CHAPTER 6
ATTRACTOR’S DIMENSIONS OF TWO
DIMENSIONAL NONLINEAR MAPS

6.1 Introduction:

The chaotic behavior is measured by various means at the accumulation point. First, the Lyapunov exponent is found to be positive at the accumulation point stating that chaos has started, secondly various fractal dimensions viz Box-counting dimension, Information dimension, Correlation dimension are calculated with the help of Generalized Correlation dimension. All these dimensions are found to have fractional value which states that the attractor at the accumulation point is chaotic.

As we have already stated that the Nicholson Bailey model[92] was developed in 1930’s to describe population dynamics of host-parasite (predator-prey) system. It has been assumed that parasites search hosts at random and that both parasites and hosts are assumed to be distributed in a non-contiguous ("clumped") fashion in the environment. However the modified version of the Nicholson-Bailey model has been discussed many times by many authors[20, 33, 35, 82]. One of the modified version has
been taken by Dutta, T.K. et al [35] where the boundedness of the predator-prey system has been taken care of. The system is considered as follows:

\[ x_{n+1} = Lx_ne^{-ay_n-x_n^2} \]

\[ y_{n+1} = x_n(1 - e^{-ay_n}) \quad (6.1.1) \]

which may be written in the form

\[ f(x,y) = (Lxe^{-ay-x^2}, x(1 - e^{-ay})) \quad (6.1.2) \]

where \( x \) represents the prey and \( y \) represents the predator system and \( L \) represents the control parameter and “a” is a constant whose value is taken to be 0.1. In this system it has been observed that the dynamic behaviour of the system takes the period-doubling route to chaos having infinite periodic behaviour at the accumulation point[35]. The accumulation point is calculated as 3.842735286416 ….

In this chapter we try to measure the chaotic behaviour at the accumulation point with the help of various tools for example calculating various kind of fractal dimensions and Lyapunov exponents which are currently present in the literature.

In section 6.2 we discuss the Lyapunov exponent for the system which is found to be positive just beyond the accumulation point.
confirming chaos. In section 6.3 the capacity dimension is calculated at the accumulation point by directly calculating the number of boxes. In section 6.4 the capacity dimension, information dimension, correlation dimension has been calculated with the help of generalised correlation sum.

6.2. Lyapunov exponents:

Lyapunov exponents are the average exponential rates of convergence or divergence of nearby orbits in phase space[]. It helps to detect the sensitivity of the initial condition. In case of k dimensional difference equation where \( k \geq 2 \) it may be defined as \( \lambda_k = \lim_{n \to \infty} \frac{1}{n} \log (k\text{th eigenvalue of } J(n)) \), where \( J(n) \) is the Jacobian of a composite function with composition up to n times of k dimensional map. If max \( \lambda_k \), \( k \geq 2 \) is negative, then the system observes periodic behaviour at that particular parameter. If it is 0 then the stability is changed at that parameter and if it is positive, then the system has become chaotic at that particular parameter.

6.2.1. Applied Method: Let us consider an initial point say \((x_0,y_0)\). As discussed in (6.2.1), Jacobian of the map \( f^n(x_0,y_0) \) is to be calculated whose eigen values will determine the lyapunov exponent. For the calculation of Jacobian of \( f^n(x_0,y_0) \), we calculate jacobian of \( f(x_0,y_0) \) say it is \( j_0 \), then Jacobian of \( f(x_1,y_1) \) say \( J_1 \), where \((x_1,y_1)=(Lx_0e^{-ay_0}-x_0, x_0 (1-e^{-ay_0}))\), and so on. Then kth Lyapunov exponent is,
\[ \lambda_k = \lim_{n \to \infty} \log k \text{th Eigen Value of } (J_n, J_{n-1}, \ldots, J_1)_{k=1,2}. \]

Out of the two Lyapunov exponents at every parameter the maximum will be considered which is crucial to detect the dynamic behaviour of the system.

**Fig 6.2.a:** Diagram of maximum Lyapunov exponent. Abscissa represents the control parameter “L” while the ordinate represents maximum Lyapunov exponent. In the above diagram the points of the abscissa where it is touched by the curve are the bifurcation points.
Table 6.2.a. Lyapunov exponent near the accumulation point.

