CHAPTER VI

SOME COMPOUND DISTRIBUTIONS

Distribution

6.1 Compound Poisson With Respect to Compounder

Generalized Beta Distribution

6.1.1 Introduction

Suppose the symbolic form

\[ \text{Poisson}(\theta) \sim \text{Generalized beta}(a, b, c) \]  

represents a compound Poisson distribution formed by ascribing to the Poisson parameter \( \theta \) the generalized beta distribution having the density (see Rainville, 1960, p.124)

\[ f(\theta| a, b, c) = \frac{e^{-c\theta} \theta^{a-1}(1-\theta)^{b-1}}{B(a, b)M(a; a+b;-c)}, \]

\( 0 \leq \theta \leq 1; \ a, b > 0; \ c > 0 \)  

where \( B(a, b) \) and \( M(a; b; x) \) denote the beta function and the confluent hypergeometric function respectively (see Johnson and Kotz, 1969, pp.8-9). In this particular problem we have restricted ourselves to the class of Poisson populations for which \( \theta \) is distributed over the finite interval \( (0,1) \) according to \( (2) \).

6.1.2 Derivation of the Distribution (1)

The probability generating function (p.g.f) of the compound distribution (1) is given by
G(s) = \int_0^1 e^{\theta(s-1)} dF(\theta)

= \frac{1}{B(a,b)M(a; a+b; -c_0)} \int_0^1 e^{-(\theta(c+1)+s)} \theta^{a-1}(1-\theta)^{b-1} d\theta

= M(a; a+b; -(c+1-s)) / M(a; a+b; -c).

From the definition of probability generating function, we have

G(s) = \sum_{r=0}^{\infty} p_r s^r,  \quad (4)

where \( p_r \) is the probability function. Comparing the coefficients of \( s^r \) from the right hand sides of (3) and (4), we get the p.f. of the compound distribution (1) as

\[ p_r = \frac{1}{\{ M(a; a+b; -c) r! \}} \left[ \frac{(a)^{[r]} (a+1)^{[r+1]} (c+1)}{(a+b)^{[r]} (a+b)^{[r+1]} 1!} \right.

+ \frac{a^{[r+2]} (c+1)^2}{(a+b)^{[r+2]} 2!} - \cdots \right]

= (1/r!) \left[ a^{[r]} / (a+b)^{[r]} \right] \left[ M(a+r; a+b+r; -(c+1)) / M(a; a+b; -c) \right],

= (1/r!) (1/e) \left[ a^{[r]} / (a+b)^{[r]} \right] \left[ M(b; a+b+r; (c+1)) / M(b; a+b; c) \right], \quad (5)

where \( a^{[r]} = a(a+1)...(a+r-1) \) and \( M(a;b;x) \) satisfies Kumar's first formula (see Johnson and Kotz, 1969, p.9)

\[ M(a;b;x) = e^{x}M(b-a;b;-x). \]  \quad (6)
Here

(161)

\[ p_0 = \frac{1}{e} \frac{M(b; a+b; c+1)}{M(b; a+b; e)} \]  \hspace{1cm} (7)

6.1.3 Properties

The probability function in (5) satisfies the recurrence relation

\[ p_{r+1} = \frac{(a+r) \cdot M(b; a+b+r+1; c+1)}{(r+1)(a+b+r)M(b; a+b+r; e)} \cdot p_r \]  \hspace{1cm} (8)

Rushton and Lang (1954) give tables of the confluent hypergeometric function \( M(a; b; x) \) which can be used to obtain the values of the probability function (5).

The factorial moments of the distribution (1) are given by

\[ \mu_k' = \frac{a^k}{(a+b)^k} \cdot \frac{M(a+k; a+b+k; c)}{M(a+k; a+b+k; c)} \]  \hspace{1cm} (9)

The mean of the distribution (1) is

\[ \mu_1' = \frac{a}{a+b} \cdot \frac{M(b; a+b+1; c)}{M(b; a+b; c)} \]  \hspace{1cm} (10)

The factorial moments satisfy the recurrence relation

\[ \mu_{k+1} = \frac{(a+k) \cdot M(b; a+b+k+1; c)}{M(b; a+b+k; c)} \cdot \mu_k' \]  \hspace{1cm} (11)

Relation (11) can be used to find the different moments of the distribution (1).
6.1.4 Two Particular Cases

(i) If c=0, (2) represents a beta distribution and (5) becomes

\[ p_r = \frac{1}{r!} \left( e^{-1} \right) \frac{a^r}{(a+b)^r} M(b; a+b+r; 1). \]

The factorial moments of (12) satisfy the recurrence relation

\[ \kappa_{(k+1)} = \frac{(a+k)}{(a+b+k)} \kappa_{(k)}, \]  \hspace{1cm} (13)

where

\[ \kappa_{(1)} = \frac{a}{a+b}. \]

(ii) If c=0, a=b=1, then (2) becomes the uniform rectangular distribution

\[ f(\theta) = 1, \quad 0 < \theta < 1, \]

and (5) becomes

\[ p_r = e^{-1} M(1; r+2; 1)/(r+1)! \quad (r=0,1,...) \]

which is a particular case of the distribution studied by Bhattacharya and Holla (1965).

