CHAPTER 1

BASIC CONCEPTS AND FUNDAMENTAL RESULTS

1.01 Introduction:

Before embarking on the exposition of our principal object we wish to keep this introductory chapter exclusively for a preliminary introduction to the background, related references, fundamental definitions, and results which are collected from dispersed literature and which are fundamental requisites needed to recall while going through the body of the thesis.

Nonlinear Population Theory [1,2,5,10,14,31,33,37,38,49,74,77,78,94,98] has emerged as one of the most sophisticated research areas in the field of Dynamical Systems and Fractal Geometry, because of its interdisciplinary nature in dynamic behaviors such as periodic cycles, periodic orbits, multiple attractors, quasi-periodicity and invariant loops, unstable equilibria with stable and unstable manifolds, chaos, strange attractors, fractal basins and so on. Population dynamics has traditionally been the dominant branch of Bio-Statistics, which in turn is an integral part of Bio-Sciences, and more recently the scope of Mathematical & Statistical Biology has greatly expanded due to their tremendous application in Biosciences.
First order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random functions. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. One of the simplest systems an ecologist can study is a seasonally breeding population in which generations do not overlap. Many natural populations, particularly among temperate zone insects (including many economically important crop and orchard pests), are of this kind. In this situation, the observational data will usually consist of information about the maximum, or the average, or the total population in each generation. The theoretician seeks to understand how the magnitude of the population in generation $t+1$ which is $x_{t+1}$ is related to the magnitude of the population in the preceding generation $t$ which is $x_t$; such a relationship may be expressed in the general form:

$$x_{t+1} = F(x_t) \ldots \quad (1.01)$$

The function $F(x_t)$ will usually be what a biologist calls “density dependent”, and a mathematician calls nonlinear; equation (1.01) is then a first-order, nonlinear difference equation [5, 74, 77, 98,].
1.02: **Population:** It is well known fact that an individual living organism of any species does not live alone in nature, they live in groups. These groups are called populations. The term “population” means that a group of individuals contains any one kind of living organism.

1.03: **Mathematical Modeling:** Mathematical modeling is a technique of translating real world problems into mathematical problems solving the mathematical problems and interpreting these solution in the language of real world. For example, a mathematical equation is a representation of growth of population, a toy train is the representation of a real train and a phototype is the representation of a future car etc. Therefore, a model is a purposeful representation of reality.

1.04: **Equilibrium points and Trajectories:** Let us consider a two dimensional system:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y)
\end{align*}
\]

(1.02)

It is convenient to plot \((x(t), y(t))\) in the xy plane with \(t\) as a parameter. Then every solution of the differential equation system (1.02) describes a curve in the xy-plane as \(t\) varies, and this curve is known as trajectory. If the trajectory through \((x_0, y_0)\) consists entirely of the points \((x_0, y_0)\), then this point is an equilibrium point or a critical point. This means, if
(x₀, y₀) is a critical point, then at this point \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} = 0 \).

This state is known as steady state.

1.05: Dynamical Systems: A Nonlinear System is a set of nonlinear equations, which may be algebraic, functional, ordinary differential, partial differential, integral or a combination of these. The system may depend on given parameters. Dynamical System is used as a synonym of nonlinear system when the nonlinear equations represent evolution of a solution with time or some variable like time.

In fact, a Dynamical System is one whose state changes with time ‘t’. Two main types of dynamical system are encountered in applications: those for which the time variable is discrete (\( t \in Z \) or \( N \)) and those for which it is continuous (\( t \in R \)).

Discrete dynamical systems can be presented as the iteration of a function, i.e.

\[ x_{t+1} = f(x_t), \quad t \in Z \text{ or } N. \]

When it is continuous, the dynamics are usually described by a differential equation:

\[ \frac{dx}{dt} = \dot{x} = X(x) \]
, where $\mathbf{x}$ represents the state of the system and takes values in the state or phase space. Sometimes, the phase space is Euclidean space or a subset thereof, but it can also be a non-Euclidean structure such as a circle, a sphere, a torus or some other differentiable manifolds.

Dynamical systems are deterministic if there is a unique consequence to every state, and stochastic or random if there is more than one consequence chosen from some probability distribution.

