Chapter – II

Transient Hydromagnetic Convective Heat and Mass transfer through a Porous Medium induced by a Traveling thermal wave on a vertical channel
1. INTRODUCTION

The time dependent thermal convection flows have applications in chemical Engineering, space technology etc. These flows can also be achieved by either time dependent movement of the boundary or unsteady temperature of the boundary. The unsteady may also be attributed due to the free stream oscillations or oscillatory flux or temperature oscillations. The oscillatory convection problems are important from the technological point of view, as the effect of surface temperature oscillations on skin friction and the heat transfer from surface to the surrounding fluid has special interest in heat transfer engineering.
The combined effects of thermal and mass diffusion in channel flows has been studied in the recent times by a few authors notably. Nelson and Wood (6,5), Lee et al(2), Miyatake and Fujii (4,3), Sparrow et al (11) and others (9,15,16,19). Nelson and Wood(6) have presented numerical analysis of developing laminar flow between vertical parallel plates for combined heat and mass transfer natural convection with uniform wall temperature and concentration boundary conditions. For along channel (low Rayleigh numbers) the numerical solutions approach the fully developed flow analytical solutions. At intermediate Rayleigh numbers it is observed that the parallel plate heat and mass transfer is higher than that for a single plate. Yan and Lin (18) have examined the effects of the latent heat transfer associated with the liquid film vapourization on the heat transfer in the laminar forced convection channel flows. Results are presented for an air-water system under various conditions. The effects of system temperature on heat and mass transfer are investigated. Recently Atul Kumar Singh et al (1) investigated the heat and mass transfer in MHD flow of a viscous fluid past a vertical plate under oscillatory suction velocity. Convection fluid flows generated by traveling thermal waves have also received attention due to applications in physical problems. The linearised analysis of these flows has shown that a traveling thermal wave can generate a mean shear flow within a layer of fluid, and the induced mean flow is proportional to the square of the amplitude of the wave. From a physical point of view, the motion induced by traveling thermal waves is quite interesting as a purely fluid-dynamical problem and can be used as a possible explanation for the
observed four-day retrograde zonal motion of the upper atmosphere of Venus. Also, the heat transfer results will have a definite bearing on the design of oil or gas-fired boilers.

Vajravelu and Debnath (14) have made an interesting and a detailed study of non-linear convection heat transfer and fluid flows, induced by traveling thermal waves. The traveling thermal wave problem was investigated both analytically and experimentally by Whitehead (17) by postulating series expansion in the square of the aspect ratio (assumed small) for both the temperature and flow fields. Whitehead (17) obtained an analytical solution for the mean flow produced by a moving source. Theoretical predictions regarding the ratio of the mean flow velocity to the source speed were found to be good agreement with experimental observations in Mercury which therefore justified the validity of the asymptotic expansion a posteriori. Ravindra (8) has analysed the mixed convection flow of a viscous fluid through a porous medium in a vertical channel. The thermal buoyancy in the flow field is created by a traveling thermal wave imposed on the boundaries. Purushothama Reddy (7) has analysed the unsteady mixed convective effects on the flow induced by imposing traveling thermal waves on the boundaries. Nagaraja (4a) has investigated the combined heat and mass transfer effects on the flow of a viscous fluid through a porous medium in a vertical channel, with the traveling thermal waves imposed on the boundaries while the concentration is maintained uniform on the boundaries. Sivanjaneya Prasad (10) has analysed heat and mass transfer effects on the flow of an
incompressible viscous fluid through a porous medium in vertical channel. Recently, Sulochana et al (13) have considered the unsteady convective heat and mass transfer through a porous medium due to the imposed traveling thermal wave boundary through a horizontal channel bounded by non-uniform walls. Tanmay Basak et al(1a) have analysed the natural convection flows in a square cavity filled with a porous matrix for uniformly and non-uniformly heated bottom wall and adiabatic top wall maintaining crust temperature of cold vertical walls Darcy – Forchheimer model is used to simulate the momentum transfer in the porous medium.

__AUTHOR'S CONTRIBUTION__

We discuss the unsteady thermal convection due to the imposed traveling thermal wave boundary through a vertical channel bounded by flat walls. The effects of free convective heat and mass transfer flow has been discussed by solving the governing unsteady non-linear equations under perturbation scheme. The velocity, the temperature and the concentration have been analysed for different variations of the governing parameters. The shear stress, the rate of heat transfer and the rate of mass transfer have been evaluated and tabulated for these sets of parameters.
2. FORMULATION OF THE PROBLEM

We consider the motion of viscous, incompressible, electrically conducting fluid through a porous medium in a vertical channel bounded by flat walls. The thermal buoyancy in the flow field is created by a traveling thermal wave imposed on the boundary wall at $y=L$ while the boundary at $y=-L$ is maintained at constant temperature $T_1$. The walls are maintained at constant concentrations. A uniform magnetic field of strength $H_0$ is applied transverse to the walls. Assuming the magnetic Reynolds to be small we neglect the induced magnetic field in comparison to the applied magnetic field. Assuming that the flow takes place at low concentration we neglect the Soret and Duffor effect. The Boussinesq approximation is used so that the density variation will be considered only in the buoyancy force. The viscous and Darcy dissipations are taken into account to the transport of heat by conduction and convection in the energy equation. Also the kinematic viscosity $v$, the thermal conducting $k$ are treated as constants. We choose a rectangular Cartesian system $O \ (x, y)$ with $x$-axis in the vertical direction and $y$-axis normal to the walls. The walls of the channel are at $y = \pm L$.

The equations governing the unsteady flow and heat transfer are

Equation of linear momentum

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \rho g - \left( \frac{\mu}{k} \right) u - \left( \sigma \mu^2 H_0^2 \right) u \quad (2.1)$$
\[
\rho_e \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \left( \frac{\mu}{k} \right) v 
\]  

(2.2)

Equation of continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(2.3)

Equation of energy

\[
\rho_e C_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + Q + 
\]

\[
+ \mu \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right) + \left( \frac{\mu}{\lambda k} + \sigma \mu^2 \right) \frac{H^2}{u^2 + v^2} 
\]

(2.4)

Equation of Diffusion

\[
\left( \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) = D_1 \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + k_{11} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) 
\]

(2.5)

Equation of state

\[
\rho - \rho_e = -\beta \rho_e (T - T_e) - \beta^* \rho_e (C - C_e) 
\]

(2.6)

where \( \rho_e \) is the density of the fluid in the equilibrium state, \( T_e, C_e \) are the temperature and Concentration in the equilibrium state, \( u, v \) are the velocity components along \( O(x, y) \) directions, \( p \) is the pressure, \( T, C \) are the temperature and Concentration in the flow region, \( \rho \) is the density of the fluid, \( \mu \) is the constant coefficient of viscosity, \( C_p \) is the specific heat at constant pressure, \( \lambda \) is the coefficient of thermal conductivity, \( k \) is the permeability of the porous medium, \( D_1 \) is the molecular diffusivity, \( k_{11} \) is the \( \beta \) is the
coefficient of thermal expansion $\beta^*$ is the and $Q$ is the strength of the constant internal heat source.

In the equilibrium state

$$0 = -\frac{\partial p_e}{\partial x} - \rho g$$

(2.7)

where $p = p_e + p_d$, $p_d$ being the hydrodynamic pressure.