Factorial moments of (14) satisfy the following recurrence relation

\[ \kappa_{(k+1)} = \frac{(k+1)}{(k+2)} \kappa_{(k)}, \]

whence

\[ \kappa_1 = 1/2, \quad \kappa_2 = 7/12, \quad \kappa_3 = 5/4 \quad \text{and} \quad \kappa_4 = 167/80 \quad \text{(approx)}. \]

Our first three moments agree with Bhattacharya and Holla (1965) for the range (0,1). The fourth moment however does not agree with them.
6.1.5 A Multivariate Generalization of the Distribution (5)

Using the approach of Bhattacharya (1966), we consider a k-dimensional discrete probability function of the form

\[
h(x_1, x_2, \ldots, x_k) = \int_0^1 \frac{e^{-k\theta} \sum_{i=1}^k x_i}{\Gamma_k \prod_{i=1}^k x_i!} \cdot f(\theta) d\theta,
\]

where \( f(\theta) \) is the density function given by (2).

Thus we obtain

\[
h(x_1, \ldots, x_k) = \int_0^1 \frac{-(k+c)\theta \sum_{i=1}^k x_i + a - 1}{\prod_{i=1}^k x_i! \Gamma(a,b) M(a; a+b; -c)} \cdot (1 - \theta)^{b-1} d\theta
\]

\[
= \prod_{i=1}^k \frac{B(x_i + \ldots + x_k + a, b)}{\{\prod_{i=1}^k x_i! B(a, b)\}} \cdot \frac{M(x_1 + \ldots + x_k + a; x_1 + \ldots + x_k + a + b; - (k+c))}{M(a; a+b;-c)}
\]

\[
= \prod_{i=1}^k \frac{B(x_i + \ldots + x_k + a, b)}{\prod_{i=1}^k x_i! B(a, b)} \cdot \frac{M(x_1 + \ldots + x_k + a; x_1 + \ldots + x_k + a + b; - (k+c))}{M(a; a+b;-c)}
\]

The multi-dimensional characteristic function (c.f) corresponding to (17) is obtained as

\[
\Phi(t_1, \ldots, t_k) = \int f(\theta) \exp \{ \theta (e^{it_1} + e^{it_2} + \ldots + e^{it_k} - k) \} \, d\theta
\]

\[
= M(a; a+b; -(\sum_{j=1}^k e^{j+k+c})/M(a; a+b;-c). \quad (18)
\]
If $k=1$, (17) reduces to the distribution (5) for which the mean and variance can be obtained from (9).

The bivariate distribution for $k=2$ is given by the probability function

$$h(x_1,x_2) = \frac{B(x_1+x_2+a,b)}{(x_1!x_2!)B(a,b)} \times \frac{M(x_1+x_2+a;x_1+x_2+a+b;-(c+2))/M(a+a+b;-c)}{x_1=0,1,2,... \quad (i=1,2) \quad (a,b>0; c>0)} \quad (19)$$

From (19), we get

$$E(X_1X_2) = \frac{a(a+1)}{(a+b)(a+b+1)} \cdot \frac{M(b;a+b+2;c)}{M(b;a+b;c)}$$

so that the coefficient of correlation is

$$r = [1+ u/v]^{-1}$$

where

$$u = (a+b)(a+b+1)M(b;a+b;c)M(b;a+b+1;c)$$

and

$$v = \{(a+1)(a+b)M(b;a+b;c)M(b;a+b+2;c)\} - \{a(a+b+1)M^2(b;a+b+1;c)\}.$$  

It is worth noting that though we started with independent Poisson variates, a correlation has been induced through a common mixing distribution of $\theta$.

6.1.6 An Application to Accident Phenomenon

An application of the foregoing results to the theory of accident proneness when the accident liabilities vary from individual to individual may be demonstrated in the following manner:
For a set of individuals exposed to accident risk, we consider two successive periods of given duration and assume that

(i) the number of accidents sustained by an individual in a given period has a Poisson distribution with parameter $\theta$

(ii) $\theta$ varies from individual to individual with an underlying distribution (2), say.

Then the overall distribution of the observed number ($x$) of accidents per individual in a given period will be given by (5). Now, if $y$ is the number of accidents sustained by an individual in the second period, we obtain, by using (5) and (19), the conditional distribution of $y$ as

$$P(y \mid x) = \frac{1}{y!} \left[ \frac{B(a+x+y, b)}{B(a+x, b)} \right] \cdot \left[ \frac{M(a+x+y; a+b+x+y; -(c+2))}{M(a+x; a+b+x; -(c+1))} \right]$$

which is the same distribution as (5) with 'a' replaced by 'a+x' and 'c' by 'c+1'. As such,

$$E(Y \mid X=x) = \left[ \frac{(a+x)}{(a+b+x)} \right] \cdot \left[ \frac{M(b; a+b+x+1; c+1)}{M(b; a+b+x; c+1)} \right].$$

where

Thus from (20) and (21), we get

$$P(y \mid 0) = \frac{1}{y!} \left[ \frac{B(a+y; b)}{B(a, b)} \right] \cdot \left[ \frac{M(a+y; a+b+y; -(c+2))}{M(a; a+b; -(c+1))} \right]$$

and
Now, from (8), we can write

\[
E(Y \mid 0) = \frac{a}{a+b} \frac{M(b; a+b+1; c+1)}{M(b; a+b; c+1)}.
\]  

