ABSTRACT

The title of the Ph.D. thesis:

“SOME ASPECTS ON NONLINEAR POPULATION MODELS AND THEIR STATISTICAL ANALYSIS”

Chaos and Fractals, a sub-discipline of nonlinear dynamical systems, have at present been riding a wave which, in its power, creativity and expanse, has become an interdisciplinary experience of the first order. This wave has also been touching distant shores far beyond the sciences. Chaos and Fractals have literally captured the attention, enthusiasm and interest of a world-wide public. To the casual observer, the colour of their essential structures and their beauty and geometric form captivate the visual senses as few other things they have ever experienced in mathematics. To the student, they are mathematics out of the realm of ancient history into the twenty-first century. And to the scientist, chaos and fractals offer a rich environment for exploring and modeling the complexity of nature.

Nonlinear Population Theory has emerged as one of the most sophisticated research areas in the field of Dynamical Systems and Fractal Geometry, because of its interdisciplinary nature in dynamic behaviors such as periodic cycles, periodic orbits, multiple attractors, quasi-periodicity and invariant loops, unstable equilibria with stable and unstable manifolds, chaos, strange attractors, fractal basins and so on.
Population dynamics has traditionally been the dominant branch of Bio-Statistics, which is in turn an integral part of Bio-Sciences, and more recently the scope of Mathematical & Statistical Biology has greatly expanded due to their tremendous application in Biosciences.

First order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random functions. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. One of the simplest systems an ecologist can study is a seasonally breeding population in which generations do not overlap. Many natural populations, particularly among temperate zone insects (including many economically important crop and orchard pests), are of this kind. In this situation, the observational data will usually consist of information about the maximum, or the average, or the total population in each generation. The theoretician seeks to understand how the magnitude of the population in generation $t+1$, $x_{t+1}$, is related to the magnitude of the population in the preceding generation $t$, $x_t$; such a relationship may be expressed in the general form.

$$x_{t+1} = F(x_t) \ldots \quad \text{(I)}$$
The function $F(x_t)$ will usually be what a biologist calls “density dependent”, and a mathematician calls nonlinear; equation (I) is then a first-order, nonlinear difference equation.

The magnificent successes in the fields of natural sciences and technology had, for many, fed the illusion that the world on the whole functioned like a huge clockwork mechanism, whose laws were only waiting to be deciphered step by step. Once the laws were known, it was believed, the evolution or development of things could— at least in principle— be ever more accurately predicted. Captivated by the breathtaking advances in the development of computer technology and its promises of a greater command of information, many have put increasing hope in these machines.

The correlation of chaos and fractal geometry is anything but coincidental. Rather, it is a witness to their deep kinship. This kinship can best be seen in the Mandelbrot set, a mathematical object discovered by Benoît Mandelbrot in 1980. It has been described by the scientists as the most complex and possibly the most beautiful— object ever seen in mathematics. Its most fascinating characteristic, however has only just recently been discovered: namely, that it can be interpreted as an illustrated encyclopedia of an infinite number of algorithms. It is a fantastically efficiently organized storehouse of images, and as such it is the example par excellence of order in chaos. In essence, chaos theory and fractal
geometry radically question our understanding of equilibria – and therefore of harmony and order- in nature as well as in other contexts. They offer a new holistic and integral model which can encompass a part of the true complexity of nature for the first time. It is highly probable that the new methods and terminologies will allow us, for example, a much more adequate understanding of ecology and climate developments, and thus they could contribute to our more effectively tackling our gigantic global problems.

Bifurcation theory is a method for studying how solutions of nonlinear problems and their stability changes as the parameter varies. The onset of chaos is often studied by the bifurcation theory. For example, in certain parameterized family of one dimensional map, chaos occurs by infinitely many period doubling bifurcations. In the case of a diffeomorphism map bifurcation occurs when one of the eigenvalues of the derivatives Df(x) equals -1. Moreover, a fixed point or a periodic point will be stable (unstable) if the absolute value of the derivative of the map / transformation at the point remain less than 1 (greater than 1).

A dynamical system can have discrete or continuous time. The discrete case is defined by a map $x_1 = f(x_0)$ that gives the state $x_1$ resulting from the initial state $x_0$ at the next time value. The continuous case is defined by a ‘flow’ $\dot{x}(t) = f(x(0))$ which gives the
state at time $t$, given that the state was at time $0$. A smooth flow can be differentiated with respect to time to give a differential equation

$$\dot{x} = f(x, t)$$

In this case we call $f(x, t)$ a vector field.

**Feigehbaum Theory:** Chaos theory began at the end of the nineteenth century (around 1890) with some great initial ideas, concepts and results of the famous mathematician, Henri Poincare. Also more recent path of the theory has many fascinating successful stories. Probably the most beautiful and important one is the route from order into chaos, i.e. the Feigenbaum universality. Mitchell J. Feigenbaum, a renowned American expert in physics, is known as the founder of the period doubling bifurcation that may be described as a universal route to chaos. Many new universal properties have been discovered by Feigenbaum for families of maps which depend on a parameter. One of his fascinating discoveries is that if a family “$f$” represents period-doubling bifurcations then there is an infinite sequence of \( \{\lambda_n\}_{n=1}^{\infty} \) of bifurcation values such that

$$\lim_{n \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta$$

where $\delta$ is a universal number known as Feigenbaum constant which does not depend at all on the form of specific family of maps. The value of $\delta$ is $4.6692016091029\ldots$ in the dissipative case and $8.72109720\ldots$ in the conservative case.
All these results have much motivated us to pursue our study and investigation.