The flow is maintained by a constant volume flux for which a characteristic velocity is defined as

$$Q = \frac{1}{2L} \int_{-L}^{L} u \, dy.$$  

(2.8)

The boundary conditions for the velocity and temperature fields are

$$u = 0, \quad v = 0, \quad T = T_1, \quad C = C_1 \quad \text{on} \quad y = -L$$

$$u = 0, \quad v = 0, \quad T = T_2 + \Delta T_e \sin(mx + nt), \quad C = C_2 \quad \text{on} \quad y = L$$

(2.9)

where $\Delta T_e = T_2 - T_1$ and $\sin(mx + nt)$ is the imposed traveling thermal wave

In view of the continuity equation we define the stream function $\psi$ as

$$u = -\psi_y, \quad v = \psi_x$$

(2.10)

Eliminating pressure $p$ from equations (2.2)&(2.3) and using the equations governing the flow in terms of $\psi$ are

$$[(\nabla^2 \psi) + \psi_x(\nabla^2 \psi)_y - \psi_y(\nabla^2 \psi)_x] = \nabla^4 \psi - \beta g (T - T_0)_y -$$

$$- \frac{V}{k} \nabla^2 \psi - \left( \frac{\sigma \mu^2 H^2}{\rho_e} \right) \frac{\partial^2 \psi}{\partial y^2}$$

(2.11)
\[ \rho C_p \left( \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right) = \lambda \nabla^2 \theta + Q + \mu \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2 + \left( \frac{\partial^2 \psi}{\partial x^2} \right)^2 + \\
\left( \frac{\mu}{\kappa} + \frac{\sigma \mu^2 H^2}{\rho_e} \right) \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right) \quad (2.12) \]

\[ \left( \frac{\partial C}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial C}{\partial y} \right) = D \nabla^2 C \quad (2.13) \]

Introducing the non-dimensional variables in (2.11)-(2.13) as

\[ x' = mx, \quad y' = y/L, \quad t' = \frac{1}{2} \frac{m^2}{t}, \quad \Psi' = \Psi / \nu, \quad \theta = \frac{T - T_e}{\Delta T_e}, \quad C' = \frac{C - C_1}{C_2 - C_1} \quad (2.14) \]

(under the equilibrium state \( \Delta T_e = T_e(L) - T_e(-L) = \frac{QL^2}{\lambda} \))

the governing equations in the non-dimensional form (after dropping the dashes) are

\[ \delta R \left( \delta (\nabla^2 \psi), \frac{\partial (\psi, \nabla^2 \psi)}{\partial (x, y)} \right) = \nabla^2 \psi + \left( \frac{G}{R} \right) \theta_x - D \nabla^2 \psi - M^2 \frac{\partial^2 \psi}{\partial y^2} \quad (2.15) \]

The energy equation in the non-dimensional form is

\[ \delta P \left( \delta \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right) = \nabla^2 \theta + \alpha + \left( \frac{PR^2 E}{G} \right) \left( \frac{\partial^2 \psi}{\partial x^2} \right)^2 + \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2 + \\
+ \left( D^{-1} + M^2 \right) \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right) \quad (2.16) \]

The Diffusion equation is

\[ \delta Sc \left( \delta \frac{\partial C}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial C}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial C}{\partial y} \right) = \nabla^2 C \quad (2.17) \]

where
The corresponding boundary conditions are

\[ \psi(+!) - \psi(-!) = 1 \]
The corresponding boundary conditions are

\[ \psi(+!) - \psi(-!) = 1 \]

\[ \frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \ y = \pm 1 \]

(2.18)

\[ \theta(x, y) = 1, \quad C(x, y) = 0 \quad \text{on} \ y = -1 \]

\[ \theta(x, y) = \sin(x + \gamma), \quad C(x, y) = 1 \quad \text{on} \ y = 1 \]

(2.19)

\[ \frac{\partial \theta}{\partial y} = 0, \quad \frac{\partial C}{\partial y} = 0 \quad \text{at} \ y = 0 \]

The value of \( \psi \) on the boundary assumes the constant volumetric flow in consistent with the hypothesis (2.8). Also the wall temperature varies in the axial direction in accordance with the prescribed arbitrary function \( \gamma \).
3. ANALYSIS OF THE FLOW

The main aim of the analysis is to discuss the perturbations created over a combined free and forced convection flow due to traveling thermal wave imposed on the boundaries. The perturbation analysis is carried out by assuming that the aspect ratio $\delta$ to be small.

We adopt the perturbation scheme and write

$$\psi(x,y) = \psi_0(x,y) + \delta \psi_1(x,y) + \delta^2 \psi_2(x,y) +$$

$$\theta(x,y) = \theta_0(x,y) + \delta \theta_1(x,y) + \delta^2 \theta_2(x,y) +$$

$$C(x,y) = C_0(x,y) + \delta C_1(x,y) + \delta^2 C_2(x,y) +$$

(3.1)

On substituting (3.1) in (2.16) - (2.17) and separating the like powers of $\delta$ the equations and respective conditions to the zeroth order are

$$\psi_{0,yyy} - M_1^2 \psi_{0,y} = -G(\theta_{0,y} + NC_{0,y})$$

(3.2)

$$\theta_{0,yy} + \frac{PE_c R^2}{G} (\psi_{0,yy})^2 + \frac{PE_c M_1^2}{G} (\psi_{0,y}^2) = 0$$

(3.3)

$$C_{0,yy} = 0$$

(3.4)

with

$$\psi_0(+1) - \Psi(-1) = 1 ,$$

$$\psi_{0,y} = 0 , \psi_{0,x} = 0 \quad \text{at} \quad y = \pm 1$$

(3.5)
\[ \theta_0 = 1 \quad C_0 = 0 \quad \text{on} \quad y = -1 \]

\[ \theta_0 = \sin(x + \eta), C_0 = 1 \quad \text{on} \quad y = 1 \]  

(3.6)

and to the first order are

\[ \psi_{1, yyy} - M_1^2 \psi_{1, yy} = -G(\theta_{1, y} + NC_{1, y} + (\psi_{0, y} \psi_{0, x} - \psi_{0, x} \psi_{0, y})) \]  

(3.7)

\[ \theta_{1, yy} = (\psi_{0, x} \theta_{0, y} - \psi_{0, y} \theta_{0, x}) + \frac{2PEcR^2}{G} (\psi_{0, yy} \psi_{1, y} + \frac{2PEcM_1^2}{G} (\psi_{0, y} \psi_{1, y})) \]  

(3.8)

\[ C_{1, yy} = (\psi_{0, x} C_{0, y} - \psi_{0, y} C_{0, x}) \]  

(3.9)

with

\[ \psi_{1(+1)} - \psi_{1(-1)} = 0 \]

\[ \psi_{1, y} = 0, \psi_{1, x} = 0 \quad \text{at} \quad y = \pm 1 \]  

(3.10)

\[ \theta_{1}(\pm 1) = 0, \ C_{1}(\pm 1) = 0 \quad \text{at} \quad y = \pm 1 \]  

(3.11)

Assuming Ec<<1 to be small we take the asymptotic expansions as

\[ \psi_0(x, y) = \psi_{00}(x, y) + Ec \psi_{01}(x, y) + \ldots \]

\[ \psi_1(x, y) = \psi_{10}(x, y) + Ec \psi_{11}(x, y) + \ldots \]

\[ \theta_0(x, y) = \theta_{00}(x, y) + \theta_{01}(x, y) + \ldots \]

\[ \theta_1(x, y) = \theta_{10}(x, y) + \theta_{11}(x, y) + \ldots \]

\[ C_0(x, y) = C_{00}(x, y) + C_{01}(x, y) + \ldots \]

\[ C_1(x, y) = C_{10}(x, y) + C_{11}(x, y) + \ldots \]  