(23)

We see that

\[
E(Y \mid 0) < E(X), \quad \text{if } \frac{p_1}{p_0} < E(X),
\]  

(24)

where \(E(X)\) is given in (10). Hence, for this application, we see that a selection of individuals with no accidents in the first period will result in the mean decrease of accidents in the second period, if the inequality in (24) holds.
Distribution

6.2. Compound Poisson With Respect to Compounder

Generalized Inverted Beta Distribution

6.2.1 Introduction

From the integral representation of the confluent hyper-

egometric function (see Erdélyi, 1953, p.255) denoted as

\[ N(a;b;c) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-\theta} \theta^{a-1}(1+\theta)^{b-a-1} d\theta, \]

\[(a,b>0, c>0) \quad (25) \]

we consider the function

\[ f(\theta) = \frac{e^{-\theta} \theta^{a-1}(1+\theta)^{b-a-1}}{\Gamma(a) N(a;b;c)}, \quad (0<\theta<\infty) \quad (26) \]

where

\[ \Gamma(a) N(a;b;0) = B(a, 1-b) = \int_0^\infty \theta^{a-1}(1+\theta)^{-(a+(1-b))} d\theta \]

\[(0 < b < 1). \quad (27) \]

Under these conditions stated \( f(\theta) \) is non-negative and, moreover, by virtue of (25), \( \int_0^\infty f(\theta) d\theta = 1 \). Thus (26) represents a density function which for \( c=0 \) and \( 0 < b < 1 \) yields an inverted beta distribution. We call (26) a generalized inverted beta distribution.

Here we consider

\[ \text{Poisson}(\theta) \wedge \text{Generalized inverted beta}(a,b,c) \quad (28) \]

6.2.2 Derivation of the Distribution (28)

The distribution is given by
6.2.3 Properties

The probability function in (29) satisfies the recurrence relation

$$P_{k+1} = \frac{(a+k)N(a+k+1;b+k+1;c+1)\cdot P_k}{(k+1)N(a+k;b+k;c+1)}\quad (k=0,1,2,\ldots).$$ (30)

where

$$P_0 = \frac{N(a;b;c+1)}{N(a;b;c)}.$$ 

The factorial moments of the distribution (29) are given by

$$\mu_r = a \cdot N(a+r;b+r;c) / N(a;b;c)$$

$$\mu_r = [N(a+r;b+r;c) / N(a;r;b+r;c)]\cdot \mu_r.$$ (31)

A recurrence relation for the factorial moments is

$$\mu_{r+1} = [N(a+r+1;b+r+1;c) / N(a+r;b+r;c)]\cdot \mu_r.$$ (32)

6.2.4 Particular cases

(i) Suppose $c=0$ and $0 < b < 1$. Then (26) becomes an inverted beta distribution and (29) becomes
where \( N(a;b;x) \) and \( B(a, 1-b) \) are defined in (25) and (27) respectively.

(ii) Let \( b=a+1 \). Then (26) reduces to the gamma density

\[
f(\theta) = \frac{c^a}{\Gamma(a)} e^{-c\theta} \theta^{a-1}
\]

and (28) becomes

\[
\text{Poisson}(\theta) \sim \text{Gamma}(a, c).
\]

6.2.4 A Multivariate Generalization of the Distribution (28)

It is considered as

\[
h(x_1, x_2, \ldots, x_k) = \int_{0}^{\infty} e^{-k \theta^{x_1+\ldots+x_k}} \frac{f(\theta)}{(\prod_{i=1}^{k} x_i !)} d\theta,
\]

\[
x_i = 0, 1, 2, \ldots
\]

\[
(i = 1, \ldots, k)
\]

where \( f(\theta) \) is the density function given by (26).

Thus we obtain

\[
h(x_1, x_2, \ldots, x_k) = \frac{a^{x_1+\ldots+x_k}}{\Gamma(a+x_1+\ldots+x_k)} \prod_{i=1}^{k} \frac{1}{(x_i !)} N(a+x_1+\ldots+x_k ; b+x_1+\ldots+x_k ; c+k),
\]

\[
x_i = 0, 1, 2, \ldots
\]

\[
(i = 1, \ldots, k)
\]

The multi-dimensional characteristic function (c.f.) corresponding to (34) is obtained as
\( \phi(t_1, t_2, \ldots, t_k) = \int_0^{\infty} f(\theta) \exp \{ \theta (t_1 \theta + \ldots + t_k \theta^{k-k}) \} d\theta \)

\[ = N(a; b; (-\sum_{j=1}^k \theta + k + c)) / N(a; b; c). \]

If \( k = 1 \), (34) reduces to the distribution (29) for which mean and variance can be obtained from (31).