**1.06: Autonomous Dynamical Systems:** An nth order autonomous dynamical system is defined by the state equation

$$\dot{\mathbf{x}} = f(\mathbf{x}) \quad \mathbf{x}(t_0) = x_0$$

where \( \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} \), \( \mathbf{x}(t) \in \mathbb{R}^n \) is the state at time \( t \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called the vector field. Since the vector field does not depend on time, the initial time may always be taken as \( t_0 = 0 \). The solution to the above equation with the initial condition \( x_0 \) at time \( t = 0 \) is called the trajectory and is denoted by \( \varphi_t(x_0) \). The mapping \( \varphi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called the flow of the system. The above dynamical system is linear if \( f(\mathbf{x}) \) is linear.
1.07: Nonautonomous Dynamical Systems: An nth order nonautonomous dynamical system is defined by the time varying state equation

\[
\dot{x} = f(x, t), \quad x(t_0) = x_0
\]

The vector field \( f \) depends on time and unlike the autonomous case, the initial time cannot arbitrarily be set to 0. The solution of the above equation passing through \( x_0 \) at time \( t_0 \) is \( \varphi(\cdot, t_0) \). If there exists \( T > 0 \) such that \( f(x, t_0) = f(x, t + T) \) for all \( x \) and all \( t \), then the system is said to be time-periodic with period \( T \). The smallest such period is called the minimal period. Thus a periodic orbit corresponds to a special type of solution for a dynamical system, namely one which repeats itself in time. A dynamical system exhibiting a stable periodic orbit is often called an oscillator.

1.08: Limit Cycles: In the area of dynamical systems, a limit-cycle on a plane or a two-dimensional manifold is a closed trajectory in phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. Oscillations are one of the most important phenomena that occur in dynamical systems. A system oscillates when it has a nontrivial periodic solution.
\( x( t + T) = x( t), \ t \geq 0 \) for some \( T > 0 \). In a phase portrait an oscillation or periodic solution looks like a closed curve. In fact, an isolated periodic orbit is called a \textit{limit cycle}. That is, a limit cycle is an isolated closed trajectory that its neighbouring trajectories are not closed – they spiral either towards or away from the limit cycle. Thus, limit cycles can only occur in nonlinear systems.

A stable limit cycle (\( \omega \)-limit cycle) is one which attracts all neighbouring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations – most of the biological processes of interest are of this kind. On the other hand, neighbouring trajectories are repelled from unstable limit cycles (\( \alpha \)-limit cycle), and half-stable limit cycles are, of course, ones which attract trajectories from one side and repel those on the other. If nearby trajectories neither approach nor recede from \( C \), it is Neutrally – stable Limit Cycle.
Fig 1.01 A Stable limit cycle
Periodic orbits in the plane are special in that they divide the plane into a region inside the orbit and a region outside it. This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems.

**Theorem 1.01: (Poincaré-Bendixson Criterion)**: Consider the system

\[ \dot{x} = f(x) \]

and let \( M \) be a closed bounded subset of the plane such that \( M \) contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix \( \left[ \frac{\partial f}{\partial x} \right] \) at this point has eigenvalues with positive
real parts. Every trajectory starting in \( M \) remains in \( M \) for all future time. Then \( M \) contains a periodic orbit of \( \dot{x} = f(x) \).

Theorem 1.02: (Negative Pointcaré-Bendixson Criterion): If, on a simply connected region of the plane, the expression \( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \) is not identically zero and does not change sign, then the system \( \dot{x} = f(x) \) has no periodic orbits lying entirely in \( D \).

1.09 Lyapunov Exponents:

The Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. Quantitatively, two trajectories in phase space with initial separation \( \delta \mathbf{z}_0 \) diverge (provided that the divergence can be treated within the linearized approximation)

\[
|\delta \mathbf{z}(t)| \approx e^{\lambda t} |\delta \mathbf{z}_0| \quad \text{where } \lambda \text{ is the Lyapunov exponent.}
\]

The rate of separation can be different for different orientations of initial separation vector. Thus, there is a spectrum of Lyapunov exponents—equal in number to the dimensionality of the phase space. It is common to refer to the largest one as the Maximal Lyapunov exponent (MLE), because it determines a notion of predictability for a dynamical system. A positive MLE is usually taken as an indication that the system is chaotic (provided some other conditions are met, e.g., phase space compactness).
Note that an arbitrary initial separation vector will typically contain some component in the direction associated with the MLE, and because of the exponential growth rate, the effect of the other exponents will be obliterated over time. The exponent is named after Aleksandr Lyapunov.