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<tr>
<th>Parameter Value</th>
<th>Maximum Lyapunov exponent</th>
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<td>3.842</td>
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<td>3.843</td>
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<td>3.84273528</td>
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<td>3.8427352865</td>
<td>0.0000492077</td>
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<td>3.842735286</td>
<td>0.0000195551039610285</td>
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Lyapunov exponent is supporting the accumulation point result up to 10 decimal places.
6.3 Box counting dimension:

Let $F$ be any non empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most $\delta$ which can cover $F$. The lower and upper box counting dimensions of $F$ respectively are given by

$$\dim_b F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

If these are equal we refer to the common value as the box counting dimension of $F$.

$$\dim_b F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad (6.3.1)$$

For the calculation purpose we construct boxes of side length $R$ which covers the attractor set. For example for a one dimensional attractor set, the boxes are line segments of length $R$. For a two dimensional attractor set the boxes will be square of side length $R$. In three dimensional case the boxes will be cubes e.t.c. Let the least number of boxes which covers the attractor set be $N(R)$. Then equation (6.3.1) may be replaced by the formula

$$\dim_b F = \lim_{R \to 0} \frac{\log N(R)}{-\log R} \quad (6.3.2)$$

We have run a computer program for 6000000 iterations for one initial value. Here $N(n,s)$ represents the number of boxes of size $1/s$ which cover
n points out of 6000000 points of the attractor set In order to use the relation (6.3.2), we need N(s)=number of boxes of size 1/s which is used to cover the whole attractor set. To calculate we apply extrapolation method in the following way.

In Grassberger’s way[], we assume that the map obeys the following formula

\[ N(n,s) = N(s) - \text{const.} \cdot n^{-a} s^{-b} \]  \hspace{1cm} (6.3.3)

where \( N(n,s) \) is the number of boxes required of size \( 1/s \) to cover \( n \) attractor points which is got after iterating the difference map at the control parameter value

\( L = 3.842735286416 \ldots \) \hspace{1cm} \( N(s) = \) total number of boxes of size \( 1/s \), to cover the whole attractor and \( a, b \) are some positive constant need to determine.

**A.1 TO CHECK WHETHER EQUATION (6.3.3) IS OBEYED OR NOT:**

If equation (6.3.3) is well fitted in our case then

\[ N(n_1,s) - N(n_2,s) = N(s) - \text{const.} \cdot n_1^{-a} s^{-b} - N(s) + \text{const.} \cdot n_2^{-a} s^{-b} \]

\[ = \text{const.} \cdot s^{-b} (n_2^{-a} - n_1^{-a}). \]

If \( n_1 \) is large enough than \( n_2 \), we can neglect \( n_1^{-a} \) with respect to \( n_2^{-a} \).
So, we can say

\[ N(n_1,s) - N(n_2,s) = N(s) - \text{const.} n_1^{-a} s^{-b} - N(s) + \text{const.} n_2^{-a} s^{-b} \]

\[ \approx \text{const.} s^{-b} (n_2^{-a}). \]  \hspace{1cm} (6.3.4)

If we fix \( n_2 \) in (6.3.4) then \( \log(N(n_1,s) - N(n_2,s)) \) vs. \( \log(n_1) \), where \( n_1 \) is very large compared to \( n_2 \), be a straight line parallel to abscissa.

In our case we set \( \log(n_1) \) as the x-axis and \( \log(N(n_1,s) - N(n_2,s)) \) in the y-axis. We vary \( n_1 \) from 1000000 to 6000000 with an increment of 100. We set \( n_2 \) as 100 and \( s = 2^{19} \), \( \log(N(n_1,s) - N(100,s)) \) as Y-axis and \( \log(n_1) \) as X-axis, then \( \log(N(n_1,s) - N(100,s)) \) vs. \( \log(n_1) \) graph is as follows:

\[ \text{Fig 6.3.a: Abscissa represents } \log(n_1) \text{ and ordinate represents } \log(N(n_1,s) - N(100,s)). \]
Again if we take $n_1$ large enough compared to $n_2$ but fix both of them and vary “s” then we have

$$N(n_1, s) - N(n_2, s) = N(s) - \text{const.} n_1^{-a} s^{-b} + \text{const.} n_2^{-a} s^{-b}$$

$$\approx \text{const.} s^{-b} (n_2^{-a}) \quad (6.3.5)$$

If we plot $\log(N(n_1, s) - N(n_2, s))$ vs $\log(s)$ then we will have a straight line whose slope should be “b”.

We have taken $\log(N(6000000, s) - N(100, s))$ vs $\log(s)$ i.e. $\log(s)$ in the X-axis and $\log(N(6000000, s) - N(100, s))$ in the Y-axis and $s=2^8, 2^9, ..., 2^{20}$.

