THE FLOW OF A NON-NEWTONIAN FLUID FILLING THE SPACE BETWEEN A ROTATING DISK AND A SATURATED POROUS MEDIUM

2.1 Introduction:

The problem of massive blowing into a petroleum reservoir, nuclear reactor, oil field operation, lubrication of high speed porous bearings and so on, where the injection or suction rates are from 10 to 100 times larger than those permitted under the classical boundary layer assumptions, was studied extensively by many investigators (see Shetland 1966). Because of the complexity of this problem for realistic flow configurations, the study of simplified flow models is helpful in illustrating some of the major physical features involved in the interaction of blowing or suction with coefficient of shear and cross-viscosity. An excellent tool for this purpose is the flow between two stationary porous disks.

Elkouh (1967) has considered the flow of a Newtonian fluid between two porous stationary disks with uniform suction or injection. He has found a series solution which is valid for small suction or injection Reynolds number. The corresponding problem for large suction has been

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investigated by Rudraiah and Chandrasekhara (1969) using the singular perturbation technique, and they have shown that the effect of large suction is to flatten the flow profiles.

The steady flow of a viscous incompressible fluid between a rotating and stationary naturally permeable disk has been discussed by Verma and Bhatt (1975) in the case when the upper disk rotates with uniform angular velocity and the lower disk is at rest, made up of a porous material having depth equal to the height between the disks. The whole flow field has been studied with the solutions of Navier-Stokes equations in the free fluid region (between two disks) and Darcy's equations in the porous region (inside the porous material). The slip condition suggested by Beavers and Joseph (1967) has been applied to the radial and transversal velocity components at the outer surface of the porous material. The method used by Lance and Roger (1962) has been utilized to obtained the solution. It has been observed that the slip in the radial direction increases with the increase of Reynolds number. Singh and Bhattacharya (1979) reconsidered the problem by assuming a particular form of pressure distribution. It has been found that due to the permeability of the stationary disk the rotational velocity of the fluid at the interface does not vanish.

Srivastava and Sharma (1992) have discussed the flow and heat transfer of an incompressible viscous fluid due to a rotating disk at a small distance from a porous medium of finite thickness when the disk and the boundaries of the porous medium are maintained at constant temperatures. They have shown that the magnitude of the velocity components increase with the increase of the permeability of the porous medium.
The steady flow of a non-Newtonian fluid between two non-porous disks one rotating and the other stationary has been studied by Srivastava (1961) and has shown that under certain conditions depending on the space between the disk and the angular velocity of the rotating disk, the stationary disk experiences a suction, but when the space between the disks is decreased sufficient, it experiences a thrust. Mital (1961) has considered the effect of uniform large suction on steady flow of a non-Newtonian liquid due to a rotating disk and a series solution for velocity components in descending powers of the suction parameter has been obtained. He has shown that beyond a certain value of the suction velocity, the radial flow of a given non-Newtonian fluid is reversed and the axial velocity at infinity is less than the suction at the disk. Rudraiah et al. (1974) have discussed the problem of a non-Newtonian fluid flow between parallel porous disks due to uniform suction at the disks for both small and large suction Reynolds numbers. In the case of small suction Reynolds number the Navier-Stokes equation have been solved by a regular perturbation technique. They have shown that the solution is valid for both suction and injection Reynolds numbers.

In this chapter we have discussed the flow due to a rotating disk at a small distance from an infinite saturated porous medium when an incompressible second-order (non-Newtonian) fluid fills the space between the disk and the porous medium. The porous medium is considered to be saturated with a viscous Newtonian fluid and flow within it is governed by the Brinkman equation (1.7). The matching conditions suggested by Williams (1978) have been applied to match the flows at the interface.
of the two regions. We have discussed the flow in the case when the Reynolds number is small. Expressions for velocity components and shearing stresses at the rotating disk have been obtained. The velocity profile for various values of the non-Newtonian parameter against the distance from the surface of the interface has been plotted and compared with those of Newtonian fluids.

