6.1 Introduction

When the ETI and DTI systems are related via an adjunction, then there is also a close relationship between their impulse responses. Namely, let ε be an ETI system, and let Δ be its adjoint dilation. It is easy to show that Δ is a DTI system[11], and therefore \( \Delta(f) = f \ominus g \), where g is the lower impulse response. Since it is the generalization, no separate proof is required for most of the results. However, proofs are given for new propositions. Examples are given for some generalizations. This generalization is helpful for developing the theory of Mathematical Morphology. This chapter gives the relation between combinatorial convexity, mathematical morphology and image processing.

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6.2 Different Structures for Morphological Operators

Moore family is defined by using a partially ordered set $L$. It also satisfies certain properties on $L$.

6.2.1 Definition Moore family

Let $L$ be a poset.

A subset $M$ of $L$ is a Moore family if every element of $L$ has a least upper bound in $M$.

\[ \forall x \in L, \exists y \in M, y \geq x \text{ and } \forall z \in M, z \geq x \Rightarrow z \geq y \]

A closure operator on $L$ is an increasing, extensive and idempotent operator from $L \rightarrow L$.

6.2.2 Proposition

Let $L$ be a poset. There is a one to one correspondence between Moore families in $L$ and closings on $L$, given as follows.

To a Moore family $M$, associate the closing defined by setting for every $x \in L$; $(x)$ is equal to the least $y \in M$ such that $y \geq x$.

To a closing, one associates the Moore family $M$ which is the invariance domain of $M = \text{Inv}$ (i.e. $M = \{ \varphi(x) / x \in L \}$).
6.2.3 Convex geometry

Let $S$ be a set, consider the family $T$ of subsets of $S$ with the following properties:

$q \in T, S \in T,$

$A, B \in T$ implies $A \cap B \in T$

This family defines a closure operator $\phi(X) = \cap\{A \in T, X \subseteq A\}.$

Every closure operator defines a family $T'$ with the above properties. Elements of $T$ or elements defined by $\phi$ will be called convex. We call the pair $(S, \phi)$ is a Convex geometry if $\phi$ verifies the anti-exchange axiom[6]

$\forall x, y \in \phi(X), x \neq y, x \in \phi(X \cup y)$ implies $y \in \phi(X \cup x)$

In the same way, if $\phi(X) \neq S, \exists p \in S \setminus \phi(X), \phi(X \cup p) = \phi(X) \cup p$

Corresponding to a partially ordered set, we have a graphical representation, known as Hasse Diagrams. So we can infer that Poset give some geometrical representation. In view of this we can define Poset Geometry.
6.2.4 Poset geometry

Let $P$ be a partially ordered set and $X$ be a subset of $P$. Define $D_P(X) = \{y \in P, y \leq x \text{ for some } x \in X\}$. $(P, D_P)$ is a convex geometry called Poset geometry which are characterized by the following:

The convex geometry $(S, \cdot)$ arises from the poset geometry on a Poset $P$ if and only if $\phi(A \cup B) = \phi(A) \cup \phi(B) \forall A, B \subseteq S$.

Definitions of Dilation and Erosion is given below. We also give the definition of Alexandroff space, in order to link it with Morphology.

6.2.5 Dilation

A dilation is defined by an operator $\delta : P(S) \rightarrow P(S)$ with the following properties: $\delta(\phi) = \phi, A, B \in P(S), \delta(A \cup B) = \delta(A) \cup \delta(B)$

6.2.6 Erosion

An erosion is defined by an operator $\varepsilon : P(S) \rightarrow P(S)$ with the following properties: $\varepsilon(S) = S, A, B \in P(S), \varepsilon(A \cap B) = \varepsilon(A) \cap \varepsilon(B)$

6.2.7 Alexandroff space

A topological space is an Alexandroff space if the intersection of any family of open sets is open (resp. the union of any family of closed sets is closed)
Let $\delta$ be a dilation on $S$. For any dilation, define a binary relation as follows:

$xRy$ is equivalent to $x \in \delta(y)$, for $x, y \in S$ or $xR'y$ is equivalent to $\delta(x) \subseteq \delta(y), \forall x, y \in S$

6.2.8 Result

Let $\delta$ be a dilation on $S$. R its binary relation canonically associated with it. Then the following are equivalent.

i) $R$ is reflexive and transitive

ii) $xRy$ is equivalent to $\delta(x) \subseteq \delta(y)$

iii) defines a dual Moore family.