The graph is as follows:

**Fig 6.3.b:** Abscissa represents $\log(s)$ and ordinate represents $\log(N(6000000, s) - N(100, s))$. 
From the graph we can see that the plotted points more or less follow a straight line path. So, we can now say that equation (6.3.3) can be used in our case also. The straight line when fitted by least square method gives the slope i.e. the value of “b” as 0.903355.

A.2 CALCULATION OF “a”:

From equation (1.3.5) putting \( n_1 = 6000000 \) and \( n_2 = 100 \) we have

\[
N(6000000, s) - N(100, s) \approx \text{const.} s^{-b} (100^a) \quad [\text{i.e. we have neglected } 4000000^a]
\]

Again putting \( n_1 = 6000000 \) and \( n_2 = 200 \) we have

\[
N(6000000, s) - N(200, s) \approx \text{const.} s^{-b} (200^a)
\]

\[
\frac{N(6000000, s) - N(100, s)}{N(6000000, s) - N(200, s)} = \left( \frac{1}{2} \right)^{-a}
\]

Therefore

\[
\frac{\log \left( \frac{N(6000000, s) - N(100, s)}{N(4000000, s) - N(200, s)} \right)}{\log (2)} = a, \text{ which should be equal for all values of } s, \text{ but in real data the value may be a little bit different. So we have taken}
\]

\[
a = \frac{\sum_{s} \log \left( \frac{N(4000000, s) - N(100, s)}{N(4000000, s) - N(200, s)} \right)}{\log (2) \times 12}, \text{ where } s = 2^9, \ldots, 2^{20} (12 \text{ values of } s).
\]

The value of “a”, we have got is 0.741963.
A.3 TO CALCULATE THE VALUE OF “b”:

Calculation of “b” is already done in section A.1

A.4 TO CALCULATE THE VALUE OF CONSTANT:

From (6.3.5) we have

$$N(200,2^{10}) - N(100,2^{10}) = \text{const.} \cdot (2^{10})^{-b} (100^{-a} - 200^{-a}).$$

Therefore const = \[
\frac{N(200,2^{10}) - N(100,2^{10})}{2^{-10b} (100^{-a} - 200^{-a})}
\]

The value of the constant is 14.9157.

Now we can calculate \(N(s) = N(n,s) + \text{const.} \cdot n^{-a} s^{-b}\), for different values of \(s,n\).

The following table is

\(N(n,s) + \text{const.} \cdot n^{-a} s^{-b}\), first column for \(s = 2^8\), next \(s = 2^9, 2^{10}, \ldots, 2^{20}\). The first row for \(n = 1000000\), next \(n = 100000, 200000, \ldots, 6000000\). Clearly in a particular column the values are almost same which verifies that it is independent of \(n\) and thus represents \(N(s)\).
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A.5 SLOPE OF THE GRAPH WHICH GIVES BOX DIMENSION:

The graph of \( \log(N(s)) \) vs. \( \log(s) \) is shown below. \( \log(N(s)) \) is taken in the Y-axis and \( \log(s) \) is taken in the X-axis. The value of \( s = 2^9, 2^{10}, \ldots, 2^{19} \).

![Graph](image)

**Fig 6.3.d:** Abscissa represents \( \log(s) \) and ordinate represents \( \log(N(s)) \).

The slope of the line fitted in least square method in the plotted points is 0.546734 with a mean deviation of 0.0891668, which clearly says that the attractor set is fractal in nature.
We also want to calculate correlation dimension, information dimension e.t.c at the accumulation point to prove the fractalness of the attractor set more concrete manner. For that purpose we give the definitions of dimensions as follows:

### 6.4. GENERALISED CORRELATION DIMENSION

We attempt to measure the dimension of the attractor at the onset of chaos, i.e. at the accumulation point. As the dimension calculation is crucial in detecting chaos. If the dimension calculated is of fraction value then the attractor will be of strange nature. If the dimension is zero, then the attractor is still periodic nature. However in our case as the procedure will be mainly numeric in nature, for that we study information dimension, correlation dimension and box counting dimension to make the result concrete.

To calculate these dimensions, we take the help of generalized correlation sum. First of all we discuss generalized dimension \([8,18,19]\), i.e. Reni’s dimension. It is defined as follows:

\[
D_q = \lim_{R \to 0} \frac{1}{q-1} \log \left( \sum_{i \in i} P_i^q \right) \log R
\]

\[(6.4.1)\]

Where \(P_i\) is the probability and is defined as \(P_i = N_i / N\) and \(R\) is the side length.
If \( q = 0 \), it gives the box counting dimension. For \( q=1 \), it gives information dimension and for \( q=2 \) it gives correlation dimension.

