CHAPTER-2

PERIODIC POINTS AND THEIR STABILITY
ON A ONE –DIMENSIONAL CHAOTIC SYSTEMS

2.1 Introduction : [1,5,6,7,32,33,58,62,64,82,111]

This chapter highlights three objectives of the quadratic iterator

\[ x_{n+1} = F(x_n) = ax_n^2 - bx_n, n = 0,1,2, \ldots \]

Where \( x_n \in [0, 4] \), \( a \) and \( b \) and are tunable parameters. Firstly, by adopting suitable numerical methods and computer programs we evaluate the period-doubling: \( 1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow \ldots \) bifurcations, a route from order into chaos. Secondly, we analyze the stability of periodic points. Thirdly, we draw the bifurcation diagram in order to support our period doubling orbits and chaotic region, and some illuminating results are obtained as the measure of chaos. We first describe some numerical methods which are necessary for our purpose.

2.2 Numerical method for obtaining periodic points

To find a periodic point of our model we apply the Newton-Recurrence formula:
\[ x_{n+1} = x_n - \frac{1}{\frac{d}{dx} g(x_n)} g(x_n), \quad \text{where} \ n = 1, 2, 3, \ldots \]  

(2.1)

[We later see that this map \( g \) equals \( F_{n-I} \), where \( I \) is the identity function]

The Newton formula actually gives the zero(s) of a map, and to apply this numerical tool, one needs a number of recurrence formulae which are given below:

Let the initial value of \( x \) be \( x_0 \). Then

\[ F(x_0) = ax_0^2 - bx_0 = x_1 \text{ (say)}, \]

\[ F^2(x_0) = F(F(x_0)) = F(x_1) = ax_1^2 - bx_1 = x_2 \text{ (say)}, \]

Proceeding in this manner, the following recurrence formula can be established:

\[ x_n = ax_{n-1}^2 - bx_{n-1}, \quad n = 1, 2, 3, \ldots \]

2.3. Numerical Method for Finding Bifurcation Values

The derivative of \( F^n \) can be obtained as follows:

\[ \frac{dF}{dx} \bigg|_{x=x_0} = 2ax_0 - b \]

Again, by the chain rule of differentiation we get

\[ \frac{d(F^2)}{dx} \bigg|_{x=x_0} = \frac{dF}{dx} \bigg|_{F(x_0)} \frac{dF}{dx} \bigg|_{x=x_0} = (2ax_1 - b)(2ax_0 - b) \text{, Where } x_1 = F(x_0) \]
Proceeding in this way we can obtain

\[
\frac{d \left( F^n \right)}{dx} \bigg|_{x=x_0}^{n=2, \ldots, \ldots, \ldots, 2a x_{n-1} - b}^\ldots^n
\]

We recall that the value of \( b \) will be the bifurcation value for the map \( F^n \)
when its derivative \( \frac{d (F^n)}{dx} \) at a periodic point equals -1. Also the Feigenbaum theory says that

\[
b_{n+2} \approx b_{n+1} + \frac{b_{n+1} - b_n}{\delta}, \text{ Where } n = 1, 2, 3, \ldots
\]

(2.2)

and \( \delta \) is the Feigenbaum universal constant. We notice that if we put

\[
I = \frac{dF^n}{dx} + 1,
\]

Then \( I \) turn out to be a function of the parameter \( b \). The bifurcation value of the parameter \( b \) of the period \( n \) occurs when \( I (b) \) equals zero. This means, in order to find a bifurcation value of period \( n \), one needs the zero of the function \( I (b) \), which is given by the Secant method, applied to the function \( I (b) \) which is given by

\[
b_{n+1} - b_n = \frac{I(b_n)(b_n - b_{n-1})}{I(b_n) - I(b_{n-1})}
\]

(2.3)

This method depends very sensitively on the initial condition.
2.4. Our model and associated universal results

Our concerned model is

\[ F(x) = ax^2 - bx \quad (2.4) \]

Where \( x \in [0,4] \), \( a \) and \( b \) are tunable parameters. To find points of period-one, it is necessary to solve the equation given by \( F(x) = x \) which gives the points that satisfy the condition \( x_{n+1} = x_n \) for all \( n \). The solutions are \( x_1^* = 0 \) and \( x_2^* = (1+b)/a \). If we fix \( a = -1 \), the function \( F \) is maximum at \( x = -b/2 \) and its maximum value equals \( b^2/4 \).

Taking this as 4, we have \( b = \pm 4 \). We again fix \( b \in [-4,-1] \) for our purpose. In this case, the fixed points of \( F \) are the intersection of the graphs of \( y = F(x) \) and \( y = x \).