If \( k = 2 \), the bivariate distribution is given by the probability function

\[ h(x_1, x_2) = \frac{a^{x_1 + x_2} N(a + x_1 + x_2; b + x_1 + x_2; c + 2)}{x_1!x_2!N(a; b; c)}, \]

\[ x_1 = 0, 1, 2, \ldots, \]

\[ i = 1, 2 \quad (a, b > 0, c > 0). \quad (35) \]

From (35), we find

\[ E(X_1X_2) = a(a+1)N(a+2; b+2; c) / N(a; b; c) \]

and the correlation coefficient as

\[ r = [1 + u/v]^{-1}, \quad (36) \]

where

\[ u = N(a; b; c)N(a+1; b+1; c) \]

and

\[ v = \{(a+1)N(a; b; c)N(a+2; b+2; c)\} - \{aN^2(a+1; b+1; c)\}. \]

6.2.5 An Application to Accident Phenomenon

As in Section 6.1.6, we obtain a simple model of accident proneness under the following assumptions:

(i) the number of accidents sustained by an individual in a fixed period (of unit length, say) is given by a Poisson distribution with parameter \( \theta \)

(ii) \( \theta \) varies from individual to individual with an underlying distribution (26).
Then the overall distribution of the observed number \( x \) of accidents per individual in a given period has a probability function (29). Now, if \( y \) corresponds to the number of accidents per individual in the next period (of unit length), then under an assumption of the same distribution for \( \Theta \), the conditional distribution of \( y \) is given by

\[
P(y \mid x) = \frac{(a+x)^y y!}{N(a+x;b+x;c+1)} \binom{N(a+x+y;b+x+y,c+2)}{y} (37)
\]

which is the same distribution as (29) with 'a' replaced by '(a+x)', 'b' by '(b+x)' and 'c' by '(c+1)'. As such

\[
E(Y \mid x) = (a+x) \cdot \frac{N(a+x+1;b+x+1;c+1)}{N(a+x;b+x;c+1)}
\]

and

\[
E(Y \mid 0) = a \cdot \frac{N(a+1;b+1;c+1)}{N(a;b;c+1)}
\]

Clearly \( E(Y \mid 0) = p_1/p_0 \).

We see that

\[
E(Y \mid 0) < E(X), \text{ if } p_1/p_0 < E(X), \quad (38)
\]

where \( E(X) \) is given by (31) for \( r = 1 \). Hence, for this application, we see that a selection of individuals with no accidents in the first period will result in the mean decrease of accidents in the second period, if the inequality in (38) holds.
6.3 Compound Poisson With Respect to Compounder Bessel Distribution

6.3.1 Introduction

Bessel distribution (see McNolty, 1964) is given by

\[ f(\theta | a, b, c) = \frac{(2/b)^c}{\Gamma(c)} \theta^{c-1} e^{-b^2/(4a)} e^{-a\theta} I_{c-1}(b\theta^{1/2}), \]  

(\( \theta > 0 \))  \hspace{1cm} (39)

where \( I_{c-1}(.) \) denotes the modified Bessel function of the first kind of order \((c-1)\).

Here we study compound Poisson with respect to compounder Bessel distribution. It is symbolically denoted as

\[ \text{Poisson}(\theta) \sim \text{Bessel distribution}(a, b, c). \hspace{1cm} (40) \]

6.3.2 Derivation of the distribution (40)

The probability generating function of (40) is given by

\[ G(s) = (2/b)^c a^{c-1} e^{-b^2/(4a)} \int_0^\infty e^{-\theta(a+1-s)} \theta^{c-1} I_{c-1}(b\theta^{1/2}) d\theta \]

\[ = \frac{(a^c / (a+1)^c) e^{-b^2/(4a)}}{(1 - s/(a+1))^{-c} / (4(a+1-s))}. \hspace{1cm} (41) \]

This can be written as

\[ G(s) = a^c / (a+1)^c e^{-b^2/(4a)} \left\{ (1 - s/(a+1))^{-c} + \right. \]

\[ \left. \frac{d}{d(1 - s/(a+1))} -(c+1) + \frac{d^2}{2!} (1 - s/(a+1))^{-(c+2)} + \ldots \right\}, \]

where \( d = b^2 / \{4(a+1)\} \). \hspace{1cm} (42)
The coefficient in (42) represents the probability function \( p_r \) of (40). Hence

\[
p_r = \frac{a^c}{(a+1)^c} \cdot e^{-b^2/(4a)} \cdot \frac{\left(\begin{smallmatrix} -c \\ r \end{smallmatrix}\right) \Gamma(r+1)}{(a+1)^r + \frac{d}{r} \left(\begin{smallmatrix} -c+1 \\ r \end{smallmatrix}\right) \Gamma(r+1) + \frac{d^2}{2!} \left(\begin{smallmatrix} -c+2 \\ r \end{smallmatrix}\right) \Gamma(r+1) + \ldots}\]

\[
= a^c \cdot e^{-b^2/(4a)} \cdot (r+c-1)^{r-1} \cdot \frac{M(c+r+1; c; d)}{(a+1)^{c+r}}, \tag{43}
\]

where \( M(a;b;c) \) is the confluent hypergeometric function (see Johnson and Kotz, 1969, p.8).