(3.12)
Substituting the expansions (3.12) in equations (3.2)-(3.4) and separating the like powers of $\text{Ec}$ we get the following

$$\theta_{00,yy} = -\alpha, \quad \theta_{00}(-1) = 1, \theta_{00}(+1) = \sin D_1$$

$$(3.13)$$

$$C_{00,yy} = 0, \quad C_{00}(-1) = 0, C_{00}(+1) = 1$$

$$(3.14)$$

$$\psi_{00,yyyy} - M_1^2 \psi_{00,yy} = -G(\theta_{00,y} + NC_{00,y}),$$

$$(3.15)$$

$$\psi_{00}(+1) - \psi_{00}(-1) = 1, \psi_{00,y} = 0, \psi_{00,x} = 0 \text{ at } y = \pm 1$$

$$\theta_{01,yy} = -\frac{PR}{G} \psi_{00,yy} - \frac{PM_1^2}{G} \psi_{00,y}^2, \quad \theta_{01}(\pm 1) = 0$$

$$(3.16)$$

$$C_{01,yy} = 0, \quad C_{01}(-1) = 0, C_{01}(+1) = 0$$

$$(3.17)$$

$$\psi_{01,yyyy} - M_1^2 \psi_{01,yy} = -G(\theta_{01,y} + NC_{01,y})$$

$$(3.18)$$

$$\psi_{01}(+1) - \psi_{01}(-1) = 0, \psi_{01,y} = 0, \psi_{01,x} = 0 \text{ at } y = \pm 1$$

$$\theta_{10,yy} = GP(\psi_{00,y} \theta_{00,x} - \psi_{00,x} \theta_{00,y}), \quad \theta_{10}(\pm 1) = 0$$

$$(3.19)$$

$$C_{10,yy} = Sc(\psi_{00,y} C_{00,x} - \psi_{00,x} C_{00,y}), \quad C_{10}(\pm 1) = 0$$

$$(3.20)$$

$$\psi_{10,yyyy} - M_1^2 \psi_{10,yy} = -G(\theta_{10,y} + NC_{10,y}) +$$

$$+ (\psi_{00,y} \psi_{00,yyyy} - \psi_{00,x} \psi_{00,yy})$$

$$(3.21)$$

$$\psi_{10}(+1) - \psi_{10}(-1) = 0, \psi_{10,y} = 0, \psi_{10,x} = 0 \text{ at } y = \pm 1$$
\[ \theta_{11,yy} = P(\psi_{00,y} \theta_{01,x} - \psi_{01,x} \theta_{00,y} + \theta_{00,x} \psi_{01,y}) \]  \hspace{1cm} (3.22)

\[ - \theta_{01,y} \psi_{01,x} \frac{2PR^2}{G} \psi_{00,y} \psi_{10,yy} - \frac{2PM_1^2}{G} \psi_{00,y} \psi_{10,y} \right) \right) , \theta_i(\pm 1) = 0 \]

\[ C_{11,yy} = Sc(\psi_{00,y} C_{01,x} - \psi_{01,x} C_{00,y} + C_{00,x} \psi_{01,y} - C_{01,y} \psi_{00,x}) \]  \hspace{1cm} (3.23)

\[ \psi_{11,yyyy} - M_1^2 \psi_{11,yy} = -G(\theta_{11,y} + NC_{11,y}) + (\psi_{00,y} \psi_{11,yyyy} - \psi_{00,x} \psi_{01,yyyy} \psi_{01,x} \psi_{01,yyyy}) \]  \hspace{1cm} (3.24)

\[ \psi_{11}(+1) - \psi_{11}(-1) = 0, \psi_{11,y} = 0, \psi_{11,x} = 0 \text{ at } y = \pm 1 \]

4. SOLUTION OF THE PROBLEM

Solving the equations (3.13)- (3.24) subject to the relevant boundary conditions we obtain

\[ \theta_{\infty}(y, t) = \left( \frac{\alpha}{2} \right)(1 - y^2) + \left( \frac{\sin(D_1)}{2} \right)y + \left( \frac{\sin(D_2)}{2} \right) \]

\[ C_{00} = 0.5(y^2 - 1) \]

\[ \psi_{\infty}(y, t) = a_3(y^3 - y) + \frac{(1 + 2a_3)}{(M_1 \text{Ch}(M_1) - Sh(M_1))} (Sh(M_1)y) \]

\[ - y Sh(M_1) + 1 + a_1 + a_5 \text{Ch}(M_1,y) + a_4 y^2 \]
\[ \theta_{01}(y,t) = a_{15}(y^2 - 1) + a_{16}(y^4 - 1) + a_{17}(y^6 - 1) \]

\[ + a_{18}(ySh(M_1y) - Sh(M_1)) - a_{19}(Ch(M_1y) - Ch(M_1)) \]

\[ - Ch(M_1)) + a_{20}(Ch(2M_1y) - Ch(2M_1)) \]

\[ \psi_{01}(y,t) = a_{23}(y^2 - 1) + a_{24}(y^4 - 1) + a_{25}(y^6 - 1) + \]

\[ + a_{26}(ySh(M_1y) - Sh(M_1)) + a_{27}(Ch(M_1y) - Ch(M_1)) \]

\[ - Ch(M_1)) + a_{28}(Ch(2M_1y) - Ch(2M_1)) \]

\[ \psi_{01}(y,t) = a_{34} + a_{35}y + a_{36}Ch(M_1y) + a_{37}Sh(M_1y) + a_{38}y^2 + a_{39}y^4 + \]

\[ + a_{40}y^6 + a_{41}yCh(M_1y) + a_{42}Ch(2M_1y) \]

\[ \psi_{10}(y,t) = b_{1}(y^2 - 1) + b_{2}(y^3 - y) + b_{3}(y^4 - 1) + b_{4}(y^5 - y) + b_{5}(y^6 - 1) + \]

\[ + b_{6}(Ch((M_1y)) - Ch(M_1)) + b_{7}(Sh(M_1y) - Sh(M_1)) \]

\[ - ySh(M_1)) + b_{8}(ySh(M_1y) - Sh(M_1)) \]

\[ C_{10}(y,t) = d_{59}(y^2 - 1) + d_{60}(y^3 - y) + d_{61}(y^4 - 1) + d_{62}(y^5 - y) + d_{63}(y^6 - 1) + \]

\[ + (d_{64} + yd_{65})(Ch((M_1y)) - Ch(M_1)) + d_{65}(Sh(M_1y) - ySh(M_1)) + \]

\[ d_{66}(ySh(M_1y) - Sh(M_1)) \]
\[ \psi_{10} = d_{80}y^2 + d_{81}y^3 + d_{82}y^4 + d_{83}y^5 + d_{84}y^6 + d_{85}y^7 + d_{86}ySh(M_1,y) + \\
+ d_{87}yCh(M_1,y) + d_{88}y^2Sh(M_1,y) + d_{89}y^3Ch(M_1,y) + d_{90}Ch(M_1,y) + \\
+ d_{91}Sh(M_1,y) + d_{92}y + d_{93} \]

where \( a_1, a_2, \ldots, a_{42}, b_1, b_2, \ldots, b_8, d_1, \ldots, d_{66} \) are constants given in the appendix

5. SHEAR STRESS, NUSSELT NUMBER AND SHERWOOD NUMBER

The shear stress on the channel walls is given by

\[ \tau = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y=\pm L} \]

which in the non-dimensional form reduces to

\[ \tau = \left( \frac{\mu U}{a} \right) \left( \psi_{yy} - \delta^2 \psi_{xx} \right) = [\psi_{00,yy} + E\psi_{01,yy} + \delta(\psi_{10,yy} + E\psi_{11,yy} + O(\delta^2))]_{y=\pm 1} \]

and the corresponding expressions are

\[ \tau_{y=+1} = b_{00} + \delta b_{91} + O(\delta^2) \]

\[ \tau_{y=-1} = b_{02} + \delta b_{93} + O(\delta^2) \]

The local rate of heat transfer coefficient (Nusselt number \( \text{Nu} \)) on the walls has been calculated using the formula

\[ \text{Nu} = \frac{1}{\theta_m - \theta_w} \left( \frac{\partial \theta}{\partial y} \right)_{y=\pm 1} \]
and the corresponding expressions are

\[
(Nu)_{y-1} = \frac{(b_{51} + \delta b_{52})}{(b_{44} - \sin(D_1) + \delta b_{45})}
\]

\[
(Nu)_{y-1} = \frac{(b_{53} + \delta b_{54})}{(b_{44} - 1 + \delta b_{45})}
\]