To be precise, let us explain Lyapunov exponent in one-dimensional case as follows:

In fact, a Lyapunov exponent is a measure of the rate of attraction to or repulsion from a fixed point in state space. We could also apply this notion to the divergence of nearby trajectories in general at any point in state space. For a one-dimensional state space, let $x_0$ be one initial point and $x$ a nearby initial point. Let $x_0(t)$ be the trajectory that arises from that initial point, while $x(t)$ is the trajectory arising from the other initial point.

The time development equation is assumed to be

$$\dot{x}(t) = f(x) \quad (i)$$

Since we assume that $x$ is close to $x_0$, we can use a Taylor series expansion to write

$$f(x) = f(x_0) + \frac{df(x)}{dx}\bigg|_{x_0}(x-x_0) + \ldots \ldots$$

We next find that the rate of change of distance between the two trajectories is given by
\[ s = x - x_0 = f(x) - f(x_0) = \frac{df(x)}{dx} \bigg|_{x_0} (x - x_0) \]  

(iii)

Where we have kept only the first derivative term in the Taylor series expansion of \( f(x) \). Since we expect the distance to change exponentially in time, we introduce the Lyapunov exponent \( \lambda \) as the quantity that satisfies

\[ s(t) = s_0 e^{\lambda t} \]  

(iv)

If we differentiate (iv) with respect to time, we find

\[ s = \lambda s_0 e^{\lambda t} \]  

(v)

Comparing equation (v) and equation (iii) yields

\[ \lambda = \frac{df(x)}{dx} \bigg|_{x_0} \]  

(vi)

Thus we see that if \( \lambda \) is positive, then the two trajectories diverge; if \( \lambda \) is negative, the two trajectories converge.

In practice, we know that the derivative of the time evolution function generally varies with \( x \); therefore, we want to find an average of \( \lambda \) over the history of a trajectory. If we know the time evolution function, we simply evaluate the derivative of the time evaluation function along the trajectory and find the average value.
In state spaces with two or more dimensions, we can associate a (local) Lyapunov exponent with the rate of expansion or contraction of trajectories for each of the directions in the state space. A system is chaotic if there exists at least one positive Lyapunov exponent.

1.10 **Self-Similarity:** If parts of a figure contain small replicas of the whole, then the figure is called **self-similar**. If the figure can be decomposed into parts which are exact replicas of the whole, then the figure is called strictly self-similar. Every part of a strictly self-similar structure contains an exact replica of the whole.

1.11 **Fractal:** The word “fractal” was derived from the Latin “fractus” meaning broken. Basically, a fractal is a geometric shape that has two special properties:

(i) The object is self-similar, (ii) The object has fractional dimension.

Fractal geometry is concerned with the properties of fractal objects, usually simply known as fractals. The subject of fractal geometry was given its name by Benoit B. Mandelbrot in the mid 1970s, and he is referred as the Father of Fractal Geometry.

1.12 **Chaos:** Generally, there is no accepted definition of Chaos. From a practical point of view, chaos can be defined as a bounded steady state behaviour that is not an equilibrium point, not periodic, and not quasi-periodic. Another important fact about the chaotic systems is that the limit
set for chaotic behavior is not a simple geometric object like cycle or torus, but is related to fractals and Cantor sets.

In fact, chaos can be defined as effectively unpredictably long time behavior arising in a deterministic dynamical system because of sensitivity to the initial conditions. The key to long-term unpredictability is a property known as sensitivity to (or sensitive dependence on) initial conditions. A map $f$ is chaotic on a compact invariant set $S$ if

(i) $f$ is transitive on $S$ (there is a point whose orbit is dense in $S$) and
(ii) $f$ exhibits sensitive dependence on $S$

1.13: Stable and Unstable Periodic Points: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. A point $\vec{x}$ in $\mathbb{R}^n$ is called a fixed point of $f$ if $f(\vec{x}) = \vec{x}$. A fixed point $\vec{x}$ is said to be stable if for every neighbourhood $U$ of $\vec{x}$, there exists a neighbourhood $V$ of $\vec{x}$ whose images $f^k(V)$ lie in $U$ for all positive integers $k$. Otherwise, $\vec{x}$ is known to be unstable. A periodic orbit of $f$ is a finite sequence of distinct points each of which is the image of the previous one and whose first point is the image of the last. Its period $k$ is the number of points in the sequence, which are called periodic points of period $k$. A periodic orbit of period $k$ (or $k$-cycle) is said to be stable or unstable according as each of its points is
stable or unstable when considered as a fixed point of \( f^{(k)} \), \( f^{(k)} \) means \( k \)-times iteration of the map \( f \).