6.3.3 Properties

A recurrence relation for the probability function (43) is given by

\[
p_{r+1} = \left[\frac{(r+c)}{(r+1)(a+1)}\right] \cdot \left[\frac{M(c+r+1; c; d)}{M(c+r; c; d)}\right] \cdot p_r, \tag{44}
\]

where

\[
p_0 = a^c \cdot e^{-b^2/(4a)} \cdot \frac{d}{(a+1)^c}.
\]

Another recurrence formula for the p.f. (43) is as follows. From (41), we have

\[
\log G(s) = \text{Constant} - c \log \left(1 - \frac{s}{a+1}\right) + \frac{b^2}{4(a+1-s)}.
\]

Now, differentiating both the sides of (45) with respect to \( s \), we get

\[
G'(s) = \left[\frac{c}{(a+1)} \left(1 - \frac{s}{a+1}\right)^{-1} + \frac{b^2}{4(a+1-s)^2}\right]G(s). \tag{46}
\]
Now, equating the coefficients of $s^r$ from both the sides of (46), we get

\[(r+1)p_{r+1} = \sum_{k=0}^{r} \left[ \frac{c}{k!(a+1)} + \frac{b^2}{4(a+1)^2} \right] \frac{(k+1)}{p_{r-k} (a+1)^k}. \quad (47)\]

We obtain below the factorial moments of the distribution (40).

We have

\[G(s+1) = (2/b)^c \sum_{k=0}^{\infty} \left( \frac{s^k}{k!} \right) M(k+c; c; b^2 / (4a)). \quad (48)\]

(see Erdélyi, 1954, formula (20), p.197 and p.386).

Differentiating (48) $r$ times with respect to $s$ and putting $s=0$, we get

\[\mu(r) = e^{-b^2/(4a)} \frac{c^r}{r!} M(r+c; c; b^2 / (4a)) \quad (r=0,1,2,...) \quad (49)\]

whence

\[\mu'_1 = e^{-b^2/(4a)} (c/a) M(c+1; c; b^2 / (4a)). \quad (50)\]

6.3.4 **A multivariate Generalization of the Distribution (42)**

As in Section 6.1.5 we consider a $k$-dimensional discrete probability function of the form
where \( f(\theta) \) is the continuous probability density function given by (39).

From (51), we obtain

\[
h(x_1, x_2, \ldots, x_k) = (2/b)^{c-1} a e^{-b^2/(4a)} \int_0^\infty \frac{(x_1^+ \ldots + x_k^+ + c + 1/2)^{k-1}}{(\prod_{i=1}^k x_i!)} f(\theta) d\theta,
\]

where \( h(x_1, x_2, \ldots, x_k) \) is the bivariate distribution for \( k=2 \) given by the probability function

\[
h(x_1, x_2) = \frac{a e^{-b^2/(4a)} \Gamma(x_1 + x_2 + c) \cdot M(x_1 + x_2 + c; b^2/(4(a+2)))}{x_1! x_2! \Gamma(c) (a+2)^{x_1 + x_2 + c}}
\]

(53)
6.3.5 An Application to Accident Phenomenon

As in Section 6.1.6, we obtain a simple model of accident proneness under the following assumptions:

(i) the number of accidents sustained by an individual in a fixed period (of unit length, say) is given by Poisson distribution with parameter $\theta$

(ii) $\theta$ varies from individual to individual with an underlying distribution (39).

Then the overall distribution of the observed number ($x$) of accidents per individual in a given period has a probability function (43). Now, if $y$ corresponds to the number of accidents per individual in the next period (of unit length), then under an assumption of the same distribution for $\Theta$, the conditional distribution of $Y$ given $X=0$ is

$$P(y | 0) = (a+1)^{c-b} / (4(a+1)) \cdot \binom{y+c-1}{y} M(y+c;c;\frac{b^2}{4(a+2)}) / [(a+2)^{y+c}]$$

which is the same distribution as (43) with $a$ replaced by $(a+1)$.

As such

$$E(Y | 0) = e^{-b^2/(4(a+1))} \cdot (c / (a+1)) M(c+1;c;\frac{b^2}{4(a+1)})$$

Clearly $E(Y | 0) = p_1 / p_0$.

We see that

$$E(y | 0) < E(X), \text{ if } p_1 / p_0 < E(X),$$

where $E(X)$ is given by (50). Hence, for this case, we see that a selection of individuals with no accidents in the first period will result in the mean decrease of accidents in the second period, if the inequality in (56) holds.
6.4 Compound Poisson With Respect to Compounder

the Rayleigh Distribution

6.4.1 Introduction

From the equation
\[ \int_0^\infty \frac{1}{\sqrt{2\pi}b} e^{-\frac{\theta^2}{2b^2}} \, d\theta = \frac{1}{2}, \]
we derive a probability density function
\[ f(\theta) = \frac{\theta}{\sqrt{2\pi}b} e^{-\frac{\theta^2}{2b^2}} \quad (\theta > 0). \quad (57) \]

Mean of the distribution is given by
\[ \mu' = \int_0^\infty \frac{\theta}{\sqrt{2\pi}b} e^{-\frac{\theta^2}{2b^2}} \, d\theta = \sqrt{2\pi}b. \quad (58) \]

Now, from (58), we write the probability density function
\[ h(\theta) = \frac{\theta e^{-\frac{\theta^2}{2b^2}}}{b^2} \quad (\theta > 0) \quad (59) \]
which is the Rayleigh distribution (see McGolty, 1964).