Proof. i) $\Rightarrow$ ii). Since $R$ is reflexive and transitive, $\delta(x) \subseteq \delta(x)$ and $\delta(x) \subseteq \delta(y)$. Therefore $\delta(x) \subseteq \delta(y)$.

ii) $\Rightarrow$ iii). Since $\delta(x) \subseteq \delta(y)$, by definition $\delta$ defines a dual Moore family.

iii) $\Rightarrow$ i). Since $\delta$ defines a dual Moore family, it is both reflexive and transitive.
6.2.9 Proposition

Let \( S \) be a set. Let \( N: S \to P(S) \) corresponding to \( \rho \) by

\[
\forall x, y \in S, x \in N(y) \iff y \in N(x) \iff x \rho y \quad \text{and} \quad N(x) = \{ y \in S / x \rho y \}
\]

Then (i) \( N \) separates \( S \) in a primary sense

(ii) \((S, N)\) is a Poset geometry.

(iii) \((S, N)\) is a To Alexandroff space.

6.2.10 Proposition

\((S, N)\) is separated in a primary sense if \( N \) verifies the following two properties.

For any family \((x_i)\), \( i \in I \) of elements and for any element \( x \in S \), verifying

\[
N(x) \subseteq \bigcup_{i \in I} N(x_i), \exists j \in I \text{ such that } N(x) \subseteq N(x_j)
\]

\(N(x) = N(y)\) is equivalent to \( x = y \) for any \( x, y \in S \)

6.2.11 Definition

\( \varphi = \circ \delta \) is called a morphological closure and

\[
\varphi(x) = \{ y \in S / \delta(y) \subseteq \delta(x) \}
\]
6.2.12 Result

Let $S$ be an infinite space and let $N$ be defined by $R$. Then $(S,N)$ is a convex geometry if and only if $\forall X \subseteq S, (S - N(x), \psi)$ is a To-Alexandroff space [1],[6] where $\psi(A) = \bigcup_{x \in A} [N(N(x) \cup y) \cap S - N(x)]$.

6.2.13 Proposition

Let $\delta: P(v) \rightarrow P(w)$ and $\varepsilon: P(w) \rightarrow P(v)$ such that

$N: v \rightarrow P(w)$ where $N(v) = \delta(\{v\}), \forall v \in V$ and $\delta = \delta_N, \varepsilon = \varepsilon_N$.

Define $\delta_N(Y) = V - \varepsilon_N(W - Y), \forall Y \in P(W)$ and $\varepsilon_N(X) = W - \delta_N(V - X), X \in P(V)$.

Then $\delta_N$ and $\varepsilon_N$ are dual by complementation of $\epsilon_N$ and $\delta_N$.

Also $\varepsilon_N$ is a dilation and $\delta_N$ is an erosion. Also $\delta_N = \delta_{\rho^{-1}}$ and $\varepsilon_N = \varepsilon_{\rho^{-1}}$ where $\rho^{-1}$ is defined as $w \rho^{-1} v \Rightarrow v \rho w$ and $v \rho w \Rightarrow w \in \delta(v) = N(v)$. 

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6.2.14 Proposition

Let $\delta$ be a dilation and $\varepsilon$ an erosion. Let $\rho$ be the relation defined as before. Then

i) $\rho$ is reflexive and transitive.

ii) $\forall \rho \ w$ is equivalent to $\delta_n(x) \subseteq \delta_n(y)$.

iii) $\delta_n$ defines a Dual Moore family.

iv) $\varepsilon_n$ defines a Moore family.

6.3 Generalized Structure for Mathematical Morphology

6.3.1 Morphogenetic field

Let $X \neq \emptyset$ and $W \subseteq P(X)$ such that i) $\phi$, $X \in W$, ii) If $B \in W$ then its complement $\overline{B} \in W$ iii) If $B_i \in W$ is a sequence of signals defined in $X$, then $\bigcup_{n=1}^{\infty} B_i \in W$.

Let $A = \{ \phi : W \rightarrow U / \phi(\cup A_i) = \lor \phi(A_i) \land \phi(\cap A_i) = \land \phi(A_i) \}$. Then $W_U$ is called Morphogenetic field [22] where the family $W_u$ is the set of all image signals defined on the continuous or discrete image Plane $X$ and taking values in a set $U$. The pair $(W_u, A)$ is called an operator space where $A$ is the collection of operators defined on $X$. 