The local rate of mass transfer coefficient (Sherwood number) \((Sh)\) on the walls has been calculated using the formula

\[
Sh = \frac{1}{C_{n} - C_{w}} \left( \frac{\partial C}{\partial y} \right)_{y=1}
\]

and the corresponding expressions are

\[
(Sh)_{y=1} = \frac{(b_{55} + \delta b_{56})}{(b_{55} - 1 + \delta b_{57})}
\]

\[
(Sh)_{y-1} = \frac{(-b_{55} + \delta b_{56})}{(b_{58} + \delta b_{57})}
\]

where \(b_4, ....... b_{90}\) constants given in the appendix.
The aim of this analysis is to discuss the flow heat and mass transfer of a viscous, electrically conducting fluid through a porous medium in a vertical channel bounded by flat walls on which a traveling thermal wave is imposed. In this analysis the viscous, Darcy dissipation, and joule heating are taken into consideration. For computational purpose we take $P = 0.71$ and $\delta = 0.01$. The channel walls are heated or cooled according as the Grashof number $G$ is positive or negative. It is observed that the temperature variation on the boundary and the dissipative effects contribute substantially to the flow filed. This contribution may be represented as perturbations over the mixed convection flow generated. These perturbations not only depend on the wall temperature and $Ec$ but also on the nature of the mixed convection flow. In general we note that the creation of the reversal flow in the flow field depends on whether the free convection effect dominates over the forced flow or vice-versa. If the free convection effects are sufficiently large as to create reversal flow the variation in the wall temperature and $Ec$ affects the flow remarkably. The velocity $u$ is exhibited in figures (1-6) for different variations in the governing parameters $G$, $D$, $N$, $Sc$, $\alpha$, and $x$. In the case of heating of the walls the actual movement of the fluid in the direction of the buoyancy while in the case of cooling of the walls for lower value of $|G| = 10^3$ the fluid moves in the buoyancy direction and for higher $|G| \geq 3 \times 10^3$ the fluid in the right region is in the upward direction. Such movement in the
Fig. 1  Variation of $u$ with $G$
$D'^1 = 2 \times 10^3, Sc = 1.3, N = 1$

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<td>$5 \times 10^3$</td>
<td>$-10^3$</td>
<td>$-3 \times 10^3$</td>
<td>$-5 \times 10^3$</td>
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Fig. 2  $u$ with $D'^1$
$G = 3 \times 10^3$

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<td>$3 \times 10^3$</td>
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Fig. 5  $u$ with $\alpha$

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<td>-4</td>
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</table>

Fig. 6  $u$ with $x + \gamma t$

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<tbody>
<tr>
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<td>$\pi/2$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
</tr>
</tbody>
</table>
opposite direction takes place for $G \geq 3 \times 10^3$. These opposing regions in the right mid half gets shrinked with increase in $|G| (<0)$ (fig.1). The magnitude of $u$ increases with $G(>0)$ and decreases with $|G| (<0)$ with maximum $u$ occurring at $y = 0.4$ for $G>0$ and at $y = -0.6$ for $G<0$. From fig.2 we find that the reversed flow occurs for $D^{-1} \geq 3 \times 10^3$ and these zones in the region abutting the left boundary $y = -1$ get widened with $D^{-1}$ for $D^{-1} \leq 3 \times 10^3$. The velocity $u$ decreases in the region $-0.8 \leq y \leq 0.2$ and increases in the region $(0.4 \leq y \leq 0.8)$ while for higher $D^{-1} \geq 5 \times 10^3$, $|u|$ enhances in the entire fluid region. Thus lesser the permeability of the porous medium greater the magnitude of $u$. The velocity experiences a retardation with an increase in the magnitude filed. The variation of $u$ with the buoyancy ratio $N$ is shown in fig.3. The buoyancy ratio $N$ measures the relative importance of the concentration buoyancy over the thermal buoyancy. $N>1=<1$ corresponds to the molecular buoyancy either greater, equal or less than then the thermal buoyancy. $N$ is negative, positive or zero according as the molecular buoyancy either opposes or absent or acting in the direction of the thermal buoyancy. It is found that when the molecular buoyancy force dominates over the thermal buoyancy force we find a reversal flow in the left region and the velocity experiences lower speeds in the region near constant temperature boundary while it moves with enhanced velocities in the region abutting the boundary experiencing a thermal wavy temperature when the forces act in the same direction. On the other hand when the two buoyancy forces act in opposing directions the fluid near $y = -1$ moves with enhanced velocities and the fluid near $y = 1$ moves with lower velocities. Also we find in this case
Fig. 7 Variation of v with G
D_1=2 \times 10^3, S_c=1.3, N = 1
i ii m iv v vi
G 10^3 3 \times 10^3 5 \times 10^3

Fig. 8 Variation of v with G
G=3 \times 10^3
I II III IV
D 1 5 \times 10^2 10^3 3 \times 10^3 5 \times 10^3
that reversal flow does not occur for any $|N|(<0)$ (fig. 3). Fig. 4 depicts $u$ with Schmidt number $Sc$. For all variations in $Sc$ we find a reversal flow in the vicinity of the left boundary and this region shrinks with an increase in $Sc$. $|u|$ reduces with $Sc$. The effect of the heat source/sink on $u$ can be seen from fig. 5. For lower strength of the heat source $\alpha$ we find a reversal flow in the vicinity of the left boundary and as $\alpha$ increases to 4 this reversal zone disappears and again reappears near the right boundary for $\alpha = 6$. In the case of heat sink the reversal flow appears in the left region and as $|\alpha|(<0)$ increases this zone shifts to the right boundary. An increase in $\alpha$ enlarges the growth of the reversal zone. An increase in the strength of the heat source enhances the axial velocity $u$ in the left region and reduces in the right region while an increase in the strength of the heat sink enhances $|u|$ (fig. 5). For different values of the phase $x+\gamma t$ of the boundary wave we find a reversal flow in the periphery of the left boundary. The behaviour of $u$ in general is oscillatory with magnitude fluctuating. This in view of the periodic wall temperature (fig. 6).

The secondary velocity $v$ is plotted in figs. (7-13) for different variations in the governing parameters. It is observed that the profiles for $v$ are asymmetric bell shaped curves with maximum at $y = -0.4$. The secondary velocity ($v$) is directed towards the boundary in the heating case and towards the midplane in the cooling case. The magnitude of $v$ increases with an increase in $|G|$ (fig. 7). It is noted that the magnitude of $v$ in the heating case is always greater than in the cooling case. The variation of $v$ with $D$...
Fig. 9  $v$ with $N$

$G = 3 \times 10^3, D = 2 \times 10^3$

I II III IV

N I 2 -0.5 -0.8

Fig. 10  $v$ with $Sc$

I II III IV

Sc 0.24 0.6 1.3 2.01
Fig. 13 \( v \) with \( x + \gamma t \)

\begin{align*}
I & \quad II & \quad III & \quad IV & \quad VI \\
\pi/4 & \quad \pi/3 & \quad \pi/2 & \quad \pi & \quad 2\pi
\end{align*}

Fig. 14 Variation of Resultant Velocity (Rt) with G

\[
G = 3 \times 10^3, \quad Sc = 1.3, \quad N = 1
\]

\begin{align*}
I & \quad II & \quad III & \quad IV & \quad V & \quad VI \\
10^3 & \quad 3 \times 10^3 & \quad 5 \times 10^3 & \quad -10^3 & \quad -3 \times 10^3 & \quad -5 \times 10^3
\end{align*}
shows that for all values of $D'^{1}$ the velocity $v$ is always towards the boundary and it decreases with increase in $D'^{1}$. Thus lesser the permeability of the medium smaller the magnitude of the velocity (fig. 8). Fig. 9 shows the influence of $v$ with buoyancy ratio $N$. We find that irrespective of the directions of the buoyancy forces the secondary velocity $v$ reduces with increases in $|N| (> < 0)$. Also $v$ increases with increase in the Schmidt number $Sc$. Thus lesser the molecular diffusivity greater the secondary velocity in the flow field.

An increase in the strength of the heat source reduces $v$ and enhances it in the case of heat sink (fig. 11). For lower values of the Hartman number $M$ the secondary velocity $v$ is directed towards the boundary. For higher $M \geq 4$ the velocity experiences a sudden transition from outward to inward direction. The magnitude of $v$ increases with $M \leq 4$ and for higher $M \geq 6$ we find a retardation in $|v|$ (fig. 12). The effect of the thermal boundary wave on $v$ may be observed from fig. 13. We find a transition from towards the boundary to towards the midplane as we move along the axial direction. Thus transition takes place at $x = \pi/2$ and $\pi$.

