Chapter 1

Introduction to Mathematical Morphology

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1.1 Introduction

Mathematical Morphology is the analysis of signals in terms of shape. This simply means that morphology works by changing the shape of objects contained within the signal. In the processing and analysis of images it is important to be able to extract features, describe shapes and recognize patterns. Such tasks refer to geometrical concepts such as size, shape, and orientation. Mathematical morphology uses concepts from set theory, geometry and topology to analyze geometrical structures in an image.

Mathematical morphology is about operations on sets and functions. It is systematized and studied under a new angle, precisely because it is possible to actually perform operations on the computer and see on the screen what happens. The need to simplify a complicated object is the basic impulse
behind mathematical morphology. Related to this is the fact that an image may contain a lot of disturbances. Therefore, most images need to be tidied up. Hence another need to process images; it is related to the first, for the border line between dirt and of other kind disturbances is not too clear.

Consider Euclidean geometry, and consider cardinalities. The set \( N \) of nonnegative integers is infinite, and its cardinality is denoted by \( \text{card}(N) = \aleph_0 \) (Aleph zero). The set of real numbers \( R \) has the same cardinality as the set of all subsets of \( N \), thus \( \text{card}(R) = 2^{\aleph_0} \). The points in the Euclidean plane have the same cardinality:

\[
\text{card}(R^2) = \text{card}(R).
\]

But the set of all subsets of the line or the plane has the larger cardinality. There are too many sets in the plane. Consider a large subclass of this huge class, a subclass consisting of nice sets. For instance, the set of all disks has a much smaller cardinality, because three numbers suffice to determine a disk in the plane: its radius and the two coordinates of its center. Similarly, four numbers suffice to specify a rectangle \([a_1, b_1] \times [a_2, b_2]\) with sides parallel to the axes; a fifth is needed to rotate it. This leads to the idea of simplifying a general, all too wild set, to some reasonable, better-behaved set. Euclidean line containing denumerably many points. Consider a line as the set of solutions in \( Q^2 \) of an equation \( a_1 x_1 + a_2 x_2 + a_3 = 0 \) with integer coefficients. Then two lines which are not parallel intersect in a point with rational coordinates. The cardinality of the set of all subsets of \( Q^2 \) is \( 2^{\aleph_0} \), so there are fewer sets to keep track of than in the real case.
So there are too many subsets in the plane. Consider digital geometry. On a computer screen with, say, 1,024 pixels in a horizontal row and 768 pixels in a vertical column there are $1,024 \times 768 = 786,432$ pixels. On such a screen a rectangle with sides parallel to the axes is the Cartesian product $R(a, b) = [a_1, b_1]_Z \times [a_2, b_2]_Z$ of two intervals.

There are only finitely many binary images. But the number of binary images must be compared with other finite numbers. Thus, although the number of binary images on a computer screen is finite, it is so huge that the conclusion must be the same as in the case of the infinite cardinal: there are too many; it is not possible to search through the whole set; for simplifying this leads, again, to image processing and mathematical morphology, with subsets of $Z^2$, or, generally, of $Z^n$, the set of all $n$-tuples of integers. When consider mathematical morphology both the cases are important. i.e., both the vector space $R^n$ of all $n$-tuples of real numbers (the addresses of points in space) and the digital space $Z^n$ (the addresses of pixels). $R^n$ and $Z^n$ form an abelian group. Therefore the space, called image carrier, is just an abelian group.

Serra (1982) lists “four principles of quantification.” These are about the ways to gather information about the external world. They apply also, but not exclusively, to image analysis.

Serra’s first principle is “compatibility under translation.” For a mapping, this means that $f(A + b) = f(A) + b$, which is expressed as $f \circ T_b = T_b \circ f$, where $\circ$
denotes composition of mappings defined by \((f \circ T_b)(x) = t(I_b(x))\), thus a kind of commutativity, writing \(T_b\) for the translation \(T_b(A) = A + b\). It means that \(f\) commutes with translations. On a finite screen like \(\{x \in \mathbb{Z}^2; 0 < x_1 < 1, 024, 0 < x_2 < 768\}\) almost nothing can commute with translations. Therefore consider the ideal, infinite, computer screen with sets of addresses equal to \(\mathbb{Z}^2\). The principle is equally useful in \(\mathbb{R}^n\) and \(\mathbb{Z}^n\).

Serra’s second principle is “compatibility under change of scale.” For a mapping this means that it commutes with homotheties (or dilatations).

The third principle is that of “local knowledge.” This principle says that in order to know some bounded part of \(f(A)\), there is no need to know all of \(A\), only some bounded part of \(A\). Mathematically speaking: for every bounded set \(Y\), there exists a bounded set \(Z\) such that \(f(A \cap Z) \cap Y = f(A) \cap Y\).