![Graph](image)

**Fig : 2.1 Graphs of \( y = F(x) \) and \( y = x \) for \( b = -4 \)**

The periodic points \( x_1^* \) and \( x_2^* \) are shown in the figure (2.1). The stability of the critical points may be determined using the following theorem:
Using stability theorem, we have \( \left| \frac{d}{dx} F(x_i^*) \right| = |b| > 1 \). Thus the fixed point \( x_i^* = 0 \) is always unstable for all \( b \in [-4,-1] \). For such small parameter values the fixed point \( x^*_2 \) is not stable anymore, it is unstable.

\( b \in -1 - b \) stable for \(-3 < b < -1\). For example, if we take the parameter value \( b = -2.9 \), then the orbit generated by the initial point \( x_0 = 1.5 \) attracted to the fixed point \( x^*_2 = 1.9 \) in the figure 2.2.

\[\text{Figure 2.2 Staircase for the initial point } x_0 = 1.5 \text{ and parameter } b = -2.9\]

Having studied the dynamics of the quadratic iterator \( F \) in detail for parameter values between -1 and -3, we continue to decrease \( b \) beyond -3. For such small parameter values the fixed point \( x^*_2 \) is not stable anymore, it is unstable.
Hence, the first bifurcation value is $b_1=-3$. To find points of period 2, we consider the iterated map $F^2(x)$. Here,

$$F^2(x) = -b(-bx-x^2)(-bx+x^2)^2$$

The periodic points of $F^2(x)$ are given by the equation

$$F^2(x) = x$$

which gives

$$x = x_1^*, x_2^*, x_{11}^*, x_{12}^*$$

where

$$x_1^* = 0, x_2^* = -1-b, x_{11}^* = 1/2(1-b - \sqrt{-3 + 2b + b^2});$$

$$x_{12}^* = \frac{1}{2}(1 - b + \sqrt{-3 + 2b + b^2})$$

These four points are the intersection of the graphs of $y = F^2(x)$ and $y = x$, in the figure 2.3. The periodic points $x_1^*, x_2^*, x_{11}^*$, and $x_{12}^*$ as marked as $A1, A2, B1$ and $B2$.

Fig : 2.3 Graphs of $y = F^2(x)$ and $y = x$ for $b = -4$
Stability of the first two fixed points is already discussed. Let us discuss the stability of the new points: $x_{11}^*$ and $x_{12}^*$. We note that these new solutions are defined only for $b \leq 3$. Moreover, at $b = -3$, $x_{11}^* = x_{12}^* = \frac{1}{2}(1 - b)$, i.e., these two solutions bifurcate from the fixed point $x_2^*$. The points $x_{11}^*$ and $x_{12}^*$ form a two-cycle, one being the image of the other.

Thus, at parameter $b = -3$, our map orbits undergo period-doubling bifurcations. Just above $b = -3$ the orbits converge to a single value of $x$. Just below $b = -3$, the orbits tend to this alteration between two values of $x$.

$$\left. \frac{dF(x)}{dx} \right|_{x = x_2^*} = b + 2 \quad (2.6)$$

tells us that function $\frac{dF(x)}{dx}$ passes through the value -1 as $b$ decreases through -3. Next we can evaluate the derivative of the second iterate function by using the chain-rule of differentiation:

$$\left. \frac{dF^2(x)}{dx} \right|_{x_1^*} = \frac{d}{dx} \left[ F(F(x)) \right] = \left. \frac{dF}{dx} \right|_{F(x)} \left. \frac{dF}{dx} \right|_x$$

If we now evaluate the derivative at one of the above two new fixed points, say $x_{11}^*$ then we find

$$\left. \frac{dF^2(x)}{dx} \right|_{x_{11}^*} = \left. \frac{dF}{dx} \right|_{x_{11}^*} \left. \frac{dF}{dx} \right|_{x_{11}^*} \left. \frac{dF^2(x)}{dx} \right|_{x_{11}^*} \quad (2.7)$$
In arriving at the last result, we made use of \( x_{12}^* = F(x_{11}^*) \) for the two fixed points. Equation (2.7) states a rather surprising and important result—

The derivative of \( F^2(x) \) are the same at both the fixed points that are actually part of the two-cycle. This result implies that both of these fixed points are either attracting or both are repelling, and that they have the same ‘degree’ of stability or instability. Again, since the derivative of \( F(x) \) equals -1 for the parameter \( b = -3 \), equation (2.7) tells us that the derivative of \( F^2(x) \) equals +1 for \( b = -3 \). As \( b \) decreases further, the derivative of \( F^2(x) \) increases and the fixed points become stable. Besides, the unstable fixed point of \( F(x) \) located at \( x_2^* \) is also an unstable fixed point of \( F^2(x) \).