Again, generalized correlation sum is given as follows:

\[
G_q(N,R) = \left[ \frac{1}{N-1} \sum_{i=1}^{N} \left( \frac{1}{N-1} \sum_{j \neq i}^{N} H(R - \|X_i - X_j\|) \right)^{-\frac{1}{q-1}} \right]^{\frac{1}{q-1}} 
\]  

(6.4.2)

Where \( H(R - \|X_i - X_j\|) \) means if \( \|X_i - X_j\| < R \), then

\( H(R - \|X_i - X_j\|) = 1 \) and if \( \|X_i - X_j\| \geq R \), then \( H(R - \|X_i - X_j\|) = 0 \).

Further, \( \lim_{N \to \infty} G_q(N,R) = G_q(R) \), and

\[
D_q = \lim_{R \to 0} \frac{\log G_q(R)}{\log R} 
\]  

(6.4.3)

To apply (1.4.2) we use the Euclidean norm i.e. if \( X_1=(x_1,y_1) \) and \( X_2=(x_2,y_2) \) then \( \|X_1 - X_2\| = (x_1 - x_2)^2 + (y_1 - y_2)^2 \).

6.4.4. Correlation dimension:

\( G_q(R) = G_q(30000,R) \) is calculated. The part of the plotted points (log \( R \), log \( G_q(R) \)) which follows equation (6.4.3) is taken. The slope of the fitted straight line in that scaling region is \( D_q \).
**Fig 6.4.a:** log R vs. log $G_q(R)$ in the scaling region

The slope of the above points when fitted with a straight line by least square method is 0.498615 with a mean deviation of 0.0754032. The data is obtained from 30000 iterated points at the parameter 3.8427352864165.

**6.4.5. Information dimension:**

If we take $q$ very near to one i.e. $q$ tends to 1 then that gives the information dimension. At $q=1.00000000001$ at the parameter 3.8427352864165 with 30000 iterations.

The log($G_q(R)$) vs. log(R) graph in the scaling region is as shown below:
The slope of the above points when fitted with a straight line by least square method is 0.51829 with a mean deviation of 0.0512648.

**6.4.6. Box counting dimension:**

For $q = 0$ it gives the box counting dimension. We have done the calculation at the parameter 3.8427352864165 with 30000 iterations.
The slope of the above points when fitted with a straight line by least square method is 0.534081 with a mean deviation of 0.0436084.

6.5  : Statistical Analysis and Conclusion:

The theory of chaos and fractals is fascinating if for no other reason than the blurring of the long-held distinction between random and deterministic phenomena. It is potentially capable of explaining very complex processes with simple, parsimonious models and essentially without error. This would seem to be sufficient reason for the Statistician to pay attention to the emerging theory and actively participate in its development. Such participation requires the statistician to be prepared to think in terms of iterations of functions rather than stochastic processes, attractors rather than spectra and embeddings in a phase space rather than time series. In return, the Statistician gets a richer
set of tools at his or her disposal, although the power and limitations of the new tools will have to be sorted out.

Although, deterministic models have been used for several decades for generating pseudo random numbers in simulation experiments, the renewed interest in them is due to their possible use in modeling actual real-world processes that have traditionally been studied through stochastic models. As yet, however, there are no known methods for “fitting” a deterministic model to an actual process. Present state of the art is at the initial stages of studying and cataloging that behavior of various models, and to use expertise and judgment to see if a particular model adequately describes an actual process under investigation.

One potentially useful tool for the statistician is the idea of fractal interpolation (Barnsley, 1988). Given a finite number of observations, this method generates a complete path interpolating the observations and in a manner consistent with self-similarity. This idea can be useful in handling missing values in the data, and it is illustrated by Chatterjee and Yilmaz (1991), [24]. Its usefulness for prediction purposes remains to be investigated.

Another recent modeling tool that seems to have been motivated by fractals and fractal dimension is the notion of fractional differencing in the ARIMA
(p, d, q) models, where d is noninteger. This idea can be used to model persistence (long memory) and antipersistence (short memory) behavior. The process (0,d, 0), -1/2 < d < ½ has been used by Mandelbrot and Van Ness (1988 [88]), Mandelbrot (1971), [87] and Matalas and Wallis (1971), [90], for simulating hydrologic data that show long-term memory.