Stability Theorem 1.04: A sufficient condition for a periodic point \( x \) of period \( k \) for a map \( f \) to be stable is that the eigenvalues of the derivative

\[ Df^k(x) \]

are less than one in absolute value.

1.14: Bifurcations: A bifurcation is a qualitative change in dynamics upon a small variation in the parameters of a system. Many dynamical systems depend on parameters. Normally, a gradual variation of a parameter in the system corresponds to the gradual variation of the solutions of the problem. However, there exists a large number of problems for which the number of solutions changes abruptly and the structure of solution manifolds varies dramatically when a parameter passes through some critical values. This kind of phenomenon is called bifurcation and these parameter values are called bifurcation values (or bifurcation points).

Bifurcation theory is a method for studying how solutions of a nonlinear problem and their stability changes as the parameter varies. The onset of chaos is often studied by bifurcation theory. For example, in certain parametrized families of one dimensional maps, chaos occurs by infinitely many period-doubling (P-D) bifurcations. In the case of a diffeomorphism \( f \), P-D bifurcations (or Flip bifurcations or Subharmonic
bifurcations) occur when one of the eigenvalues of the derivative $Df(x)$ equals -1.

1.15: Feigenbaum Theory of Bifurcations

The Feigenbaum theory has opened up many outstanding research problems in the field of nonlinear dynamical systems. Period-Doubling (P-D) bifurcations, as a universal route to chaos, is one of the most exciting discoveries of the last few years in the field of nonlinear dynamical systems. Recently there has been much interest in the chaotic behaviour of simple dynamical systems in a variety of fields and an increasing effort has been devoted to elucidate it both theoretically and experimentally. Since, in many cases, chaotic behaviour appears after the period-doubling phenomenon when a parameter, say $\lambda$, is varied, it has been one of the important problems to analyze and describe the phenomenon theoretically.

The period-doubling phenomenon implies a series of bifurcations in any of which a periodic orbit loses its stability and is replaced by a new stable one with the double period. These bifurcations accumulate at a certain point $\lambda = \lambda_c$, where chaotic behaviour begins to appear.

Recently, however, Feigenbaum has shown theoretically that there exist some universal properties in the period-doubling phenomenon of one-dimensional mappings of the form

$$x_{n+1} = f(x_n, \lambda),$$

where $\lambda$ is a controllable parameter.
The essence of his theory consists in his observation that the configuration of periodic points belonging to a periodic orbit is represented by a set of universal functions when the orbit is maximally stable. These functions are roughly \((-1)^k \beta_k f^{(2^k \ell)} (X / \beta_k \lambda_{k+n})\) with sufficiently large \(k\) and obey a functional recursion formula, where \(f^{(j)}\) stands for the \(j\)th iterate of \(f\), \(\beta_j\) is a scaling factor and \(\lambda = \lambda_j\) is the maximal stability point of a \(2^j\ell\)-periodic orbit. Note that in addition to the bifurcation points of periodic orbit denoted by \(\lambda_k\) for \(2^k\ell\) periodic orbit, those maximal stability points \(\lambda\) also converge to \(\lambda_c\) in the limit \(k \to \infty\). Then, one important conclusion follows, viz. that the convergence rate of \(\lambda_k\), what is called Feigenbaum ratio, is universal, since it can be shown that the rate is determined as an eigenvalue associated with a linear functional equation which is derived from a fixed point equation for the recursion formula mentioned above.

Typical examples that we often encounter are the maps of the form:

- \(f: [0,1] \to [0,1]\) defined by \(f(x) = \lambda x (1-x)\) or \(f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\) defined by \(f(x) = x + \mu \sin (2\pi x) + \alpha \mod 1\).

Even for these simple maps this turns out to be highly non-trivial and also extremely interesting from both of mathematical and application point of view. There is enough motivation for studying such dynamical systems.

To discuss the Feigenbaum Universality, let us consider a one-dimensional map of the interval
\[ x_{n+1} = f(x_n, \lambda) \]

where \( \lambda \) is a control parameter. We are interested in the maps with quadratic maxima where, as \( \lambda \) increases, a stable fixed point give birth to a stable 2-cycle (pitchfork bifurcation), which then gives birth to a stable 4-cycle, and so on until at \( \lambda = \lambda_{\infty} \) all cycles of order \( 2^n \) are unstable and the invariant set of the map consists of \( 2^\infty \) points. (The bifurcation diagram in this range is therefore a complete binary tree. The logistic map belongs to this category).