The resultant velocity ($Rt$) is exhibited in figs (14)-(18). The resultant velocity gradually rises from rest to attain a maximum $y = 0.4$ and then falls to the prescribed value at $y = 1$. $Rt$ enhances in the heating case and reduces in the cooling case. For $D'^{1} \leq 3 \times 10^{3}$ we find that the resultant velocity reduces in the region $0.8 \leq y \leq 0.2$ and enhances in the region $(0.4 \leq y \leq 0.8)$ while for higher $D'^{1} \geq 5 \times 10^{3}$, we find an enhancement in $Rt$ in the entire fluid region (fig. 15). When the molecular buoyancy force dominates over
Fig. 17  $R_t$ with $Sc$
$D^{-1} = 2 \times 10^3, G = 2 \times 10^3, N = 1$
I  II  III  IV
Sc  0.24  0.6  1.34  2.01

Fig. 18  $R_t$ with $x + \gamma t$
$Sc = 1.3, N = 1, D^{-1} = 2 \times 10^3$
I  II  III  IV  V
$x + \gamma t$  $\pi/4$  $\pi/3$  $\pi/2$  $\pi$  $2\pi$
Fig 19  Variation of temperature (θ) with G

\[ D^3 = 2 \times 10^3, \text{Sc} = 1.3, N = 1 \]

\[ \begin{array}{cccccc}
I & II & III & IV & V & VI \\
G & 10^3 & 3 \times 10^3 & 5 \times 10^3 & -10^3 & -3 \times 10^3 & -5 \times 10^3 \\
\end{array} \]

Fig 20  θ with D^{-1}

\[ G = 3 \times 10^3, \text{Sc} = 1.3, N = 1 \]

\[ \begin{array}{cccccc}
I & II & III & IV \\
D^{-1} & 5 \times 10^2 & 10^3 & 3 \times 10^3 & 5 \times 10^3 \\
\end{array} \]
the thermal buoyancy force the resultant velocity enhances in the entire fluid region when the two buoyancy forces act in the same direction while in the case of the forces acting in opposite directions, the resultant velocity enhances in the left region and reduces in the right region. Fig. 17 shows that \( R_t \) experiences an enhancement in the left region and reduces in the right region with increase in \( S_c \). With increase in the phase \( x+\gamma t \) of the boundary wave \( R_t \) periodically varies in view of the periodic variation of the boundary wave (fig. 18).

The behaviour of the temperature distribution \( (\theta) \) for Prandtl number \( P = 0.71 \) is exhibited in fig (19)-(24). The perturbed temperature in general is positive and hence contributes to the enhancement of the actual temperature in the field. Fig 1a depicts the temperature \( \theta \) for different \( G(\geq 0) \). The temperature profiles exhibit that for all \( G(\geq 0) \) the temperature gradually enhances from its prescribed value on the left boundary to attain maximum in the mid plane and later falls to its prescribed value at \( y = 1 \). The temperature reduces in both heating and cooling cases. Also the temperature \( \theta \) enhances with \( D^{-1} \). Thus lesser the permeability of the porous medium greater the temperature in the entire fluid region (fig. 20). Irrespective of the directions of the buoyancy forces the temperature enhances with increase in the buoyancy ratio \( |N|(\geq 0) \) (fig. 21). The influence of the heat sources/sinks on the temperature (fig. 22) shows that \( \theta \) is positive for \( \alpha > 0 \) and \( \alpha = -2 \) and negative for \( |\alpha| \geq 4 \). This implies that for the strength of the heat sink greater than or equal
Fig. 23  \( \theta \) with \( M \)
\[ \alpha = 2, \text{Sc} = 1.3, N = 1, x + \gamma t = \frac{\pi}{4} \]

I  II  III
M  2   4   6

Fig. 24  \( \theta \) with \( x + \gamma t \)
\[ D^{-1} = 2 \times 10^3, G = 3 \times 10^3, \text{Sc} = 1.3, N = 1 \]

\( x + \gamma t \)  \( \pi/4 \)  \( \pi/3 \)  \( \pi/2 \)  \( \pi \)  \( 2\pi \)
Fig. 25 Variation of Concentration with $G$

$D^{-1} = 2 \times 10^3, Sc = 1.3, N = 1, \alpha = 2$

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<th>III</th>
<th>IV</th>
<th>V</th>
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<tr>
<td>$G$</td>
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<td>$-10^3$</td>
<td>$-3 \times 10^3$</td>
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</table>

Fig. 26 $C$ with $D^{-1}$

$G = 3 \times 10^3$

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</table>
\[ \frac{D^4}{\rho} = 2 \times 10^3 G = 3 \times 10^3 \alpha = 2, Sc = 1.3 \]

\[ M = 2, \alpha = 2, N = 1, x + y t = \frac{\pi}{4} \]

Fig. 27 C with N
Fig. 28 C with Sc

M = 2, \alpha = 2, N = 1, x + y t = \frac{\pi}{4}

I II III IV

Sc 0.24 0.6 1.3 2.01

I II III IV

N 1 2 -0.5 -0.8
Fig. 29  C with M
Sc=1.3, n=1, α = 2
I  II  III
M  2   4   6

Fig. 30  C with x + γt
D'1=2×10^3, G=3×10^3, SC=1.3
I  II  III  IV  V
x+γt  π/4  π/3  π/2  π  2π
to 4 the actual temperature is less than the equilibrium temperature. The temperature enhances in magnitude with increase in the strength of either heat sources or heat sink. Also \( \theta \) enhances with increase in \( M \) (fig.23). The temperature varies periodically with increase in the phase \( x+\gamma t \) of the boundary wave (fig.2). Thus we may conclude that the presence of the traveling thermal wave on any boundary wall changes \( \theta \) periodically in the fluid region.

The concentration distribution \( (C) \) for different variations in the governing parameters is exhibited in figs.(25)-(30). We find that the concentration is positive for all variations. This means that the actual concentration is less than the equilibrium concentration. The magnitude of \( C \) decreases in the heating case and enhances in the cooling case with maximum attained at \( y = -0.4 \). As the permeability of the porous medium decreases \( |C| \) enhances in the entire fluid region (fig.26). Also \( |C| \) decreases with increase in \( |N| (>0) \) irrespective of the directions of the buoyancy forces (fig.27). \( |C| \) decreases marginally with increase in \( \text{Sc} \). Also an enhancement in \( M \) leads to an increment in \( |C| \) in the field (fig.29). As the \( x+\gamma t \) of the boundary curve increases we find that \( |C| \) varies periodically (fig.30).

The shear stress (\( \tau \)) on the boundaries \( y \) at \( =\pm 1 \) have been evaluated for different variations in \( G,D \) and \( x+\gamma t \) and are shown in tables 1\&2. We find that the shear stress is negative at both the boundaries for all variations. The magnitude of stress at \( y = \pm 1 \) enhances with increase in \( D^{-1} \). Thus lesser the permeability of the medium larger the
Table 1
Shear Stress ($\tau$) at $y = 1$
P=0.71

<table>
<thead>
<tr>
<th>G</th>
<th>I</th>
<th>II</th>
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<th>V</th>
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<td>$3 \times 10^3$</td>
<td>-531.977</td>
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<td>-98.3075</td>
<td>-534.201</td>
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<td>$5 \times 10^3$</td>
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<td>-1211.23</td>
<td>-2211.19</td>
<td>-198.124</td>
<td>-1519.31</td>
<td>-1326.81</td>
<td>-1329.23</td>
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<td>-525.617</td>
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<table>
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<td>$\pi$</td>
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Table 2
Shear Stress ($\tau$) at $y = -1$
P=0.71

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<td>2.48314</td>
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<td>$-10^3$</td>
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<td>4.1392</td>
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### Table 3
Average Nusselt Number (Nu) at \( y = 1 \)