The 2-cycle fixed points of \( F^2(x) \) continue to be stable fixed points until parameter value \( b_2 = -3.449489742783 \ldots \) we have values of \( x_{11}^* \) and \( x_{12}^* \) as 1.517638092051063 \ldots and 2.931851652577893 \ldots respectively at \( b_2 = -3.449489742783 \ldots \) Also for this value of \( b \)

\[
\frac{dF^2(x)}{dx} \bigg|_{x=1.517638092051063 \ldots} = -1, \quad \frac{dF^2(x)}{dx} \bigg|_{x=2.931851652577893 \ldots} = -1 \quad (2.8)
\]

The above results guarantee that if a system is stable or unstable at a periodic point, then the system is so at any other periodic point. So our study will be completed if we study the dynamics at any of the periodic
points. We can find that for values of $b$ smaller than $b_2$, the derivative is more negative than -1. Hence for $b$ values smaller than $b_2$, the 2-cycle points are repelling fixed points. We find that for values just smaller than $b_2$, the orbits settle into a 4-cycle, that is, the orbit cycles among 4 values which we can label as

$$x_{21}^*, x_{22}^*, x_{23}^*, x_{24}^*$$

These points are nothing but the intersection of the graphs of $y = F^4(x)$ and $Y = x$.

To determine these periodic points analytically, we need to solve an eight degree equation, namely $F^4(x) = x$ which is manually cumbersome and time consuming. Therefore, for finding periodic points, bifurcation values of $F^4$ as well as for higher iterated map functions, we have to write a computer program. We write here a C-program for our purpose.

Using the relation (2.4), an approximate value $b_3^1$ of $b$ is obtained. Since the Secant method needs two initial values, we use $b_3$ and a slightly large value, say $b_3 + 10^4$ as the two initial values to apply this method and ultimately obtain $b_3$. In the like manner, the same procedure is employed to obtain the successive bifurcation values $b_4, b_5$, etc. to our requirement. Through our numerical mechanism, we obtain some periodic points and bifurcation values. In the Table 1.1 Period doubling cascades are shown:
2.5. Period doubling cascade [25.32]

Table 2.1

<table>
<thead>
<tr>
<th>Period</th>
<th>One of the Periodic points</th>
<th>Bifurcation Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_1 = 2.0000000000000...$</td>
<td>$b_1 = -3.0000000000000...$</td>
</tr>
<tr>
<td>2</td>
<td>$x_2 = 1.517638090205...$</td>
<td>$b_2 = 3.449489742783...$</td>
</tr>
<tr>
<td>4</td>
<td>$x_3 = 2.905392825125$</td>
<td>$b_3 = -3.544090359552...$</td>
</tr>
<tr>
<td>8</td>
<td>$x_4 = 3.138826940664...$</td>
<td>$b_4 = 3.564407266095...$</td>
</tr>
<tr>
<td>16</td>
<td>$x_5 = 1.241736888630...$</td>
<td>$b_5 = -3.568759419544...$</td>
</tr>
<tr>
<td>32</td>
<td>$x_6 = 3.178136193507...$</td>
<td>$b_6 = -3.569691609801...$</td>
</tr>
<tr>
<td>64</td>
<td>$x_7 = 3.78152098553...$</td>
<td>$b_7 = -3.569891259378...$</td>
</tr>
<tr>
<td>128</td>
<td>$x_8 = 3.178158223315...$</td>
<td>$b_8 = -3.569934018374...$</td>
</tr>
<tr>
<td>256</td>
<td>$x_9 = 3.3.178160120824...$</td>
<td>$b_9 = -3.569943176048...$</td>
</tr>
<tr>
<td>512</td>
<td>$x_{10} = 1.696110052289...$</td>
<td>$b_{10} = -3.569945137342...$</td>
</tr>
<tr>
<td>1024</td>
<td>$x_{11} = 1.696240778303...$</td>
<td>$b_{11} = -3.569945557391$</td>
</tr>
</tbody>
</table>

Based on these values, the ratios of successive separations of bifurcation points are given by,

$$\frac{b_2 - b_1}{b_3 - b_2} \approx \frac{b_4 - b_3}{b_5 - b_4} \approx \frac{b_6 - b_5}{b_7 - b_6} \approx \cdots \approx \frac{b_{k} - b_{k-1}}{b_{k+1} - b_{k}}$$
And have a particular scaling associated with them. We see that

\[ \delta_1 = \frac{b_2 - b_1}{b_3 - b_2} = 4.751446218163496..., \delta_2 = \frac{b_3 - b_2}{b_4 - b_3} = 4.6562510178075..., \]

\[ \delta_3 = \frac{b_4 - b_3}{b_5 - b_4} = 4.66824223480187..., \delta_4 = \frac{b_5 - b_4}{b_6 - b_5} = 4.66877394728035..., \]

\[ \delta_5 = \frac{b_6 - b_5}{b_7 - b_6} = 4.6691320114184..., \delta_6 = \frac{b_7 - b_6}{b_8 - b_7} = 4.6691829948741..., \]

\[ \delta_7 = \frac{b_8 - b_7}{b_9 - b_8} = 4.669198318862..., \delta_8 = \frac{b_9 - b_8}{b_{10} - b_9} = 4.6692000279164..., \]

and so on.