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6.3.2 Morphological space

The triplet \((X, W_\mu, A)\) consisting of a set \(X\), a morphogenetic field \(W_\mu\) and an operator \(A\) (or collection of operators) defined on \(X\) is called a Morphological space.

Note: If \(X = \mathbb{Z}^2\) then it is called Discrete Morphological space

6.3.3 Definition

Let \((X, W_\mu, A)\) be a morphological space and \((W_\nu, A)\) be an operator space in \((X, W_\mu, A)\).

If \(X\) is a class of concave functions then \((X, W_\mu, A)\) is called concave morphological space. If \(X\) is a class of convex functions then \((X, W_\mu, A)\) is called convex morphological space.[22]

6.3.4 Proposition

Every convex morphological space has * property.
6.4 Results in Generalized Structure

6.4.1 Definition * property

Let \((X, W_u, A)\) be a morphological space and \((W_U, A)\) be an operator space in \((X, W_u, A)\).

Let \(x(\alpha) \in X\), then \(x(\alpha)\) has at least one maxima or minima in \(X\).

6.4.2 Proposition

Every convex morphological space is optimizable.

6.4.3 Definition

Let \((X, W_u, A)\) be a morphological space and \((W_U, A)\) be an operator space in \((X, W_u, A)\).

If \(\phi\) is an operator in \(A\) and in particular if \(\phi\) satisfies (or defines a rule) in \(W_U\), then the operator space \((W_U, \phi)\) is called a geometrical space and \(\phi\) defines a morphological geometry in \(W_U\).

6.4.4 Proposition

Let \((X, W_u, A)\) be a morphological space and \((W_U, A)\) be an operator space in \((X, W_u, A)\).

Suppose that \(\varphi \in A, S \in W_U\). \(X_1, X_2 \in W_U\) \(\Rightarrow X_1 \cap X_2 \in W_U\) and
\( \phi(S) = \cap \{ X_1 \in W_u \mid S \subseteq X_1 \} \) then \( \phi \) defines a morphological geometry, known as convex geometry if

\[ \forall x, y \in \phi(S), x \neq y, x \in \phi(S \cup Y) \Rightarrow y \notin \phi(S \cup x) \] . \( (W_u, \phi) \) is called a convex geometrical space.[22]

Also if \( \phi(S) \neq X \) then \( \exists z \in X - \phi(S) \) and \( \phi(S \cup z) = \phi(S) \cup z \).

**6.4.5 Definition  Poset Geometry**

Let \( (P, \leq) \) be a poset and \( X \) be a subset of \( P \). Define

\[ T_p(X) = \{ y \in P \mid y \leq x, \text{ for some } x \in A \} \].

Let \( (X, W_u, A) \) be a morphological space and \( W_u = P, A = T_p \). Then the operator space is \( (P, T_p) \) defines a geometry known as poset geometry and \( (P, T_p) \) is called a poset geometrical space.

**6.4.6 Proposition**

Let \( (X, W_u, A) \) be a morphological space and let \( (W_u, \phi) \) be a poset geometrical space [22] in \( (X, W_u, A) \). Then \( (W_u, \phi) \) is called a convex geometrical space if\( \phi(X_1 \cup X_2) = \phi(X_1) \cup \phi(X_2), \forall X_1, X_2 \in X \).
6.4.7 Proposition

Let \((X,W_\mu,A)\) be a morphological space. Then for \(\gamma(x) \in W_\mu\), \((X,\gamma(x))\) is called an anti matroid if \((X,\gamma(x))\) satisfies the following.

i) \(\varnothing \in \gamma(x)\), \(\gamma(x)\) is closed under union.

ii) For \(S \in \gamma(x), S \neq \varnothing, \exists x \in S\) such that \(S - x \in \gamma(x)\).

6.4.8 Proposition

Let \((X,W_\mu,A)\) & \((Y,W_\mu,A)\) be a morphological spaces. The pair \((A,A)\) is called an adjunction iff \(A(X) \leq Y \iff X \leq A(Y)\) where \(\tilde{A}\) is an inverse operator of \(A\).

