ii) The conjugate of \([(\eta,\xi),(\eta',\xi')]\) is \([(\eta,\xi),(\xi',\eta')]\).

**Theorem 4.1.3**

Let \( <X,*> \) be a fuzzy semi inner product space. Considering
\( X \) as a real vector space the function \( ||.|| : X \rightarrow R^*(I) \) defined
by \( ||x|| = \sqrt{[x*x]} \) is a fuzzy norm on \( X \).

**Proof:**

(i) \( x*x > \bar{0} \) for every nonzero \( x \)

ie, \( [x*x] > \bar{0} \)

\( ||x||^2 > \bar{0} \)

\( ||x|| > \bar{0} \)

(ii) \( ||x+y||^2 = \sqrt{[(x+y)*(x+y)]} \)

\( = \sqrt{[x*(x+y)+y*(x+y)]} \)
\[
\leq \left[ x*(x+y) \right] \oplus \left[ y*(x+y) \right]
\]
\[
\leq \left[ x*x \right]^{1/2} \left[ (x+y)*(x+y) \right]^{1/2} \oplus \\
\left[ y*y \right]^{1/2} \left[ (x+y)*(x+y) \right]^{1/2}
\]

ie, \[x+y\|^2 \leq \|x\| \|x+y\| \oplus \|y\| \|x+y\|
\]

\[\|x+y\|^2 \leq (\|x\| \oplus \|y\|) \|x+y\|\]

ie, \[x+y\| \leq \|x\| \oplus \|y\|
\]

(iii) Let \(t \in \mathbb{R} \& t \neq 0\)

Consider \[\|tx\|^2 = [tx*tx]\]

\[\|tx\|^2 \leq |t| \left[ x*tx \right] \]

\[\|tx\|^2 \leq |t| \|x\| \|tx\| \]

ie, \[\|tx\| \leq |t| \|x\| \]

(also \[\|x\| = \| \frac{1}{t} \cdot tx \| \leq \frac{1}{|t|} \|tx\|\]

ie, \[|t| \|x\| \leq \|tx\| \]

(2)
by (1) & (2) $\|tx\| = |t|\|x\|$

When $t = 0$, $tx = 0$, $|t| = 0$

hence $\|tx\| = 0 = |0|\|x\|$

ie, $\|$ is a fuzzy norm on $X$.

Note 4.1.4

Let $<X,*>$ be a fuzzy semi inner product space. If $\|$ is the fuzzy norm generated from the fuzzy semi inner product $*$, then the fuzzy semi inner product space is denoted by $<X,*,\|$.

Theorem 4.1.5

On $C'(I)$ define $*$ by

$$[(\eta_1,\xi_1), (\eta_1',\xi_1')] * [(\eta_2,\xi_2), (\eta_2',\xi_2')] =$$

$$[(\eta_1,\xi_1), (\eta_1',\xi_1')] * [(\eta_2,\xi_2), (\xi',\eta_2')]$$

then $<C'(I),*,\[]>$ is a fuzzy semi inner product space.
Proof:

(i) \[
\left\{ (\eta_1, \xi_1), (\eta_1', \xi_1') \right\} * \left\{ (\eta_2, \xi_2), (\eta_2', \xi_2') \right\} * (\eta_3, \xi_3), (\eta_3', \xi_3') \right\} = \left\{ (\eta_1, \xi_1), (\eta_1', \xi_1') \right\} \left[ (\eta_2, \xi_2), (\eta_2', \xi_2') \right\] + \left\{ (\eta_2, \xi_2), (\eta_2', \xi_2') \right\} \left[ (\eta_3, \xi_3), (\eta_3', \xi_3') \right\]

also if \( t \in \mathbb{R} \) then

\[
\left\{ t(\eta_1, \xi_1), (\eta_1', \xi_1') \right\} \left\{ (\eta_2, \xi_2), (\eta_2', \xi_2') \right\} = t\left\{ (\eta_1, \xi_1), (\eta_1', \xi_1') \right\} \left[ (\eta_2, \xi_2), (\eta_2', \xi_2') \right\] \]