The ratios tend to a constant as \( k \) tends to infinity: more formally

\[ \lim_{k \to \infty} \left( \frac{b_k - b_{k-1}}{b_{k+1} - b_k} \right) = \delta = 4.669201... \]

The nature of \( \delta \) is universal i.e. it is the same for a wide range of different iterators

2.6. Bifurcation diagram

The different behaviors of a system for different values of the parameter can be qualitatively analyzed by using a bifurcation diagram, which is created by plotting the asymptotic orbits of the maps \( y \) axis) generated for different values of the parameter \( x \) axis). A bifurcation diagram is essentially a diagram of attractors, because almost all initial points are attracted to the points shown in the figure of our model, provided a sufficient
Numbers of transients have been thrown away. Fixed points and periodic points are trivial attractor, while the darkened vertical segments are chaotic attractors. Just beyond \( b = -3.96995 \) (approx) the system becomes chaotic. However, the system is not chaotic for all parameter values \( b \) smaller than \(-3.56995\) (approx). If we zoom into the details of the bifurcation diagram by changing to smaller and smaller scales both in \( x \) and in \( b \), we see that within the chaotic region, there are many periodic windows, that is, lucid intervals where only periodic orbits exist instead of chaotic output.
Figure 2.5 Bifurcation diagram for $b$ in the range $-4.0 \leq b \leq 2.9$

2.7 STATISTICAL ANALYSIS: [30, 51]

2.5.1 Coefficients of variation: At the first instance, the method used for describing the dynamic behaviour of the equation is through a “Bifurcation diagram”, and this method gives a global view of the long term behaviour of the model over a range of parameter values and allows a simultaneous comparisons of periodic and chaotic behaviours that a model may exhibit with changing parameters.

Here, we calculate the coefficients of variation for different parameter ranges as follows:

(i) In between 2$^{nd}$ and 3$^{rd}$ bifurcations, we consider 10 equidistant points with the step length 0.005 as
-3.4494897, -3.4544897, -3.4594897, -3.4644897, -3.4694897, -3.4744897,

Then mean, \( m_1 = -3.471437 \), standard deviation, \( \sigma_1 = 0.143614066163 \) and coefficient of variation, \( v_1 = 6.294997201869. \)

We obtain analogous results as follows:

(ii) In between 3\(^{rd}\) and 4\(^{th}\) bifurcation points, we consider 10 equidistant points:
range: -3.54409- (-3.56444) with the step length 0.012.
Then \( m_2 = -3.54904 \), \( \sigma_2 = 0.034467375879 \) and \( v_2 = 1.330581218315. \)

(iii) In between 4\(^{th}\) and 5\(^{th}\) bifurcation points, we consider 10 equidistant points:
range: -3.564407- (-3.56875) with the step length 0.002.
Then \( m_3 = -3.56673 \), \( \sigma_3 = 0.005744562647 \) and \( v_3 = 0.215369948882 \)

(iv) In between 5\(^{th}\) and 6\(^{th}\) bifurcation points, we consider 10 equidistant points:
range: -3.568759- (-3.569691) with the step length 0.0007.
Then \( m_4 = -3.568795 \), \( \sigma_4 = 0.002010596926 \) and \( v_4 = 0.074786472737. \)
(v) In between 6\textsuperscript{th} and 7\textsuperscript{th} bifurcation points, we consider 10 equidistant points:

range: -3.5696916 – (- 3.56993401) with the step length 0.000025

Then, \( m_5 = -3.569716 \), \( \sigma_5 = 0.000071807033 \), and \( v_5 = 0.002667291936 \)

and so on.

From the above, we find that \( v_1 > v_2 > v_3 > v_4 > v_5 > v_6 > \ldots \ldots \)

As we move from 2\textsuperscript{nd} periodic orbit to 4\textsuperscript{th} periodic orbit, from 4\textsuperscript{th} periodic orbit to 8\textsuperscript{th} periodic orbit, ….. we can conclude that our periodic orbits are more consistent, more uniform, more stable and more homogeneous. This means that we have a most prominent and reliable universal route from the regular region to the chaotic region.

2.8. Conclusion

The study of chaos in population models is quite interesting. Although there are so many methods for finding bifurcation values, we have developed our own numerical mechanism for establishing Feigenbaum tree of bifurcation values leading to chaotic region the study of which is intrinsically marvelous. Our method seems to be applicable to all the chaotic models.

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