6.4.9 Proposition

Let \((X,W_\mu,\delta)\) & \((Y,W_\mu,e)\) be a morphological spaces with operators dilation and erosion on \(A\). Then \(\delta(X) \leq Y \iff X \leq e(Y)\).
6.4.10 Proposition (For lattice)

Let \((X, W_u, A) \& (Y, W_v, A)\) be a morphological spaces. The pair \((A, \tilde{A})\) is called an adjunction iff \(\forall u, v \in X, \exists \) an adjunction \((l_{u,v}, m_{v,u})\) on \(U\) such that

\[ A(x(u)) = \bigvee_{v \in X} m_{v,u}(x(v)) \]

and

\[ A(y(v)) = \bigwedge_{u \in X} l_{v,u}(y(u)), \forall u, v \in X, x, y \in W_U. \]

6.4.11 Definition

The operator \(\phi = \varepsilon \circ \delta\) defines a closure called morphological closure and \(\phi^* = \delta \circ \varepsilon\) defines a kernel, called morphological kernel.

6.4.12 Lemma

Let \((X, W_u, A)\) be a morphological space.

\(\phi^*(S) = \cup \{X_1 \in W_U / X_1 \subseteq S\}\) defines a kernel operator in \(A\). The pair \((X, \phi^*(S))\) is an anti matroid if \(\phi^*\) satisfies the axiom:

For \(\phi^*(S) \neq \varnothing, \exists z \in \phi^*(S), \phi^*(S - z) = \phi^*(S) - z\).

Proof:

Since \(\phi^*(S) \in W_U\) where \(W_U\) is a morphogenetic field in a morphological space \((X, W_u, A)\) \(\phi^*(S)\) is an anti matroid.
Direct proof.

Since $\phi^*(S - z) \subseteq S - z$, so $z \in \phi^*(S - z)$. From monotonicity, $S - z \subseteq S \Rightarrow \phi^*(S - z) \subseteq \phi^*(S)$. Therefore

$\phi^*(S - z) \subseteq \phi^*(S)$

Conversely, $\phi^*(S) - z \subseteq S - z \Rightarrow \phi^*(\phi^*(S) - z) \subseteq \phi^*(S - z)$.

Therefore $\phi^*(S) - z \subseteq \phi^*(S) - z \Rightarrow \phi^*(S - z) = \phi^*(S - z)$.

6.4.13 Theorem

Let $(X, W_\mu, A)$ be a morphological space. $(X, \phi)$ defines a convex geometry iff $(X, \phi^*)$ is an antimatroid.

6.4.14 Definition Separation

Let $(X, W_\mu, A)$, $(X, W_\mu, \tilde{A})$ be morphological spaces. Let $(A, \tilde{A})$ be adjunctions. $(X, A)$ is separated in a primary sense if $A$ verifies the following two properties.

Let $x \in X$, $A(x) \subseteq \bigcup_{x_i \in \tilde{W}_\mu} A(x_i) \Rightarrow \exists j \in I$ such that $A(x) \subseteq A(x_j)$

$A(x) = A(y) \Rightarrow x = y \forall x, y \in X$ and $\phi = A \circ \tilde{A}$ defines a morphological closure.
6.4.15 Theorem

Let \((X, W_u, A)\) be a morphological space and \(\phi = A \circ \tilde{A}\) be the morphological closure. Then the following statements are equivalent.

A separates \(X\) in a primary sense.

\((W_u, \phi)\) is a morphological geometrical space.

\((W_u, \phi)\) is a poset geometrical space.

6.4.16 Theorem

Let \((X, W_u, A)\) be a morphological space and let \(X\) be an infinite set and \(\phi = A \circ \tilde{A}\) be the morphological closure. Then the following statements are equivalent.

1) A separates \(X\) in a primary sense.

2) \((W_u, \phi)\) is a morphological geometrical space.

3) \((W_u, \phi)\) is a poset geometrical space.

\((W_u, \phi)\) is a \(T_0\) Alexandroff space.
Proof:

3) $\Rightarrow$ 4)

Let $(W_U, \phi)$ be a poset geometrical space. $\forall y \in \phi(y), \exists B \subseteq Y , B$ being a finite set such that $y \in \phi(B) \Rightarrow y \in \phi(z)$ for some $z \in Y . \therefore (W_U, \phi)$ is an Alexandroff space

$\therefore \forall x , y \in \phi(y) , x \neq y, x \in \phi(x \cup y) \Rightarrow y \notin \phi(x \cup y)$,

is a $T_0$ space.

4) $\Rightarrow$ 1)

Let $(W_U, \phi)$ is a $T_0$ Alexandroff space.

$\therefore \phi = A \circ \tilde{A}$, $\phi(Y) = \left\{ y \in X / \tilde{A}(y) \subseteq \tilde{A}(Y) \right\}$.

$\therefore \forall x \neq y, \tilde{A}$ separates X in a primary sense.