(ii) \[
(\eta_1, \xi_1), (\eta_1', \xi_1') \right\} \left\{ (\eta_1, \xi_1), (\eta_1', \xi_1') \right\} = \left\{ (\eta_1, \xi_1), (\eta_1', \xi_1') \right\} \right\} \left[ (\eta_1, \xi_1), (\xi_1', \eta_1') \right\]

\[
= \lim_{n \to \infty} ((\eta_1, \xi_1), (\eta_1', \xi_1'))((\eta_1, \xi_1), (\xi_1', \eta_1'))
\]

\[
= \lim_{n \to \infty} ((\eta_1, \xi_1), (\eta_1', \xi_1'))^2 \Phi((\eta_1, \xi_1), (\xi_1', \eta_1'))^2, (\bar{0}, \bar{0}))
\]
where \( \{((n_{1n}, \xi_{1n}), (n'_{1n}, \xi'_{1n}))\} \subseteq [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] \)

let \( [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] \neq 0 \)

then \([(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] * [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] = 0 \)

iff \( \lim_{n \to \infty} ((\eta_{1n}, \xi_{1n})^2 * (\eta'_{1n}, \xi'_{1n})^2) = 0 \)

iff \( \lim_{n \to \infty} ((\eta_{1n}, \xi_{1n})^2 = 0 \) & \( \lim_{n \to \infty} (\eta'_{1n}, \xi'_{1n})^2 = 0 \)

iff \( \lim_{n \to \infty} \left[\left((\eta_{1n}, \zeta_{1n}), (\eta'_{1n}, \zeta'_{1n})\right) - \left((\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})\right)\right] = 0 \)

iff \( \left\{\left((\eta_{1n}, \zeta_{1n}), (\eta'_{1n}, \zeta'_{1n})\right)\right\} \not\subseteq \left[(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})\right] \)

but this is not the case

hence \( [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] * [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] > 0 \)

when \( [(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] \neq 0 \)

(iii) Consider \( \left[[(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})] * [(\eta_{2}, \zeta_{2}), (\eta'_{2}, \zeta'_{2})]\right]^2 \)

\[ = \left[[(\eta_{1}, \zeta_{1}), (\eta'_{1}, \zeta'_{1})]x[(\eta_{2}, \zeta_{2}), (\zeta'_{2}, \eta'_{2})]\right]^2 \]

\[ = \lim_{n \to \infty} \left[\left((\eta_{1n}, \zeta_{1n}), (\eta'_{1n}, \zeta'_{1n})\right)x((\eta_{2n}, \zeta_{2n}), (\zeta'_{2n}, \eta'_{2n}))\right]^2 \]
where \( \left\{ ((\eta_{1n}, \xi_{1n}),(\eta'_{1n}, \xi'_1)) \right\} \in \left[ (\eta_1, \xi_1), (\eta'_1, \xi'_1) \right] \) and

\[
\left\{ ((\eta_{2n}, \xi_{2n}),(\eta'_{2n}, \xi'_2)) \right\} \in \left[ (\eta_2, \xi_2), (\eta'_2, \xi'_2) \right]
\]

i.e., \( \left[ [(\eta_1, \xi_1),(\eta'_1, \xi'_1)] \ast [(\eta_2, \xi_2),(\eta'_2, \xi'_2)] \right]^2 = \]

\[
\lim_{n \to \infty} \left[ ((\eta_{1n}, \xi_{1n}),(\eta_{2n}, \xi_{2n}) \ast (\eta'_{1n}, \xi'_{1n}),(\eta'_{2n}, \xi'_{2n})) \right]^2 = 
\]

\[
\lim_{n \to \infty} \left[ ((\eta_{1n}, \xi_{1n})^2 \Phi (\eta'_{1n}, \xi'_{1n}),(\eta_{2n}, \xi_{2n})^2 \Phi (\eta'_{2n}, \xi'_{2n})) \right]
\]

\[
\leq \lim_{n \to \infty} \left\{ \left[ ((\eta_{1n}, \xi_{1n})^2 \Phi (\eta'_{1n}, \xi'_{1n})) \times ((\eta_{1n}, \xi_{1n}),(\eta_{1n}, \xi_{1n})) \right] \times 
\left[ ((\eta_{2n}, \xi_{2n})^2 \Phi (\eta'_{2n}, \xi'_{2n})) \times ((\eta_{2n}, \xi_{2n}),(\eta_{2n}, \xi_{2n})) \right] \right\}
\]
Theorem 4.1.6

Every fuzzy normed space can be made into a fuzzy semi inner product space.

Proof:

Let $X$ be a fuzzy normed space. By fuzzy Hahn-Banach theorem corresponding to every $x_0 \in X$ there exists a bounded fuzzy linear map $f_{x_0}$ on $X$ such that $f_{x_0}(x_0) = \|x_0\|^2$ and

\[
\|f_{x_0}\| \leq \|x_0\|
\]

define $f_y(x) = x^*y$

then

(i) $f_y(x_1 + x_2) = (x_1 + x_2)^*y = f_y(x_1) + f_y(x_2)$
\[ = x_1 \ast y + x_2 \ast y \]

\[ f_y(\alpha x) = (\alpha x) \ast y = \alpha f_y(x) = \alpha (x \ast y) \]

(ii) \( f_y(y) = \|y\|^2 > 0 \) ie, \( y \ast y > 0 \), when \( y \neq 0 \)

(iii) Consider \( \square(f_y(x) \square \leq \|f_y\| \|x\| \leq \|y\| \|x\| \)

ie, \( \square(f_y(x) \square^2 \leq \|y\|^2 \|x\|^2 \)

\[ = f_y(y) f_x(x) = \square(f_y(y) \square \square(f_x(x) \square). \]

4.2 FUZZY ORTHOGONAL SET

Definition 4.2.1

A subset \( A \) of a fuzzy semi inner product space \( <X, \ast, \| \|> \) is said to be fuzzy orthogonal in \( X \) if \( x \ast y = \overline{0} \) for every \( x, y \in A \).