2.2 Formulation of the Problem:

We consider the steady flow of a second order fluid between an impervious rotating disk and a saturated porous medium at a small distance 'd' from it and also the flow through the porous medium by choosing cylindrical polar coordinates (r, θ, z) with the origin at the interface (see figure 2.1). The disk at z = d is rotating in its own plane with a constant angular velocity Ω about an axis r = 0 and hence the flow generated by it is axi-symmetric. In this problem we consider the case when the thickness of the porous medium is infinite in extent i.e. much larger than the width of the gap between the rotating disk and the interface. Taking u, v, and w as the velocity components in the directions of r, θ and z respectively the boundary conditions are given by

\[ u = 0, \quad v = r\Omega, \quad w = 0 \quad \text{at} \quad z = d, \quad (2.1) \]
\[ u \to 0, \quad u \to 0, \quad \text{as} \quad z \to -\infty \quad (2.2) \]

In region I, (free region) the equations of motion in cylindrical polar coordinates, (in axi-symmetrical case), are
\[ \rho(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r}) = \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{zz}}{r}, \] (2.3)

\[ \rho(u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r}) = \frac{\partial \tau_{r0}}{\partial r} + \frac{\partial \tau_{z0}}{\partial z} + \frac{2}{r} \tau_{r0}, \] (2.4)

\[ \rho(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z}) = \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r} \tau_{rz}, \] (2.5)

The equation of continuity for both the region is
\[ \frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \] (2.6)

where \( \tau_{rr}, \tau_{00}, \tau_{zz}, \tau_{r0}, \tau_{0z}, \tau_{rz} \) are the physical components of stress tensor \( \tau_{ij} \).

The matching conditions at the interface of the two regions are given by (1.40) and (1.41) which in case of a second order fluid can be written as:
\[ u(o^+) = \phi u(o^-), \quad v(0^+) = \phi v(0^-), \quad w(o^+) = \phi w(o^-) \text{ at } z = 0, \] (2.7)
\[ [\tau_{zz}]_{0^+} = \lambda \phi[\tau_{zz}]_{0^-}, \quad [\tau_{z0}]_{0^+} = \lambda \phi[\tau_{z0}]_{0^-} \text{ at } z = 0. \] (2.8)

Following Von-Karman' (1921) we assume the following form of velocity components and pressure for the flow between the disk and the porous medium [in the region-I]:
\[ u = r \Omega f(\eta), \quad v = r \Omega g(\eta), \quad w = -2d \Omega f(\eta), \] (2.9)
\[ p = -\mu \Omega P_1(\eta). \]
For the flow in the porous medium in the region II, we assume
\[
\begin{align*}
\bar{u} &= r\Omega F'(\eta), \quad \bar{v} = r\Omega G(\eta), \quad \bar{w} = -2d\Omega F(\eta), \\
\bar{p} &= -\mu\Omega p'(\eta),
\end{align*}
\]
(2.10)

where \( \eta = z/d \) and a prime denotes differentiation with respect to \( \eta \). The above form of velocity components satisfy identically the equation of continuity in both the regions.

Substituting the form of velocity components and pressure from (2.9) into the equations (2.3) - (2.5), we get the following ordinary non-linear differential equations in the directions of \( r, \theta \) and \( z \) respectively for the region I:
\[
\begin{align*}
R_e [f^2 - g^2 - 2f'f''] &= f'' + 2\alpha_1 R_e [g^2 + 2f'' - f'f'], \\
2R_e [fg - fg'] &= g'' + 2\alpha_1 R_e [2fg'' - f'g' - fg'''], \\
4R_e^{f'''} &= P_1 - 2f'' + 4\alpha_1 R_e [f^{m} - 3f'''],
\end{align*}
\]
(2.11, 2.12, 2.13)

where \( R_e = \frac{\rho \Omega d^2}{\mu} \) is the Reynolds number and \( \alpha_1 = \frac{\nu_2}{d^2} \).