Here we study Poisson compound distribution with respect to compounder
the Rayleigh distribution:

\[ \text{Poisson}(\theta) \overset{\theta}{\sim} \text{Rayleigh}(b). \quad (60) \]

6.4.2 Derivation of the Distribution (60)

The probability generating function of the distribution (60) is given by
\[ G(s) = \int_0^\infty e^{\theta(s-1)} \cdot \frac{e^{-\frac{\theta^2}{2b^2}}}{b^2} \, d\theta \]
(178)

\[
= \frac{1}{b^2} \sum_{k=0}^{\infty} \left( \frac{s^k}{k!} \right) \int_0^\infty \theta^{k+1} e^{-\theta^2/(2b^2)} e^{-\theta} d\theta
\]

\[
= e^{-b^2/4} \sum_{k=0}^{\infty} \left( \frac{s^k}{k!} \right) \Gamma(k+2) b^k D_{-(k+2)}(b)
\]

(see Erdélyi, 1954, formula (24), p. 146) where \(D_n(x)\) is the
parabolic cylinder function (see Erdélyi, 1953, p. 267).

From (61), we get the probability function of the
compound distribution (60) as

\[
p_r = e^{-b^2/4} (r+1) b^r D_{-(r+2)}(b),
\]

(62)

where \(D_n(x)\) is the parabolic cylinder function.

Alternatively,

\[
p_r = \int_0^\infty e^{-\theta} (\theta^r/r!) h(\theta) d\theta,
\]

(63)

where \(h(\theta)\) is given by (59).

From (63), we get

\[
p_r = \frac{1}{r! b^2} \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{k!} \right\} \int_0^\infty \theta^{r+k+1} e^{-\theta^2/(2b^2)} d\theta
\]

\[
= \frac{1}{r!} \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{k!} \right\} (s2b)^{r+k} \Gamma(\frac{r+k+2}{2})
\]

\[(r=0, 1, 2, \ldots). \]

(64)

6.4.3 Probability Generating Function and Factorial
Moments of the Distribution (60)

The probability generating function of the distribution
(60) is given by

\[
G(s) = \frac{1}{b^2} \int_0^\infty \theta e^{-\theta^2/(2b^2)} e^{-\theta(1-s)} d\theta.
\]

(65)
Putting $\theta(1-s)=t$ and $c=1/\{2b^2(1-s)^2\}$ in (65), we get

$$G(s) = (2c) \int_0^\infty te^{-ct^2}dt = 2e^{1/(8c)}D_{-2}(\sqrt{2c}^{-1}), \quad (66)$$

(see Erdélyi, 1954, formula (24), p.146) where $D_n(x)$ is the parabolic cylinder function (see Erdélyi, 1953, p.267).

Alternatively,

$$G(s) = \left(\frac{1}{b^2}\right) \sum_{k=0}^\infty \{(s-1)^k/k!\} \int_0^\infty \theta^{k+1} e^{-\theta^2/(2b^2)} d\theta$$

$$= \sum_{k=0}^\infty \{(s-1)^k/k!\} (\sqrt{2b})^k \Gamma\{(k/2)+1\}. \quad (67)$$

The rth factorial moment of (60) is given by

$$\mu_r = (\sqrt{2b})^r \Gamma\{(r/2)+1\}. \quad (68)$$

The factorial moments satisfy the recurrence relation

$$\mu_{r+2} = (\sqrt{2b})^2 ((r/2)+1) \mu_r, \quad (69)$$

where $\mu_0=1$ and $\mu_1=b\sqrt{r/2}$. 
Distribution
6.5 Compound Poisson with Respect to Compounder
the Maxwell Boltzman Distribution

6.5.1 Introduction
From the normal distribution
\[ dF(\theta) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\theta^2/(2\sigma^2)} d\theta \quad (-\infty < \theta < \infty) \quad (70) \]

having the second central moment
\[ \mu_2 = \int_{-\infty}^{\infty} \theta^2 \frac{1}{\sqrt{2\pi} \sigma} e^{-\theta^2/(2\sigma^2)} d\theta, \]

we derive the probability density function
\[ h(\theta) = \sqrt{2/\pi} \frac{\theta^2 e^{-\theta^2/(2\sigma^2)}}{\sigma^3} \quad (\theta > 0). \quad (71) \]

Putting \( b=1/(2\sigma^2) \) in (71), we get
\[ h(\theta) = 4bJ(b/\pi) \theta^e e^{-b\theta^2} \quad (\theta > 0). \quad (72) \]
This is known as the Maxwell Boltzman distribution (see McNolty, 1964).

Here we study compound Poisson with respect to compounder the Maxwell Boltzman distribution. The distribution is symbolically denoted as
\[ \text{Poisson } (\theta) \wedge \text{Maxwell Boltzman } (b) \quad (73) \]

6.5.2 Derivation of the Distribution (73)
The probability generating function of the distribution (73) is given by
\[ G(s) = 4bJ(b/\pi) \int_{0}^{\infty} e^{s(\theta-1)} \theta^2 e^{-b\theta^2} d\theta \]
The probability function of (73) is given by

\[ p_r = \sqrt{2/\pi} e^{1/(8b)} (r+1)(r+2)(\sqrt{2b})^{-r} D_{-(r+3)}((\sqrt{2b})^{-1}). \]  

(75)