The condition for a fixed point is that \( x = f(x, \lambda) \) corresponding to a 1 cycle. In order to decide the stability of the fixed point, the approximate linear map

\[ x(\lambda) \]

Fig 1.03 A typical bifurcation diagram
\[ \delta x_{n+1} \approx f'(x, \lambda) \delta x_n \]

whose solution is
\[ \delta x_n \approx (f'(x, \lambda))^{n-1} \delta x_0 \]

A fixed point is stable if \( |f'(x, \lambda)| < 1 \) and unstable if \( |f'(x, \lambda)| > 1 \) and the value \( \lambda_1 \) where \( f'(x, \lambda_1) = -1 \) signals a bifurcation. After the value \( \lambda = \lambda_1 \) is passed, the point \( x(\lambda) \) becomes unstable and there appear around it two points say, \( x_{11}(\lambda) \) and \( x_{12}(\lambda) \) forming a stable periodic trajectory of period 2. The differences \( |x_{11}(\lambda) - x_{12}(\lambda)| \) and \( |x_{12}(\lambda_1) - x_{21}(\lambda_1)| \) are of order \( (\lambda - \lambda_1)^{\frac{1}{2}} \) as \( (\lambda - \lambda_1) \to 0 \), while \( |x(\lambda) - x(\lambda_1)| = 0(\lambda - \lambda_1) \).

Thus, under one period-doubling bifurcation the previously stable fixed point \( x_0 \) becomes unstable, and a stable periodic trajectory of period 2 appears near it. As the parameter is further increased, the original fixed point \( x_0 \) continues to exist as an unstable fixed point, and all the remaining points are attracted towards the stable periodic trajectory of period 2. This happens up to some value \( \lambda = \lambda_2 \) at which the periodic trajectory of period loses stability in such a way that

\[ \left. \frac{df(x, \lambda_1)}{dx} \right|_{x=x_{11}} = \left. \frac{df^2(x, \lambda_2)}{dx} \right|_{x=x_{21}} = -1 \]

We can then repeat the same arguments and find that the periodic trajectory of periods 2 becomes unstable and a periodic trajectory of period 4 appears
near it. As $\lambda$ is increased, an infinite sequence $\{\lambda_n\}$ of parameter values emerges such that at $\lambda = \lambda_n$ there is a loss of stability of the periodic trajectory of period $2^{n-1}$ and a periodic trajectory of period $2^n$ arises. By now, we can imagine what happens when $\lambda = \lambda_\infty = \lim \lambda_n$; the map $f(x, \lambda_\infty)$ has an invariant set $F$ of Cantor type surrounded by infinitely many unstable periodic trajectories of periods $2^n$. Moreover, all the points except those belonging to these unstable trajectories and their inverse images are attracted to $F$ under the action of $f(x, \lambda_\infty)$. Feigenbaum University in its simplest form means that the sequence $\{\lambda_n\}$ behaves in a universal manner, that is, $\lambda_\infty - \lambda_n \sim C\delta^{-n}$, where the constant $C$ depends on the family $f$, while $\delta$ is universal and is known as the Feigenbaum constant. The value of $\delta$ is $4.6692016091029\ldots$ in the dissipative case and $8.721097200\ldots$ in the conservative case.

Moreover, the structure of the attractor $F$, in particular, its Hausdorff dimension, and the behaviour of the integrates $F^{(n)}$ in a neighbourhood of $\lambda = \lambda_\infty$ do not depend on $f$. This means that all this behaviours are of universal nature.

1.16: Using $\delta$ to make predictions:

At a practical level, the existence of a universal number such as $\delta$ allows us to make quantitative predictions about the behaviour of a nonlinear system, even if we cannot solve the equations describing the system. More
importantly, this is true even if we do not know what the fundamental equations for the system are, as is often the case. For example, if we observe that a particular system undergoes a period doubling bifurcation from period 1 to period 2 at a parameter value $\lambda_1$, and from period 2 to period 4 at a value $\lambda_2$, then we can use $\delta$ to predict that the system will make a transition from period 4 to period 8 at $\lambda_3$ given by

$$\lambda_3 = (\lambda_2 - \lambda_1)/\delta + \lambda_2 \quad (1.03)$$

Observing the first two period-doublings give no guarantee that a third will occur, but if it does occur, the above equation (1.03) gives us a reasonable prediction of the parameter value near which we should look to see the transitions.