Definition 4.2.2

A fuzzy orthogonal set \( A \) in a fuzzy semi inner product space
is said to be complete if there exist no other fuzzy orthogonal set properly containing $A$.

**Proposition 4.2.3**

A fuzzy orthogonal set $A$ in $\langle X,*\rangle$ is complete iff for any $x$ such that $x \perp A$, $x$ must be zero.

**Proof:**

Suppose $A$ is complete and $x$ is a nonzero element of $X$ such that $x \perp A$, clearly this is contradictory because the fuzzy orthogonal set $A \cup \{x\}$ contains $A$, properly and contradicts the maximality of $A$.

Conversely suppose the above condition is satisfied. That is $x \perp A$ implies $x = 0$. If $A$ is not complete, there exist some fuzzy orthogonal set $B$ such that $B$ properly contains $A$. In such case there exists an $x \in B-A$, where $x \perp A$ and $x \neq 0$, this is a contradiction. ie, $A$ is complete.
Theorem 4.2.4

Let \( \langle X, * \rangle \) be a fuzzy semi inner product space.

(i) There exists a complete fuzzy orthogonal set in \( X \).

(ii) Any fuzzy orthogonal set can be extended to a complete fuzzy orthogonal set.

Proof:

It is clear if (ii) can be proved, this will imply (i). By virtue of the fact that in any fuzzy semi inner product space, fuzzy orthogonal sets must exist for, any nonzero vector \( x \), \( \{x\} \) is a fuzzy orthogonal set. Hence we shall prove (ii).

Let \( A \) be a fuzzy orthogonal set and \( \mathcal{A} \) be the collection of all fuzzy orthogonal sets containing \( A \). Then \( \mathcal{A} \) is partially ordered by set inclusion. Let \( T \) be a totally ordered subset of \( \mathcal{A} \). Let

\[
T = \{ A_\alpha \} \quad \alpha \in \Lambda, \text{ for any } \alpha, \quad A_\alpha \subseteq \bigcup \limits_\alpha A_\alpha \text{ also } A \subseteq \bigcup \limits_\alpha A_\alpha
\]
Let $x, y \in \bigcup_{\alpha} A_\alpha \Rightarrow$ there exist $A_\alpha \in A_\alpha$ such that $x \in A_\alpha$ and $y \in A_\beta$

Since $T$ is totally ordered either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$ suppose the former inclusion holds, then we can say $x, y \in A_\beta$

then $x \perp y$, hence $\bigcup_{\alpha} A_\alpha \in A$.

Then $\bigcup_{\alpha} A_\alpha$ is an upper bond for $T$ in $A$.

Hence by Zorn's lemma there must exist a maximal element in $A$. Because of the maximality no other fuzzy orthogonal set containing this maximal element.

4.3. GENERALIZED FUZZY SEMI INNER PRODUCT

Definition 4.3.1

A $C'(I)$ module $E$ is called a generalized fuzzy semi inner product space if

(i) There is a submodule $M$ of $E$ which is a fuzzy semi inner product space, and

(ii) there is a nonempty set $\alpha$ of fuzzy linear operators on $E$ which has the following properties.
(a) $\alpha \in E \subseteq M$

(b) if $Tx = 0$, $\forall T \in \alpha$ then $x = 0$

A generalized fuzzy semi inner product space is represented by the triple $(E, \alpha, M)$.

Remark 4.3.2

Every fuzzy semi inner product space is a generalized fuzzy semi inner product space.

Proposition 4.3.3

Let $(E, \alpha, M)$ be a generalized fuzzy semi inner product space and $x \in E$

(a) if $Tx*y = \overline{0}$, $\forall y \in M$, $T \in \alpha$, then $x = 0$

(b) if $Tx*Tx = \overline{0}$, $\forall T \in \alpha$, then $x = 0$.

Proof:

(a) $Tx*y = \overline{0}$, $\forall y \in M$

in particular $Tx*Tx = \overline{0}$
\[ T_x = 0, \forall T \in \alpha \]

ie, \( x = 0 \).

(b) \( T_x \cdot T_x = 0 \Rightarrow T_x = 0, \forall T \in \alpha \Rightarrow x = 0 \).