Alternatively,

\[ p_r = \int_0^\infty e^{-\theta r/\theta!} 4b \sqrt{b/\pi} \theta^2 e^{-b\theta^2} d\theta \]

\[ = (1/r!) 4b \sqrt{b/\pi} \int_0^\infty \sum_{k=0}^{\infty} \{(-1)^k/k!\} \theta^{r+k+2} e^{-b\theta^2} d\theta \]

\[ = (1/r!) (2/\sqrt{\pi}) \sum_{k=0}^{\infty} \{(-1)^k/k!\} \Gamma\left(\frac{r+k+3}{2}\right)/(4b)^{r+k}. \]  

(76)

6.5.3 Probability Generating Function and Factorial Moments of the Distribution (73)

The probability generating function of the distribution (73) is given by

\[ G(s) = \int_0^\infty 4b \sqrt{b/\pi} \theta^2 e^{-b\theta^2} e^{-\theta(1-s)} d\theta. \]  

(77)

Putting \( t = \theta(1-s) \) and \( c = \{b/(1-s)^2\} \) in (77), we get

\[ G(s) = \int_0^\infty \frac{4b \sqrt{b/\pi}}{(1-s)^3} t^2 e^{-ct} e^{-t} dt \]

\[ = 2 \sqrt{2/\pi} e^{1/(8c)} D_{-3}((\sqrt{2c})^{-1}). \]  

(78)
Alternatively,

\[ G(s) = \int_0^\infty e^{\theta(s-1)} \sin(b/\pi) \theta^2 e^{-b^2} d\theta \]

\[ = \left( \frac{2}{\sqrt{\pi}} \right) \sum_{k=0}^{\infty} \left\{ \frac{(s-1)^k}{k!} \right\} \Gamma \left( \frac{k+3}{2} \right) (\sqrt{b})^k. \] (79)

The rth factorial moment of (73) is given by

\[ \mu_r = \left( \frac{2}{\sqrt{\pi}} \right) \Gamma \left( \frac{r+3}{2} \right) (\sqrt{b})^r. \] (81)

The factorial moments satisfy the recurrence relation

\[ \mu_{r+2} = \left\{ b(r+3)/2 \right\} \mu_r, \] (82)

where \( \mu_0 = 1 \) and \( \mu_1 = 2(\sqrt{b\pi})^{-1}. \)
6.6 Compound Binomial With Respect to Compounder
the Generalized Beta Distribution

6.6.1 Introduction

Suppose

\[ \text{Binomial}(n, \phi) \overset{\phi}{\underset{\Phi}{\sim}} \text{Generalized Beta}(a, b, c) \quad (83) \]

represents a compound binomial distribution formed by ascribing the generalized beta distribution

\[ f(\phi|a, b, c) = \frac{e^{-c\phi} \phi^{a-1}(1-\phi)^{b-1}}{B(a, b)M(a; a+b;-c)} \]

\[ (0 \leq \phi \leq 1; a, b > 0; c > 0) \quad (84) \]

where \( B(a, b) \) and \( M(a; b; x) \) denote the beta function and the confluent hypergeometric function respectively (see Johnson and Kotz, 1969, p.8), to the parameter \( \phi \) of the binomial distribution.

In the following sections we derive the distribution (83) and study some of its properties. We also consider two particular cases, one of which is the Polya-Eggenberger distribution in particular and the other represents a discrete rectangular distribution. Further, we study a k-dimensional form of the distribution (83). An application of these results is suggested.

6.6.2 Derivation of the Distribution (83)

The probability function of the distribution (83) is given by

\[ p_r = \int_0^1 \binom{n}{r} \phi^r (1-\phi)^{n-r} \frac{e^{-c\phi} \phi^{a-1}(1-\phi)^{b-1}}{E(a, b)M(a; a+b;-c)} \, d\phi . \]
Incidentally, we derive the following relation for the confluent hypergeometric function $M(a;b;-c)$:

$$M(a;b;-c) = \sum_{x=0}^{n} \binom{n}{x} \frac{a^x}{b^n} \frac{(b-a)^{n-x}}{M(a+x;b+n;-c)} \frac{M(a+x;b+n;-c)}{M(a;a+b;-c)}.$$  \hspace{1cm} (35)

\[ (r=0,1,\ldots,n). \]

6.6.3 Particular cases

(i) If $c=0$, (35) reduces to the compound distribution

$$p_r = \binom{n}{r} \frac{a^r}{b^{n-r}} \frac{1}{(a+b)^{n}} \frac{1}{M(a+r;a+b+n;-c)}.$$ \hspace{1cm} (87)

which is Binomial$(n, \Phi)$ $\Phi$ Beta $(a,b)$.

This is a particular case of the Pólya-Eggenberger distribution (see Johnson and Kotz, 1969, pp.229-230). This distribution can be explained in terms of random drawings of coloured balls from an urn in the following way:

Initially it is supposed that there are $a$ white balls and $b$ black balls in the urn. One ball is drawn at random, and then replaced, together with another ball of the same colour. If this procedure is repeated $n$ times, and $r$ represents the total number of times a white ball is drawn, then the distribution of $r$ is given by (87).

(ii) If $c=0$, $a=b=1$, (35) becomes the discrete rectangular distribution (see Johnson and Kotz, 1969, pp.238-240)

$$p_r = \frac{1}{(n+1)}, \quad r=0,1,\ldots,n.$$ \hspace{1cm} (88)
which represents compound binomial with respect to the compound Rectangular (0, 1) distribution.