Similarly substituting the form of velocity components and pressure from (2.10) into the equation (1.7), we get the following differential equation in the directions of \( r, \theta, \) and \( z \) respectively for the region II:
\[
\begin{align*}
F'' - \sigma^2 F' &= 0, \\
G'' - \sigma^2 G &= 0, \\
P'_1 - 2F'' + 2\sigma^2 F &= 0,
\end{align*}
\]
(2.14, 2.15, 1.16)
where \( \sigma = \frac{d}{\sqrt{k}} \). Inside the porous medium the velocity is considered small, therefore we neglect the square of the velocity. Boundary conditions (2.1) and (2.2) in terms of \( f, g, F \) and \( G \) take the forms:

\[
\begin{align*}
 f &= f = 0, \quad g = 1 \quad \text{at } \eta = 1, \\
 F' &\to 0, \quad G \to 0 \quad \text{as } \eta \to -\infty
\end{align*}
\]

(2.17) \hspace{1cm} (2.18)

The matching conditions (2.7) and (2.8) at the interface can be written as:

\[
\begin{align*}
 f(o) &= \phi F'(o), \quad g(o) = \phi G(o), \quad f(o) = \phi F(o) \\
 f'(o) + 2\alpha_1 \Re \{3f(o)f''(o) - f(o)f''(o)\} &= \lambda \phi F''(o) \\
 g'(o) + 2\alpha_1 \Re \{3f(o)g'(o) - f(o)g''(o)\} &= \lambda \phi G'(o)
\end{align*}
\]

(2.19) \hspace{1cm} (2.20)

2.3 Solution of the Problem:

Inspection reveals that the four equations (2.11), (2.12), (2.14) and (2.15) involves four unknown functions \( f, g, F, G \). Further (2.11) and (2.14) are third order and (2.12), (2.15) are second order ordinary differential equations, hence by solving them we shall get ten unknown constants which can be determined as we have five boundary conditions in (2.17) and (2.18) and five matching conditions in (2.19) and (2.20). Thus the problem is well posed and any numerical method may be used to solve these equations. After determining \( f \) and \( F \) we get the pressure from equations (2.13) and (2.16). The flow in the porous medium at a large distance from the interface is given by the value of \( F \) as \( \eta \to -\infty \) which is induced by the relation of the disk and is not prescribed as a
The solutions of (2.14) and (2.15) satisfying the boundary conditions (2.18) are given by

\[ F'(\eta) = A e^{\eta} \]  \hspace{1cm} (2.21)
\[ F(\eta) = A e^{\eta} + C \] \hspace{1cm} (2.22)
\[ G(\eta) = B e^{\eta} \] \hspace{1cm} (2.23)

where \( A, B \) and \( C \) are constants of integration to be determined from conditions (2.19) and (2.20). In this problem we consider the distance 'd' between the rotating disk and porous interface to be small, hence Reynolds number becomes small. For small values of \( Re \), a regular perturbation scheme can be developed for equations (2.11) and (2.12) by expanding \( f \) and \( g \) in powers of \( Re \) as :

\[ f = \sum_{n=0}^{\infty} R_e^n f_n, \quad g = \sum_{n=0}^{\infty} R_e^n g_n \]  \hspace{1cm} (2.24)

Since \( f \) and \( g \) have to be matched with (2.21), (2.22) and (2.23) at the interface, the constants \( A, B \) and \( C \) have also to be expanded in powers of \( Re \) as :

\[ A = \sum_{n=0}^{\infty} R_e^n A_n, \quad B = \sum_{n=0}^{\infty} R_e^n B_n, \quad C = \sum_{n=0}^{\infty} R_e^n C_n \]  \hspace{1cm} (2.25)

This method of expansion has been adopted by Srivastava (1964) in solving similar problem for a second-order fluid and it is found
that the series are highly convergent. Substituting (2.24) and (2.25) in (2.17), (2.19) and (2.20) we get the following conditions:

\[g_0(1) = 1, \quad f_n(1) = f_n'(1) = g_{n+1}(1) = 0, \quad \text{(2.26)}\]

\[f_n'(0) = \phi A_n, \quad g_n(0) = \phi B_n, \quad f_n(0) = \frac{A_n}{\sigma} + C_n, \quad \text{(2.27)}\]
and

\[f_n''(0) = \lambda \phi \sigma A_n, \quad g_n'(0) = \phi B_n, \quad \text{(2.28)}\]
for \(n = 0, 1, 2, 3, \ldots\)