Serra’s fourth principle of quantification is that of “semi continuity.” It means that if a decreasing sequence \(\langle A_j \rangle\) of closed sets tends to a limit \(A\), thus \(A = \cap A_j\), then \(f(A_j)\) tends to \(f(A)\). Thus if \(A_j\) is close to \(A\) in some sense and \(A_j\) contains \(A\), then \(f(A_j)\) must be close to \(f(A)\). To express this property as semi continuity, one must define a topology. In this thesis an attempt is made to derive some meaningful results by introducing some topological properties to the theory of morphological operators.
Over the last 10-15 years, the tools of mathematical morphology have become part of the mainstream of image analysis and image processing technologies. The growth of popularity is due to the development of powerful techniques, like granulometries and the pattern spectrum analysis, that provide insights into shapes, and tools like the watershed or connected operators that segment an image. But part of the acceptance in industrial applications is also due to the discovery of fast algorithms that make mathematical morphology competitive with linear operations in terms of computational speed. A breakthrough in the use of mathematical morphology was reached, in 1995, when morphological operators were adopted for the production of segmentation maps in MPEG-4.

J. Serra and George Matheron worked on image analysis. Their work lead to the development of the theory of Mathematical Morphology. Later Petros Maragos contributed to enrich the theory by introducing theory of lattices. Firstly the theory is purely based on set theory and operators are defined for binary cases only. Later, the theory extended to Gray scale images also. He also gave a representation theory for image processing. Heink J. Heijmans gave an algebraic basis for the theory. Heink J. Heijmans extended the theory to Signal processing also. He also defined the operators for convex structuring elements. Rein Van Den Boomgaard introduced Morphological Scale space operators. In this thesis, an attempt to link some topological concepts to operators is made.
Morphological scale space operators can be linked with Fractals. A general Morphological algebraic structure is also introduced in this thesis. An attempt to characterize morphological convex geometries, using the definition of Moore family is made in this thesis.

The Moore family stands for the family of closed objects. There exist interrelationships between Moore family, adjunctions and Morphological transforms. Adjunctions are pairs of operators which satisfy, some mathematical property. In mathematical Morphology Dilation and erosion are fundamental operators. These operators form an adjunction between two spaces. These operators are dual operators. All morphological adjunctions can be defined using a general rule.

1.2 Birth of Mathematical Morphology

Mathematical morphology (MM) originates from the study of the geometry of binary porous media such as sandstones. It can be considered as binary in the sense it is made up of two phases: the pores embedded in a matrix. This led Matheron and Serra to introduce in 1967 a set formalism for analyzing binary images.

Mathematical morphology is a non-linear theory of image processing. Its geometry-oriented nature provides an efficient method for analyzing object shape characteristics such as size and connectivity, which are not easily accessed by linear approaches.
Mathematical Morphology (MM) is associated with the names of Georges Matheron and Jean Serra, who developed its main concepts and tools. (Matheron, 1975; Serra, 1982; Serra, 1988), They created a team at the Paris School of Mines. Mathematical Morphology is heavily mathematized. In this respect, it contrasts with different experimental approaches to image processing.

MM stands also as an alternative to another strongly mathematized branch of image processing, the one that bases itself on signal processing and information theory. Main contributors in this area are Wiener, Shannon, Gabor, etc. These classical approaches has a lot of applications in telecommunications. Analysis of the information of an image is not similar to transmitting a signal on a channel. An image should not be considered as a combination of sinusoidal frequencies, nor as the result of a Markov process on individual points. The purpose of image analysis is to find spatial objects. Hence images consist of geometrical shapes with luminance (or colour) profiles. This can be analyzed by their interactions with other shapes and luminance profiles. In this sense the morphological approach is more relevant.

MM has taken concepts and tools from different branches of mathematics like algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc.
1.3 Image Processing using Mathematical Morphology

Mathematical morphology is theoretically based on set theory. It contributes a wide range of operators to image processing, based on a few simple mathematical concepts. MM started by considering binary images and usually referred to as standard mathematical morphology. It also used set-theoretical operations like the relation of inclusion and the operations of union and intersection.

In order to apply it to other types of images, for example grey-level ones (numerical functions), it was necessary to generalize set-theoretical notions. Using the lattice-theory it is generalized. The notions are, the partial order relation between images, for which the operations of supremum (least upper bound) and infimum (greatest lower bound) are defined. Therefore the main structure in MM is that of a complete lattice. All the basic morphological operators are defined by using this framework. Nowadays, most morphological techniques combine lattice-theoretical and topological methods.

The computer processing of pictures led to digital models of geometry. Azriel Rosenfeld has contributed in this field after having contributed to digital geometry and image processing for 40 years. Mathematical morphology is perfectly adapted to the digital framework.
The operators are particularly useful for the analysis of binary images, boundary detection, noise removal, image enhancement, shape extraction, skeleton transforms and image segmentation. The advantages of morphological approaches over linear approaches are

1) Direct geometric interpretation, 2) Simplicity and 3) Efficiency in hardware implementation.

An image can be represented by a set of pixels. A morphological operation uses two sets of pixels, i.e., two images: the original data image to be analyzed and a structuring element which is a set of pixels constituting a specific shape such as a line, a disk, or a square. A structuring element is characterized by a well-defined shape (such as line, segment, or ball), size, and origin. Its shape can be regarded as a parameter to a morphological operation.