<table>
<thead>
<tr>
<th>( G )</th>
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<th>VI</th>
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<tr>
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<td>2.7317</td>
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<td>2.3484</td>
<td>71.0063</td>
<td>5.3375</td>
<td>5.3694</td>
<td>5.3733</td>
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<tr>
<td>( 5 \times 10^3 )</td>
<td>0.2313</td>
<td>1.1777</td>
<td>20.4743</td>
<td>3.6239</td>
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<tr>
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<td>2.7418</td>
<td>3.7773</td>
<td>-8.2413</td>
<td>26.6571</td>
<td>27.1455</td>
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<tr>
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<td>4.9521</td>
<td>4.9566</td>
</tr>
<tr>
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<td>13.4311</td>
<td>3.1992</td>
<td>3.2247</td>
<td>3.2277</td>
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### Table 4
Average Nusselt Number (Nu) at \( y = -1 \)

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<th>IV</th>
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<th>VI</th>
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<td>-4.2422</td>
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<td>-9.1496</td>
<td>-3.2649</td>
<td>-3.2763</td>
<td>-3.2775</td>
</tr>
<tr>
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<td>-2.2792</td>
<td>-11.3414</td>
<td>-3.9235</td>
<td>-3.9441</td>
<td>-3.9466</td>
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<tr>
<td>( -5 \times 10^3 )</td>
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<td>-7.3658</td>
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<td>-2.9016</td>
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<table>
<thead>
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<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
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<tbody>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>-0.5</td>
<td>-0.8</td>
</tr>
<tr>
<td>( 10^3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-0.5</td>
<td>-0.8</td>
</tr>
<tr>
<td>( 3 \times 10^3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-0.5</td>
<td>-0.8</td>
</tr>
<tr>
<td>( 2 \times 10^3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>-0.8</td>
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<td>( 2 \times 10^3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-0.5</td>
<td>-0.8</td>
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Table. 5
Average Nusselt Number (Nu) at y=1
P=0.71

<table>
<thead>
<tr>
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Table. 6
Average Nusselt Number (Nu) at y=-1
P=0.71

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<td>-4.4692</td>
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<td>2</td>
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### Table 7
Average Nusselt Number (Nu) at y = 1

\[ P = 0.71 \]

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</tr>
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<td>-6.3791</td>
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<td>9.02619</td>
<td>2.96569</td>
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### Table 8
Average Nusselt Number (Nu) at y = -1

\[ P = 0.71 \]

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\[ x + \gamma t \]

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\[ x + \gamma t \]
Table 9
Sherwood Number (Sh) at y = 1
P = 0.71

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<th>VI</th>
<th>VII</th>
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<td>0.7682</td>
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<td>0.7683</td>
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<td>0.2841</td>
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Table 10
Sherwood Number (Sh) at y = -1
P = 0.71

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<td>2.7832</td>
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Table 9
Sherwood Number (Sh) at y = 1
P = 0.71

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<td>1</td>
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<td>1</td>
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Table 11
Sherwood Number (Sh) at y=1
P = 0.71

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<td>9.0357</td>
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Table 12
Sherwood Number (Sh) at y = -1
P = 0.71

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<td>-16.0632</td>
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Table 13
Sherwood Number (Sh) at y=1
P=0.71

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<td>0.7689</td>
<td>0.7686</td>
<td>0.7687</td>
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<td>0.7671</td>
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<td>0.7663</td>
<td>0.7682</td>
<td>0.7689</td>
</tr>
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<td>0.7692</td>
<td>0.7695</td>
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<tr>
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Table 14
Sherwood Number (Sh) at y = -1
P=0.71

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Table 15
Sherwood Number (Nu) at y = -1
P=0.71

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<tbody>
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<td>0.7687</td>
<td>0.7681</td>
<td>0.7688</td>
</tr>
<tr>
<td>$3x10^3$</td>
<td>0.7671</td>
<td>0.7697</td>
<td>0.7689</td>
<td>0.7665</td>
<td>0.7695</td>
</tr>
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<td>0.7695</td>
<td>0.7726</td>
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</tr>
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<td>0.77069</td>
<td>0.7684</td>
<td>0.7726</td>
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Table 16
Sherwood Number (Nu) at y = 1
P=0.71

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<th>IV</th>
<th>V</th>
</tr>
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<td>13.953</td>
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<tr>
<td>$-5x10^3$</td>
<td>0.5553</td>
<td>0.7868</td>
<td>-6.7882</td>
<td>-0.3934</td>
<td>0.3911</td>
</tr>
</tbody>
</table>

| $x+y t$ | | | | | |
| $\pi/4$ | | | | | |
| $\pi/3$ | | | | | |
| $\pi/2$ | | | | | |
| $\pi$  | | | | | |
| $2\pi$ | | | | | |
magnitude of $\tau$ at both the boundaries. As the phase of the traveling boundary wave increases the magnitude of $\tau$ varies periodically at both the boundaries (tables 1 & 3).

The average Nusselt number which measures the local rate of heat transfer is shown in tables (3)-(8) for variations in $G, D, N, M, \alpha$ and $x+\gamma t$. The magnitude of $Nu$ reduces with $|G|(< >0)$ in both the cases of heating and cooling of the channel walls. $|Nu|$ at $y=\pm 1$ enhances with $D^{1}$. Thus lesser the permeability of the porous medium larger the magnitude of the $Nu$. When the molecular buoyancy force dominates over the thermal buoyancy force $|Nu|$ enhances with $N$ irrespective of the directions of the forces. In a given porous medium $|Nu|$ reduces with $M$. With increase in the strength of the heat sources/sinks we find that the magnitude of $Nu$ increases for any $|G|$ or $D$(table. 3-6). From table 7&8 we observe that $|Nu|$ fluctuates in magnitude as the phase of the traveling thermal wave increases.

The Sherwood number (Sh) which measures the rate of mass transfer is exhibited in tables 9-16 for different variations of parameters. We find that in a given porous medium the rate of mass transfer at $y=\pm 1$ decreases in the heating case and increases in the cooling case. Also lesser they permeability of the medium greater the magnitude of $Sh y=\pm 1$. Irrespective of the directions of the buoyancy forces we find a marginal enhancement in $|Sh|$ with increase in $|N|(< >0)$. An increase in the strength of the magnetic field or Schmidt number $Sc$ enhances $|Sh|$ at both the walls (Table 12&14). Also
an increment in $x+\gamma t$ periodically varies the magnitude of $|Sh|$. This is in view of the traveling thermal wave imposed on the walls (tables 15&16).

7. REFERENCES


APPENDIX

\[ M_1^2 = D^{-1} + M^2 \]

\[ D_1 = x + y \]

\[ Ch(M_1) = Cosh(M_1) \]

\[ Sh(M_1) = Sinh(M_1) \]

\[ a_1 = -G\alpha \]

\[ a_2 = G(Sin(D_1) - 1) + N/2 \]

\[ a_3 = -\frac{a_1}{6M_1^2} \]

\[ a_4 = -\frac{a_2}{2M_1^2} \]

\[ a_{s1} = -\frac{(1+2a_2)}{(M_1 Ch(M_1) - Sh(M_1))} \]

\[ a_{s2} = -\frac{2a_4}{M_1 Sh(M_1)} \]

\[ a_6 = -PM_1^2(4a_4^2 Ch^2(M_1) + 9a_3^2) \]

\[ a_7 = -PM_1^2(4a_4^2 - 18a_3^2) \]

\[ a_8 = 9a_3^2 \]

\[ a_9 = 2PM_1^4a_3^2 Ch(M_1) \]

\[ a_{10} = \frac{8PM_1^7a_3^2}{Sh(M_1)} \]

\[ a_{11} = \frac{4a_3^2}{Sh^3(M_1)} \]

\[ a_{12} = M_1^2a_4^2 \]

\[ a_{13} = a_6 + (a_{12} - a_{11})/2 \]
\[ a_{14} = \frac{(a_{12} + a_{13})}{2} \qquad ; \qquad a_{15} = \frac{a_{13}}{2} \]

\[ a_{16} = \frac{a_{7}}{2} \qquad ; \qquad a_{17} = \frac{a_{8}}{30} \]

\[ a_{18} = \frac{a_{10}}{M^2_1} \qquad ; \qquad a_{19} = \frac{1 + M^2_1}{M^2_1} \]