1) $\Rightarrow$ 2)

Let $\tilde{A}$ separates X in a primary sense. Since $\tilde{A}(x) \subseteq \tilde{A}(y)$

$\phi(y) = A \circ \tilde{A}(y)$ and

$\phi(y) = \left\{ y \in X / \tilde{A}(y) \subseteq \tilde{A}(Y) \right\}$. $\phi(Y) = \cap \{ Y_i \in W_U / Y \subseteq Y_i \}$

$\Rightarrow \{ y \in X / yRx \forall x \in Y \}$
\( (W_u, \phi) \) is a morphological geometrical space.

2) \( \Rightarrow \) 3)

Let \((W_u, \phi)\) is a morphological geometrical space. \( \Rightarrow \phi(Y) \) is an ideal of \( X \).

\( \Rightarrow \phi(Y) \) is closed and \( R \) is an order relation. \( \therefore (W_u, \phi) \) is a poset geometrical space.

Hence the result.

6.4.17 Definition Self Conjugate Operator Space

An operator space \((W_u, A)\) is called self conjugate if it has a negation.

Example A clodum \( V \) has conjugate \( a^* \) for every \( a \) such that \((avb)^* = a^* \land b^* \) and \((a*b)^* = (a^* \land b^*) \) \([23]\)

Example If \( V \) is a blog \([4]\) then it becomes self conjugate by setting

\[
a^* = \begin{cases} \frac{1}{a}, \text{when} V \inf < a < V \sup & \text{[23]} \\
V \sup, \text{when} V \inf = a \\
V \inf, \text{when} V \sup = a 
\end{cases}
\]

Example If \( X \) is a concave class then \( A^* x (t) = x(-t) \) where \( A^* = A \land (A \lor) \).

6.4.18 Definition Self Conjugate Morphological Space

If the operator space \((W_u, A)\) is self conjugate then the morphological space \((X, W_u, A)\) is called a self conjugate morphological space.
6.4.19 Definition Operable Functions

Let (X, W, A) be a morphological space. The collection K (X, W, A) of operable functions consists of all real valued morphologically operable functions x(t) defined on X such that x(t) has finite operatability with respect to A. A morphologically operable function x ∈ K iff |x| ∈ K, i.e., iff |A(x(α))| ≤ A|x(α)|

6.4.20 Definition Morphological Transform Systems

Let (X, W, A) be a perfect morphological space and K = K (X, W, A) be an operatable space. K is called a morphological transform system if

\[ A[x_r(t)] = X(α) \circ T(α) \]

Remark Since K is an operatable space,

1) \[ A[x(t) + y(t)] = X(α) + Y(α) \]

2) \[ A[x_r(t)] = X(α) \circ T(α) \]

6.4.21 Definition Morphological Slope Transform System

If A = Av in the previous definition, then K is called a Morphological slope transform system where Av is the upper slope transform.

Let (X, W, A) be a self conjugate morphological space. If X is a concave class then A* (x(t)) = x(-t) where A* = A ∩ (A v) and A ∩ is the lower slope transform. Also Av (∨ x_c) = \[ \sum_{v \in c} A_v(x_c) \]
6.4.22 Proposition (Characterization of Slope Transforms).*

A Slope transform is an extended real valued function $A\nu$ (or $A\wedge$) defined on a Morphogenetic field $W_u$ such that

1. $A\nu(\phi) = 0$
2. $A\nu(x_c) \geq 0 \forall x_c \in W_u$
3. $A\nu$ is countably additive in the sense that if $(x_c)$ is any disjoint sequence [or sampling signal] then $A\nu(\lor x_c) = \sum_{x_c} A\nu(x_c)$

Remark $A\nu$ takes $+\infty$ i.e $A\nu(x_c) = \infty$ if $x(t) = \infty$

$A\nu(\alpha) > -\infty, \forall \alpha$ unless $x(t) = -\infty, \forall t$

If $x=\infty$ then $A\nu = -\infty$

6.4.23 Proposition Let $K$ be a morphological transform system. Let $X$ be a class of concave functions. Let $x(\alpha) \in X$ with each $x(\alpha)$ has an invertible derivative. Then $A_\nu(x(\alpha)) = L(x(\alpha))$ where $L$ is the Legendre transform and $A\nu$ is the upper slope transform.

Algebraic structures are important for defining Morphological operators. Many properties of the algebraic structure may applicable to these operators as well.


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6.5 References


6) Mathematical Morphology and Poset Geometry, Alain Bretto and Enzo Maria Li Marzi.