An ARIMA (0, d, 0) process $x_t$ is defined as

$$ (1 - B)^d x_t = \xi_t, \quad \text{where } \xi_t \text{ is a white noise process and } B \text{ is the backward shift operator.} $$

Noting that,

$$ (1 - B)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-B)^k $$

$$ = 1 - dB - (1/2) d (1 - d) B^2 - (1/6) d (1 - d) (2 - d) B^2 - \ldots $$

it is easy to see the slowly decaying weights on past observations. Thus,

ARIMA (0, d, 0) provides long-term persistence behavior, whereas

ARIMA (p, 0, q) describes short-run persistence. Granger (1980), [46], Granger and Joyeux (1980), [46], Hosking (1981), [65], Geweke and Porter-Hudak (1983), and Carlin and Dempster (1989), [19] discuss ARIMA (p, d, q) with noninteger d, thus generalizing the procedures of Box and Jenkins (1970). These authors have investigated different
properties of fractionally differenced models and estimation procedures, and provided some sampling theory.

There are many important questions about the new theory that are currently unresolved, and it is likely that some of these issues may never be resolved. The question of choosing between deterministic versus stochastic modeling of a process under study is one such issue, but stochastic modeling presently has a clear advantage because of its rich variety of model-fitting tools. On the other hand, given that we wish to use a deterministic model, it would be imperative to have some means for deciding which specific model to use. Although methods or guidelines for this purpose are not yet available, we can expect progress in this direction as a larger variety of models are studied.

Another immediate problem in the application of the new theory is the possibility that the observed data includes a random component of environmental noise. In experimental settings, it may be possible to know the sources of noise and minimize it, but this may not be possible when data pertains to real world phenomena. Thus a basic question is how to recognize the presence of noise and how to separate it from the deterministic effect. In the presence of noise, say $\xi_t$, one could contemplate a model such as
$$x_{t+1} = f_w ( x_t , x_{t+1} , \ldots \ldots \ldots , x_{t-L} , \xi_t )$$ and if noise is additive, this would become

$$x_{t+1} = f_w ( x_t , x_{t+1} \ldots \ldots , x_{t-L} ) + \xi_t$$

If $\xi_t$ is a stochastic process, then the advantage of deterministic modeling over the classical stochastic approach would disappear.

Unlike the deterministic approach, stochastic modeling involves an attempt to separate structure (pattern) in the data from lack of structure (nonpattern), and for this reason, it naturally allows for the presence of noise and other random elements. Although, deterministic modeling can deal with very small amounts of noise, it provides no means for recognizing its existence in the first place. Controversy and research in this area are likely to continue for the foreseeable future.

Present state of the art of dynamical modeling is such that the ability to make general statements about the long-term behavior of a dynamic process requires the assumption that transients have died out and motion has reached the attractor. In experimental settings, it may be possible to wait until this happens. In many real-world processes, on the other hand, we may be perpetually observing transient states due to changes in this environment or even changes in system parameters (bifurcations). More importantly, we do not yet have methods for determining if observed data include such changes or transients.
In summary, then, there are various distinct situations in which we must currently turn to stochastic models, even if a deterministic model is desired. These situations include:

(i) Initial conditions are unknown.
(ii) Process cannot be observed without random error or noise.
(iii) Process cannot be observed long enough.
(iv) Inability to fit a deterministic model even when situations (i) to (iii) are not at issue.

Consequently, stochastic models are likely to continue to serve as prototypes for nonlinear deterministic models.

Here we have looked at several ways of quantitatively characterizing a chaotic system, in essence, looking for some quantitative way of specifying how chaotic a system is. Various fractal dimensions focus on the geometric structure of attractors in state space. If the system has at least one positive Lyapunov exponent, we say that the system’s behaviour is chaotic. If the attractor exhibits scaling with (in general) a noninteger dimension (scaling index), then we say the attractor is strange. Again, we have calculated different dimensions by applying direct formulae and deduce some of them from the generalised correlation sum, but all these results are not in conformity. This essentially needs more sophisticated
mechanism to evaluate these dimensions more accurately. This is, in fact, a challenging research problem. Moreover, we would like to conclude by pointing out that no single quantifier has emerged as the best way to characterise a nonlinear system. All of the quantifiers proposed to date require considerable computational effort to extract their numerical values from the data.

The numerical techniques used in this chapter to calculate Lyapunov exponent and other fractal dimensions may be used to calculate the same in higher dimensional models.

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