\[ a_{20} = \frac{a_{14}}{4M^2_1} \qquad ; \qquad a_{23} = 0 \]

\[ a_{24} = 0 \qquad ; \qquad a_{25} = 0 \]

\[ a_{26} = \frac{ScSoa_{16}}{NM^2_1} \qquad ; \qquad a_{27} = -\frac{ScSo(M^2_1 + 1)}{NM^3_1} \]

\[ a_{28} = 0 \qquad ; \qquad G_1 = -GgScSo \]

\[ a_{29} = G_1 a_{13} \qquad ; \qquad a_{30} = G_1 a_7 \]

\[ a_{31} = G_1 a_8 \qquad ; \qquad a_{32} = G_1 a_{10} \]

\[ a_{33} = G_1 a_{14} \qquad ; \qquad a_{38} = \frac{a_{36}}{12M^2_1} \]

\[ a_{41} = \frac{a_{32}}{2M^3_1} \qquad ; \qquad a_{37} = -\frac{a_{41} Sh(M_1)}{Ch(M_1)} \]

\[ a_{42} = \frac{a_{43}}{12M^4_1} \]

\[ a_{43} = (a_{51} Ch(M_1) - a_{51}^2) \left( \frac{SidD_1 - 1}{2} \right) + CosD_1 (M_1 a_{51} Ch(M_1) - 3a_3) \]

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\[ a_{44} = 2\alpha(a'_{22}Ch(M_1) - a'_2) + \left(\frac{\text{S}i\text{d}D_1 - 1}{2}\right)(a_{31}Sh(M_1 - a'_3) + \text{C}o\text{s}D_1(M_1a_4Ch(M_1) - 3a_4)) \]

\[ a_{45} = 2\alpha(a'_{31}Sh(M_1) + a'_3) + a'_4\text{C}o\text{s}(D_1) \]

\[ a_{46} = -2\alpha a'_4 + a'_3\left(\frac{\text{S}i\text{d}D_1 - 1}{2}\right) - 3a_4\text{C}o\text{s}D_1 / 2 \]

\[ a_{47} = -2\alpha a'_3 \quad ; \quad a_{48} = a_{52}\left(\frac{\text{S}i\text{d}D_1 - 1}{2}\right) - M_1a_4\text{C}o\text{s}D_1 / 2 \]

\[ a_{49} = -2a'_3\left(\frac{\text{S}i\text{d}D_1 - 1}{2}\right) + a_4\text{C}o\text{s}D_1 / \text{S}h(M_1) \]

\[ a_{50} = -2\alpha a'_1 + 7a_2\text{C}o\text{s}D_1 / \text{S}h(M_1) \]

\[ a_{52} = -0.5a'_1\text{C}h(M_1) - 0.5a'_4 \quad ; \quad a_{51} = \text{S}e\alpha 3 / 2 \]

\[ a_{54} = 0 \quad ; \quad a_{55} = \text{S}e\alpha'_3 / 2 \]

\[ a_{56} = \text{S}c(1 - 0.5a_3, \text{S}h(M_1)) \quad ; \quad a_{57} = 0.5\text{S}e\alpha'_3 \]

\[ a_{58} = 0 \quad ; \quad a_{59} = 0.5\text{S}e\alpha'_5 \]

\[ a_{60} = 0 \quad ; \quad b_1 = \frac{a_{43}}{2} \]

\[ b_2 = \frac{a_{45}}{6} \quad ; \quad b_3 = \frac{a_{45}}{12} \]

\[ b_4 = \frac{a_{46}}{20} \quad ; \quad b_5 = \frac{a_{47}}{30} \]
\[ b_6 = \frac{a_{48}}{2} - \frac{2a_{30}}{M_1^2} \quad ; \quad b_7 = \frac{a_{49}}{M_1^2} \]

\[ b_8 = \frac{a_{50}}{M_1^2} \quad ; \quad b_{15} = -(b_2 + b_4) \]

\[ b_{16} = 2b_1 \quad ; \quad b_{17} = 3b_2 \]

\[ b_{18} = 4b_3 \quad ; \quad b_{19} = 5b_{14} \]

\[ b_{20} = 6b_4 \quad ; \quad b_{21} = M_1b_6 + b_8 \]

\[ b_{22} = M_1b_7 \quad ; \quad b_{23} = -M_1b_8 \]

\[ b_{24} = G_1b_{15} - 2a_3(M_1a_{51}chM_1 + 3a_3) + 6a_3(a_{32}chM_1 + a_3') \]

\[ b_{25} = G_1b_{16} + 4a_4a_4' - 6M_1a_{32}a_3'chM_1 + 6a_3(a_3'shM_1 + a_3') \]

\[ b_{26} = a_1b_17 - 3a_3a_4' + 12a_3'a_4 \]

\[ b_{27} = a_1b_18 - 3a_3a_4' \]

\[ b_{28} = a_1b_{19} \quad ; \quad b_{29} = G_1b_{20} \]

\[ b_{30} = G_1b_{22} + 2a_4M_1A_{32} - \frac{M_1^2a_{32}chM_1}{shM_1} - M_1^3A_{32}a_3'shM_1 - 6a_3A_{32} \]

\[ b_{31} = G_1b_{21} + \frac{4a_4a_4'}{shM_1} - M_1^2a_{51}(M_1a_{51} - 3a_3) + \frac{2a_4M_1^2a_3'}{shM_1} - 6a_3a_{51} \]
\[ b_{32} = \frac{-b_{24}}{2M_1^2}; \quad b_{33} = \frac{-b_{25}}{6M_1^2} \]

\[ b_{34} = \frac{-b_{26}}{12M_1^2}; \quad b_{35} = \frac{-b_{27}}{20M_1^2} \]

\[ b_{36} = \frac{-b_{28}}{30M_1^2}; \quad b_{37} = \frac{-b_{29}}{42M_1^2} \]

\[ b_{38} = \frac{b_{30}}{2M_1^3}; \quad b_{39} = \frac{b_{31}}{2M_1^3} \]

\[ b_{41} = -(2b_{33} + 4b_{35} + 7b_{37} + b_{38}shM_1)/(M_1chM_1 - shM_1) \]

\[ a_{36} = -[(2a_{38} + 4a_{40} + 2M_1a_{28}shM_1)]/(M_1shM_1) \]

\[ b_{44} = b_{42} + Ec b_{43} \]
\[ b_{45} = \frac{-8}{5} b_3 - \frac{12}{7} b_5 + b_6 \left( \frac{2shM_1}{M_1} - 2chM_1 \right) + 2b_4shM_1 \]

\[ b_{46} = -\alpha + \frac{2}{2} \left( \frac{\sin D - 1}{\sin D} \right); \quad b_{47} = \alpha + \frac{\sin D - 1}{2} \]
\[ b_{48} = 2a_{13} + 4a_{15} + 6a_{17} + a_{19}(shM_1 + M_1chM_1) - a_{10}M_1shM_1 + 2M_1a_{20}sh(2M_1) \]
\[ b_{49} = 2b_1 + 4b_3 + 6b_5 + M_1b_4shM_1 + b_4(shM_1 + M_1chM_1) \]
\[ b_{50} = 2b_2 + 4b_4 + b_4(M_1chM_1 - shM_1) \]
\[ b_{51} = b_{46} + Ec b_{48} \]; \quad b_{52} = b_{49} + b_{50} \]
\[ b_{53} = b_{47} - Ec b_{48} \]; \quad b_{54} = b_{50} - b_{49} \]
\[ b_{55} = 0 \]
\[ b_{60} = \frac{4}{3} a_{23} - \frac{8}{5} a_{24} - \frac{12}{7} a_{25} + a_{26} \left( \frac{2cM_1}{M_1} - \frac{shM_1}{M_1^2} - shM_1 \right) + \\
+ a_{27} \left( \frac{2shM_1}{M_1} - 2cM_1 \right) + a_{28} \left( \frac{sh(2M_1)}{2M_1} - 2cM(2M_1) \right) \\
\]