We can also use $\delta$ to predict the parameter value to which the period-doubling sequence converges and at which point chaos begins. To see how this works, we first write an expression for $\lambda_4$ in terms of $\lambda_3$ and $\lambda_2$, in analogy with equation . [we are, of course, assuming that the same $\delta$ value describe each ratio. This is not exact, in general, but it does allow us to make a reasonable prediction.]

$$\lambda_4 = (\lambda_3 - \lambda_2)/\delta + \lambda_3 \quad (1.04)$$

We now use equation (1.03) in equation (1.04) to obtain

$$\lambda_4 = (\lambda_2 - \lambda_1) (1/\delta + 1/\delta) + \lambda_2 \quad (1.05)$$
If we continue to use this procedure to calculate $\mu_5, \mu_6$ and so on, we just get more terms in the sum involving powers of $(1/\delta)$. We can sum the series to obtain the result,

$$\lambda_{\infty} \approx \lambda_{n+1} + (\lambda_{n+1} - \lambda_n) / \delta - 1 \quad (1.06)$$

1.17 : Feigenbaum Size Scaling:

As part of numerical investigation of simple mapping functions such as the logistic Map, it is recognized that each successive period doubling bifurcation is just a smaller replica with more branches, of course, of the bifurcation just before it. This observation suggested that there might be a universal size scaling in the period doubling sequence. The ‘size’ ratio, designated as the Feigenbaum $\alpha$ (alpha) value and is defined by

$$\alpha = \lim d_n / d_{n+1} = 2.5029 \ldots \ldots$$

where $d_n$ is the ‘size’ of the bifurcation pattern of period $2^n$ just before it gives birth to period $2^{n+1}$

The relation between $\alpha$ and $\delta$ is given by the equation

$$\alpha (\alpha + 1) = \delta$$

Remark: Similar results and phenomena can be carried out with higher dimensional nonlinear maps and differential equations .[39, 40, 42 ]
1.18: The Doubling Transformation:

The renormalization group (RG) methods were very successful in resolving delicate issues of statistical mechanics. More recently, the fundamental philosophy of RG has been directed towards understanding some basic aspects of nonlinear differential equations. The incentive was found from the universality of Feigenbaum number $\delta$ (the common convergence ratio) and the Feigenbaum number $\alpha$ (the scaling ratio) for many one-dimensional iterated maps. The essential idea is that to find $\alpha$ and $\delta$, we need to concentrate our attention only on the behaviour of the iterated function $f$ near some value $x_c$ (known as the critical value) for which it has a maximum i.e. we mainly concentrate on the trajectories that involve $x_c$ (the super cycles). As the period doubling sequence proceeds, we shift our attention to $f^{(n)}$. It turns out that if we rescale our graphs by the factor $\alpha$ for each period doubling, then $f^{(n)}$ near $x_c$ approaches a universal function (i.e. it is the same for a wide class of iterated map functions $f$). It is seen that the existence of this universal function, the existence of size scaling near $x_c$ and the shape of $f$ near $x_c$ are all that we need to find the values of $\alpha$ and $\delta$. The potential of this research direction lies in the capability to determine a characteristic scaling
exponent with a relatively simple calculation upon understanding a transformation that relates two parameters.

By applying the RG method we now explain the doubling transformation in connection with the Feigenbaum Universality in the space of functions. We consider a one parameter family of smooth maps $f(x, \lambda)$ of $[-1,1]$ into itself and assume that $x_c(\lambda) = 0$ is the unique critical point and is a maximum point for all $\lambda$. This condition is no restriction of generality, because any unimodal family $f(x, \lambda)$ can be brought to this form by means of a suitable conjugation $\tilde{f}(x, \lambda) = S^{-1}_\lambda \cdot f(x, \lambda) \cdot S_\lambda$. The properties of existence or nonexistence of periodic trajectories and the character of their stability (attracting, repelling) do not change under a conjugation $\tilde{f} = S^{-1} \cdot f \cdot S$ where $S$ is a diffeomorphism of the interval onto itself.