6.6.4 Properties

The probability function (85) satisfies the recurrence relation

\[ p_{r+1} = \left[ \frac{(n-r)(r+a)}{(r+1)(n-r-1+b)} \right] \cdot \left[ \frac{M(a+r+1; a+b+n; -c)}{M(a+r; a+b; -c)} \right] p_r. \] (89)

The \( r \)th factorial moment of (83) is given by

\[ \kappa_r = (n)_r \left[ \frac{a[r]}{(a+b)[r]} \right] \cdot \left[ \frac{M(b; a+b+r+1; c)}{M(b; a+b; c)} \right], \] (90)

where \((n)_r = n(n-1)...(n-r+1)\) and \(M(a,b,c) = e^{cM(b-a; b; -c)}\).

From (90), we have

\[ \kappa'_1 = n \cdot \frac{a}{(a+b)} \cdot \frac{M(b; a+b+1; c)}{M(b; a+b; c)}. \] (91)

The factorial moments satisfy the recurrence relation

\[ \kappa_{r+1} = \frac{(n-r)(a+r)}{(a+b+r)} \cdot \frac{M(b; a+b+r+1; c)}{M(b; a+b+r; c)} \cdot \kappa_r \] (92)

6.6.5 A multivariate Generalization of the Distribution (85)

We consider a \( k \)-dimensional discrete probability function of the form

\[ h(x_1, \ldots, x_k) = \prod_{i=1}^{k} \left( \frac{n}{x_i} \right) \int_{\frac{x_i}{n}}^{1} x_1^{x_i} \cdots x_k^{x_k} \cdot (1-\phi)^{kn-(x_1+\cdots+x_k)} f(\phi) d\phi, \]

\[ x_i = 0, 1, 2, \ldots \]

\[ (i=1, 2, \ldots, k) \] (93)

where \( f(\phi) \) is a continuous p.d.f given by (84).
From (93), we obtain

$$h(x_1, \ldots, x_k) = \prod_{i=1}^k \left( \binom{n}{x_i} \frac{B(x_1 + \ldots + x_k + a, kn + b - (x_1 + \ldots + x_k))}{B(a, b)} \right) \left[ \frac{M(x_1 + \ldots + x_k + a; a+b+nk;-c)}{M(a; a+b;-c)} \right]. \quad (94)$$

If $k=1$, (94) reduces to (85).

The bivariate distribution for $k=2$ is given by the probability function

$$h(x_1, x_2) = \binom{n}{x_1} \binom{n}{x_2} \frac{\left[ \frac{x_1 + x_2}{a} \right]^{2n-x_1-x_2}}{(a+b)^{2n}} \left[ \frac{M(x_1 + x_2 + a; a+b+2n;-c)}{M(a; a+b;-c)} \right]. \quad (95)$$

From (95), we obtain

$$E(X_1X_2) = \frac{n^2 a(a+1) \cdot M(b; a+b+2; c)}{(a+b)(a+b+1) \cdot M(b; a+b; c)} \quad (96)$$

and the correlation coefficient is

$$r = \left[ 1 + \frac{v}{u} \right]^{-1} \quad (97)$$

where

$$u = (a+1)M(b; a+b; c)\{(a+b+1)M(b; a+b+1; c) - (a+1)M(b; a+b+2; c)\}$$

and

$$v = (a+1)^2 n M(b; a+b; c)M(b; a+b+2; c) - a(a+b+1)n M^2(b; a+b+1; c).$$

The conditional distribution $h(x_2/x_1)$ is obtained by dividing (95) by (85), and from that we obtain

$$h(x_2/0) = \binom{n}{x_2} \frac{\left[ \frac{x_2}{a} \right]^{b+n}}{(a+b+n)^{[n]}} \cdot \frac{M(x_2+a; a+b+2n;-c)}{M(a; a+b+n;-c)} \quad (98)$$
which is same as (85) with $b$ replaced by $b+n$.

Thus we have

$$E(X_2 / 0) = n(a / (a+b+n)) \frac{M(b+n; a+b+n+1; c)}{M(b+n; a+b+n; c)}. \quad (99)$$

### 6.6.6 An Application

Application of the above results to the theory of statistical quality control technique, when the number of defective items found in routine sampling inspection varies from lot to lot (see Johnson and Kotz, 1969, p. 189), may be demonstrated in the following way:

We consider a lot by lot production system, where each lot is subjected to two sampling inspections (first by the producer, defective items, if any, being replaced by non-defective ones, and then by the consumer). Here we assume that

(i) the number of defectives found in a sample from a lot follows binomial distribution with parameters $n$ (sample size) and $p$ (proportion of defectives per lot)

(ii) $p$ varies from lot to lot with the generalized beta (84) as an underlying distribution.

Then, by what has been shown in Section 6.6.2, the overall distribution of the observed number ($X$) of defectives per sample is given by (85). Now, if $Y$ is the number of defectives found in a sample during the second inspection,
then under the assumption of the afore-said distribution (84) for p, equation (99) gives

\[ E(Y|0) = n \left( a/ (a+b+n) \right) \left( M(b+n; a+b+n+1; c) \right) / M(b+n; a+b+n; c) \]

It represents the number of defectives to be expected per second sample from lots where first samples had no defectives.

It would be of much interest to compare numerically \( E(Y|0) \) with \( E(X) \), given by (91), in many practical situations.