\[ b_{57} = \frac{4}{3} b_8 - \frac{8}{5} b_{10} - \frac{12}{7} b_{12} + b_{13} \left( \frac{2shM_1}{M_1} - 2cM_1 \right) + b_{14} \left( \frac{2cM_1}{M_1} - \frac{2shM_1}{M_1^2} - 2shM_1 \right) \\
\]

\[ b_{58} = b_{55} + Ec \]

\[ b_{59} = 0 \]

\[ b_{60} = 2a_{23} + 4a_{24} + 6a_{25} + (a_{26} + M_1 a_{28}) shM_1 + M_1 a_{26} ch(M_1) + 2M_1 a_{28} sh(2M_1) \\
\]

\[ b_{61} = 2b_9 + 4b_{10} + 4b_{11} \\
\]

\[ b_{62} = 2b_8 + 6b_{12} + (M_1 b_{13} + b_{14}) shM_1 + M_1 b_{14} chM_1 \\
\]

\[ b_{63} = b_{61} + b_{62} \]

\[ b_{64} = b_{61} - b_{62} \\
\]

\[ b_{65} = b_{59} + Ec b_{60} \]

\[ b_{66} = M_1^2 a_{35} shM_1 + 6a_3 \\
\]

\[ b_{67} = 2a_4 (1 - M_1 coth(M_1)) \]

\[ b_{68} = b_{66} + b_{67} \\
\]

\[ b_{69} = -b_{66} + b_{67} \\
\]

\[ b_{70} = [(M_1 a_{39} + a_{41}) M_1 shM_1 + M_1 a_4 ch(M_1) + M_1^2 a_{41} ch(M_1) + 2a_8 (1 - M_1 ch(M_1))] \\
\]

\[ b_{71} = 4a_{39} (3 - M_1 coth(M_1)) + 6a_{40} (5 - M_1 coth(M_1)) \\
\]

\[ b_{72} = b_{70} + b_{71} \]

\[ b_{73} = b_{71} - b_{70} \\
\]

\[ b_{74} = 6b_{35} + 20b_{36} + 42b_{37} + 4b_{38} (M_1 shM_1 + M_1^2 chM_1 + shM_1) + b_{41} M_1^2 sh(M_1) \\
\]

\[ b_{75} = 2b_{33} (1 - M_1 coth(M_1)) + 4b_{34} (3 - M_1 coth(M_1)) + 6b_{36} (5 - M_1 coth(M_1)) + \\
+ b_{39} (M_1^2 sh(M_1) + chM_1 - M_1^2 coth(M_1) ch(M_1)) \\
\]

\[ b_{76} = b_{74} + b_{75} \]

\[ b_{77} = b_{76} - b_{74} \\
\]
\[ b_{78} = A_{52}^1 M_1 sh(M_1) + 2a_4^1 \]

\[ b_{79} = a_{51}(M_1, chM_1 - shM_1) + 2a_3^1 \]

\[ b_{80} = b_{78} + b_{79} \]

\[ b_{81} = b_{79} - b_{78} \]

\[ b_{82} = Ma_{36}^1 sh(M_1) + 2a_{33}^1 + 4a_{39}^1 + 2M_1 a_{42}^1 sh(2M_1) + \frac{a_{37}^1 M_1}{shM_1} \]

\[ b_{83} = -\frac{a_{37}^1}{shM_1} + a_{41}^1 M_1 sh(M_1) \]

\[ b_{84} = b_{82} + b_{83} \]

\[ b_{85} = b_{83} - b_{82} \]

\[ b_{86} = 2b_{32}^1 + 4b_{34}^1 + 6b_{36}^1 + b_{30}^1 (M_1, chM_1 + shM_1) + M_1 b_{46}^1 shM_1 \]

\[ b_{87} = 2b_{37}^1 + 4b_{35}^1 + 6b_{37}^1 + M_1 b_{44}^1 sh(M_1) + b_{41}^1 (M_1, chM_1 - shM_1) \]

\[ b_{88} = b_{86} + b_{87} \]

\[ b_{89} = b_{86} + Ecb_{72} \]

\[ b_{90} = b_{88} + Ecb_{72} \]

\[ b_{91} = b_{89} + Ecb_{73} \]

\[ b_{93} = b_{81} + Ecb_{85} \]

\[ d_{42} = -(a_1^1 chM_1 + a_4^1)/2 \]

\[ d_{43} = -(a_5^1 chM_1 + a_1^1 shM_1 + a_4^1)/2 \]

\[ d_{44} = \frac{a_4^1}{2} \]

\[ d_{45} = a_3^1/2 \]

\[ d_{46} = 0 \]

\[ d_{47} = \frac{a_{52}^1}{2} \]

\[ d_{48} = \frac{a_{51}^1}{2} \]

\[ d_{50} = 0 \]

\[ d_{51} = d_{42} \]

\[ d_{52} = d_{41} \]

\[ d_{53} = d_{44} \]

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\[d_{34} = d_{45}; \quad d_{55} = d_{46}\]
\[d_{36} = d_{47}; \quad d_{53} = d_{48}\]
\[d_{38} = d_{50}; \quad d_{59} = \frac{d_{51}}{2}\]
\[d_{40} = \frac{d_{52}}{6}; \quad d_{61} = \frac{d_{53}}{12}\]
\[d_{42} = \frac{d_{54}}{20}; \quad d_{63} = \frac{d_{55}}{30}\]
\[d_{64} = \frac{d_{56}}{M_1^2}; \quad d_{65} = \frac{d_{57}}{M_1^2}\]
\[d_{66} = d_{49}M_1; \quad d_{67} = d_{58}M_1\]

\[d_{68} = -(d_{60} + d_{62} + d_{65} + d_{67}ch(M_1))\]
\[d_{69} = -(d_{59} + d_{61} + d_{63} + d_{64}ch(M_1))\]
\[d_{70} = G(b_{15} + Nd_{60} + b_{42})\]
\[d_{71} = G(b_{16} + Nd_{68} + b_{43})\]
\[d_{72} = G(b_{17} + Nd_{59} + b_{44})\]
\[d_{73} = G(b_{18} + Nd_{60} + b_{45})\]
\[d_{74} = G(b_{19} + Nd_{61})\]
\[d_{75} = G(b_{20} + Nd_{62})\]
\[d_{76} = G(b_{21} + Nd_{63})\]
\[d_{77} = G(b_{22} + Nd_{64})\]
\[d_{78} = G(b_{23} + Nd_{65})\]
\[d_{79} = GNd_{66}\]
\[d_{80} = -b_{70} / 2M_1^2\]
\[d_{81} = -b_{71} / 6M_1^2\]
\[d_{82} = -b_{72} / 12M_1^2\]
\[d_{83} = -b_{73} / 20M_1^2\]
\[d_{84} = -b_{74} / 30M_1^2\]
\[d_{85} = -b_{75} / 42M_1^2\]
\[d_{86} = b_{76} / 2M_1^3\]
\[d_{87} = \frac{d_{77}}{2M_1^3}\]

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\[ d_{88} = \frac{d_{78}}{2M_1^3} ; \quad d_{99} = \frac{d_{79}}{2M_1^3} \]
\[ d_{90} = -2d_{40} + 6d_{84} + d_{86}(shM_1 + MchM_1) + d_{89}(2chM_1 + M_1shM_1)/M_1shM_1 \]
\[ d_{91} = (d_{94} - d_{95})/(M_1chM_1 - shM_1) \]
\[ d_{94} = d_{91}sh(M_1) + d_{92} \]
\[ d_{95} = d_{91}M_1chM_1 + d_{92} \]
\[ d_{96} = 2d_{86} + 6d_{87} + 12d_{82} + 30d_{44} + 42d_{85} \]
\[ d_{97} = -2M_1(-2d_{89} - 4d_{42} - 6d_{84} - M_1d_{86} - d_{85}chM_1)chM_1d \]
\[ d_{98} = [M_1(d_{86} + d_{87} + d_{85}) + 2d_{89}]chM_1 \]
\[ d_{99} = (M_1^2d_{86} + (4 + M_1^2)d_{87} + (2(1 + M_1^2))d_{88} + 3M_1d_{90})shM_1 \]