Suppose that an infinite sequence of period-doubling bifurcations take place for the family $f(x ; \lambda)$ as $\lambda$ increases; let $\lambda_1, \lambda_2, \ldots$ denote the sequence of bifurcation values of the parameter. We recall that a stable periodic trajectory of period $2^n$ appears for $\lambda = \lambda_n$. Let us consider in a little more detail what happens in the interval $\lambda_n < \lambda < \lambda_{n+1}$, the map $f^\lambda(x, \lambda)$ has a stable fixed point $\lambda(\lambda)$ for these values of the parameter. In fact, there
are $2^n$ such points. For $\lambda$ close to $\lambda_n$ the derivative $\left. \frac{\partial f^{2^n}(x;\lambda)}{\partial x} \right|_{x=x(\lambda)}$ is close to +1, while for $\lambda \to \lambda_{n+1}$, $\left. \frac{\partial f^{2^n}(x;\lambda)}{\partial x} \right|_{x=x(\lambda)} \to -1$ and at $\lambda = \lambda_{n+1}$ there is a loss of stability of the periodic trajectory. As $\lambda$ varies, a value $\lambda_n < \lambda_n < \lambda_{n+1}$ appears such that $\left. \frac{\partial f^{2^n}(x;\lambda_n)}{\partial x} \right|_{x=x(\lambda_n)} = 0$. In this case $f^{2^n}(x;\lambda_n)$ is said to have a superstable fixed point. By our assumption $\alpha(\lambda_n) = 0$.

We proceed to explain the main idea, due to Feigenbaum [21,22,23,32,33] for proving universality. By successively doubling the family of maps and carrying out parameter renormalization and scale transformations, we obtain in the limit a family of maps that is invariant under the transformations. Interestingly, the limiting family does not depend on the original one, but is entirely determined by the transformations listed above.

We define the doubling transformation $T$ acting in the space of maps of $[-1,1]$ into itself as follows:

Let $f(x)$ be an even unimodel map of $[1-1]$ into itself and $x=0$ the maximum point of $f$. We write

$$\alpha = \alpha(f) = -\frac{f(0)}{f(f(0))}$$
Under the action of \( f \), the interval \( [-\alpha^{-1}, \alpha^{-1}] \) is mapped onto \( [f(\alpha^{-1}), f(0)] \), and this, in turn, is mapped onto \( [f(f(0)), f(f(\alpha^{-1}))] \).

We assume the following conditions:

\[ \alpha > 0, \quad f(f(\alpha^{-1})) < \alpha^{-1}, \quad \alpha^{-1} < f(\alpha^{-1}), \quad f(0) > 0. \]

Then, \( [f(f(0)), f(f(\alpha^{-1}))] \subseteq [-\alpha^{-1}, \alpha^{-1}] \) and \( [f(\alpha^{-1}), f(0)] \cap [-\alpha^{-1}, \alpha^{-1}] = \emptyset. \)

Thus, \( h(x) = -\alpha f(\alpha^{-1}x) \) is again a unimodal map of \([-1,1]\) into itself, and \( h(0) = f(0) \). We now define the doubling transformation \( T \) by setting

\[ (Tf)(x) = -\alpha f(\alpha^{-1}x), \quad \alpha = \frac{f(0)}{f(f(0))}. \]

If \( T \) is defined for some map \( f \), then also for maps close to \( f \). We now explain how \( T \) is connected with the universality phenomenon for a sequence of period doubling bifurcations.

We consider the space of maps \( f(x) \) (not necessarily even) of \([-1,1]\) into itself such that \( f(x) \in C^1([-1,1]), x = 0 \) is a maximum point and \( f(0) = \text{const} \), to be definite, \( \text{const} = 1 \). This space is invariant under \( T \). If turns out that \( T \) has a fixed point \( g(x) \) in it and the spectrum of the linearized transformation \( DT(g) \) at the fixed point lies inside the unit disk except for a
single eigenvalue greater than 1. This is the Feigenbunm constant \( \delta = 4.6692 \ldots \) Therefore, a one-dimensional unstable separatrix \( \Gamma'(g) \) passes through the fixed point \( g \) consisting of maps that recede from \( g \) under the action of \( T \), and a stable separatrix \( \Gamma''(g) \) of codimension 1 consisting of maps attracted to \( g \) under the action of \( T \). The unstable separatrix \( \Gamma'(g) \) corresponds to the eigenvalue \( \delta \).