**Chapterization:**

The Thesis consists of eight chapters along with a bibliography.

Chapter-1 deals with exclusively for a preliminary introduction to the background, related references, fundamental definitions, and results which are collected from dispersed literature and which are fundamental requisites needed to recall while going through the body of the thesis.

Chapter-2 is basically concerned to highlight three objectives of the quadratic iterator

\[ x_{n+1} = F(x_n) = a_n x_n^2 - b_n, \quad n=0,1,2,\ldots \]

where \( x_n \in [0, 4] \), \( a \) and \( b \) are tunable parameters. Firstly, by adopting suitable numerical methods and computer programs we evaluate the period-doubling: \( 1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow \ldots \) bifurcations, a route from order to chaos. Secondly, we analyze the stability of periodic points. Thirdly, we draw the bifurcation diagram in order to support our period doubling orbits and chaotic region, and some illuminating results are obtained as the measure of chaos. By applying the technique of the Coefficient of variation in Statistics it is shown that there exists a most prominent and reliable universal route from the regular region to the chaotic region.
Time series analysis guarantees the existence of different periodic orbits and finally leading to the chaotic region.

In Chapter-3, we study the chaotic model: \( f(x) = ax^2 - bx \), where \( a \) and \( b \) are adjustable parameters. For our purpose we take \( x \in [0,4] \), \( a=-1 \) and \( b \in [-4,-1] \). Two techniques are adopted, namely (i) Lyapunov Exponents and (ii) Time-series Analysis, in order to confirm the periodic orbits of periods \( 2^0, 2^1, 2^2 \ldots \), as the parameter varies in a suitable region and the existence of the chaotic region. Some statistical analysis is put forward in conformity with our dynamical properties.

Chapter-4 is concerned with the control of chaos. It is known that the frame of a chaotic attractor is given by infinitely many unstable periodic orbits, which coexist with the strange attractor and play an important role in the system dynamics. There are many methods available for controlling chaos. The periodic proportional pulses technique is interesting one. In this chapter it is aimed to apply the periodic proportional pulses technique to stabilize unstable periodic orbits embedded in the chaotic attractor of the nonlinear dynamics. We are successful in controlling chaos with our model.

Chapter-5: We consider the Nicholson Bailey model:

\[
x_{n+1} = Lx_ne^{-ay_n}, y_{n+1} = x_n(1 - e^{-ay_n}),
\]
where $x_{n+1}$ represents the number of hosts (or prey) at stage $n$ and $y_{n+1}$ represents number of parasites (or predator) at $n$th stage. The difference equation can also be written in the function form as follows: $f(x,y) = (Lxe^{-ay}, x(1 - e^{-ay}))$

whose dynamical behavior is analyzed. It has been observed that the steady state occurs when there is no predator and prey for certain control parameters and after that certain region of control parameter, the natural equilibrium state never occurs. However another modified version of the model has been taken taking care of the unboundedness of the prey system which has been restricted to some extent. It has been observed that the model follows the stability of period-doubling fashion obeying Feigenbaum universal constant $\delta$ and at last attains infinite period doubling declaring chaos in the system. The bifurcation points have been calculated numerically and after that the accumulation point i.e. onset of chaos is calculated based on the experimental values of bifurcation points.

The sole objective of Chapter-6 is to study different fractional dimensions. The chaotic behavior is measured by various means at the accumulation point. First, the Lyapunov exponent is found to be positive at the accumulation point stating that chaos has started, secondly various fractal dimensions viz. Box-counting dimension, Information dimension, Correlation dimension are calculated with the help of Generalized
Correlation dimension. All these dimensions are found to have fractional value which states that the attractor at the accumulation point is chaotic.

The salient feature of Chapter 7 is that a non-linear ecological map which may be called as modified Nicholson-Bailey map is considered, which becomes chaotic in nature with the increase of the control parameter. As in most of the cases chaos is an unwanted phenomenon, so controlling of chaos becomes a necessary part of our principal study. First of all Chau’s method is applied on this map and the chaotic region is controlled forming periodic trajectories. Again OGY method is applied on the map to have chaos controlled. Lastly, the model has been modified to Chau’s form which generates a set of fixed points that has been stabilized by OGY method.

In Chapter-8 we want to concentrate to some motivations of the results described in this monograph. These motivations give rise to many interesting research problems in the field of our study. In the last few years, a wide range of new phenomena in the field of Dynamical systems and Chaos has been discovered and this has remarkably enlarged this field innovating a promising dimension for future researches. At the end, there is an exhaustive bibliography which is closely related to our research enterprise. +++