We have mentioned above that a period-doubling bifurcation takes place when the derivative at the fixed point of the mapping passes through -1. Let \( \Sigma_1 \) be the hypersurface of codimension 1 that consists of the maps in the function space having derivative -1 at the fixed point. If the family of maps intersects \( \Sigma_1 \) transversally, then a period-doubling bifurcation takes place.

Let
\[
\Sigma_2 = T^{-1}\Sigma_1, \ldots, \Sigma_k = T^{-1}\Sigma_{k-1}
\]

When the family of maps \( f(x; \lambda) \) intersects the surface \( \Sigma_k \), a stable periodic trajectory of period \( 2^k \) is created from the stable periodic trajectory of period \( 2^{k-1} \), that is, the parameter values corresponding to the intersections of the family of maps with the surfaces \( \Sigma_k \) are the bifurcation parameter values. For, if the family \( f(x; \lambda) \) intersects \( \Sigma_k \) then \( T^{k-1} f \)
intersects $\Sigma_i$ and this means that a period-doubling bifurcation takes place and a periodic trajectory of period $2^k$ appears.

The unstable separatrix $\Gamma^u(g)$ intersects $\Sigma_i$ transversally, hence it intersects all the $\Sigma_k(k \geq 2)$ transversally. The surfaces $\Sigma_k$ converge to $\Gamma^u(g)$ and asymptotically for large $k$ the distance between $\Sigma_k$ and $\Gamma^u(g)$ is $\delta$ times smaller than that between $\Sigma_k$ and $\Gamma^u(g)$. Therefore, the bifurcation parameter values for any family of maps $f(x; \lambda)$. In some neighbourhood of $\Gamma^u(g)$ satisfy the relation $\lambda_{\infty} - \lambda_k \sim \text{const} \delta^k$ where constant depends on the family of mappings and is determined by the derivative $\frac{\partial f(x; \lambda_k)}{\partial \lambda} \bigg|_{\lambda=\lambda_k}$.

The map $g$ is universal in the sense that $\lim_{\lambda \to \infty} T^i f(x, \lambda) = g$. The principal properties of $T$ that lead to universality are:

1. $T$ has a fixed point $g$,
2. The linearized transformation $DT(G)$ has only one eigenvalue $\delta$ greater than 1 in modulus: $\delta = 4.6692$...
3. The unstable separatrix corresponding to $\delta$ interests the surface $\Sigma_i$ transversally.

At the present time the properties 1, 2 and 3 of $T$ can be regarded as proven. Proofs were obtained by Lanford [73] and by Campanino and Epstein [18]. The course of the arguments is as follows: We assume that $\bar{g}$ is an even
polynomial of degree 2m. Then \(-\alpha \tilde{g}(\alpha^{-1}x)\) is a polynomial of degree \(4m^2\). The problem is to find a polynomial \(\tilde{g}\) close to \(-\alpha \tilde{g}(\alpha^{-1}x)\). One approach is to consider a shortened polynomial \(\tilde{g}_1(x)\), that is, a polynomial obtained by forcibly throwing out the higher powers in \(-\alpha \tilde{g}(\alpha^{-1}x)\). Then one looks for a polynomial \(\tilde{g}\) such that \(\tilde{g} = \tilde{g}_1\). In another approach the values of \(\tilde{g}(x)\) and \(-\alpha \tilde{g}(\alpha^{-1}x)\) at previously chosen points are compared. In Lanford’s paper [41], he gave results of computations to an accuracy sometimes as great as \(10^{-40}\), that is, to 40 decimal places. After obtaining a polynomial \(\tilde{g}(x)\) approximating the true fixed point of \(T\) with great accuracy we can linearize our problem in a neighbourhood of \(\tilde{g}(x)\) and show again rigorously, by Newton’s method that there is a fixed point of the whole doubling transformation in a suitable neighbourhood. Even the singularities of \(g(x)\) when it is extended analytically into the complex plane are studied in [41]. The terms in the expansion of \(g\) are as follows:

\[
g(x) = 1 - 1.52763 x^2 + 0.104815 x^4 - 0.0267057 x^6 + \ldots \alpha = \alpha(g) = 2.50290 \ldots.
\]

The quantity \(\alpha\) is also a universal constant, which characterizes the scale change associated with doubling of the maps. The Feigenbaum Universality has created an innovative research field in Mathematics where
all research workers in applicable sciences can carry out interdisciplinary research together.