Substituting (2.24) in (2.11) and (2.12) and equating like powers of \(R_e\) on both sides of the equations, we get the following system of linear differential equation:

\[f_0''' = 0\]
\[f_0'' = 0\]
\[f_1''' = f_0'^2 - 2f_0f_0'' - 2\alpha_1 (2f_0'f_0''' - f_0f_0^{iv}) - 2\alpha_2 g_0^2 - g_0^2\]
\[g_2'' = 2 (f_0'g_1 - f_0g_1') + f_1'g_0 - f_1g_0'' + 2 (2f_0'g_1'' - f_0''g_0') - f_0g_1''' + 2 f_1'g_0'' - f_1''g_0' - f_1g_0''\] \(\text{(2.29)}\)

Solving the system of equations in (2.29) satisfying the boundary conditions (2.26) and matching conditions (2.27) and (2.28), we get the following solutions:

\[f_0(\eta) = f_0'(\eta) = 0\]
\[g_0(\eta) = (1-b) + b\eta\]
\[A_0 = C_0 = 0, \quad B_0 = (1/\phi)(1-b) \quad \text{(2.30)}\]
\[ f_1'(\eta) = 1/12[6(1-b)^2(1-\eta)^2 + 4b(1-b)(1-\eta)^3 - b^2 \{ \eta^4 + 12\alpha\eta^2 - (1 + 12\alpha)\} - 12a(1-\eta)] \]

\[ f_1(\eta) = -1/60[10(1-b)^2(\eta^3 - 3\eta + 2) + 5b(1-b)(\eta^4 - 4\eta + 3) + b^2 \{ \eta^5 + 20\alpha\eta^3 - 5(1 + 12\alpha)\eta + 4(1 + 10\alpha)\} - 30a(\eta^2 - 2\eta + 1)] \]

\[ g_1(\eta) = 0 \] (2.31)

\[ A_1 = \frac{1}{12\phi} \left[ 6 - 14b + 11b^2 + 12b^2\alpha(1-b) \right] \]

\[ C_1 = \frac{A_1}{\sigma} - \frac{1}{120} \left[ 40 - 80b + 58b^2 - 15b^3 - 20b^2\eta(3b - 4) \right] \]

\[ B_1 = 0 \]

\[ g_2(\eta) = (1-\eta) \left[ \frac{5}{12} b(1-b)^3 + \frac{1}{15} b^2(1-b)^2(9 - 5\alpha) + \frac{1}{45} b^3(1-b)(14 + 30\alpha) \right. \]

\[ + \frac{1}{630} b^4(40 - 420\alpha^2 + 357\alpha) + \frac{6\alpha}{\lambda\sigma} \left\{ \frac{1}{2} b^2(1-b)^2 \right. \]

\[ + \frac{1}{3} \left. b^3(1-b) + \frac{1}{12} b^4(1+12\alpha - ab^2) \right\} - \frac{1}{12} (1-b)^3(\eta^4 - 6\eta^2 + 5) \]

\[ - \frac{1}{15} b(1-b)^2 \{ \eta^5 - 10\eta^2 + (9 - 5\alpha)(1-\eta^3) \} \]

\[ - \frac{1}{45} b^2(1-b) \{ (\eta^6 - 15\eta^2 + 14) + 15\alpha(9 - 5\alpha)(\eta^4 - 3\eta^3 + 2) \} \]

\[ - \frac{1}{630} b^3 \{ (2\eta^7 - 42\eta^2 + 40) + 63\eta^5 + 420(\alpha\eta^3 - \eta^2 - \alpha) + 357 \} \]
\[
\frac{1}{12} a [3b^2(1+4\alpha)(1-\eta)+b \{(\eta^4-4\eta^3+6\eta^2+8\eta-11)-12\alpha(1-\eta^2)\}+4(\eta^3-3\eta^2+2)]
\]

\[A_2 = C_2 = 0\]

\[B_2 = -\frac{1}{1260\phi} [525(1-b)^4+84b(1-b)^3(9-5\alpha)+28b^2(1-b)^2(14+30\alpha)
+2b^3(1-b)(40-420\alpha^2+357\alpha)] \quad (2.32)\]

\[-105a(3b^2+12b^2\alpha-11b-12b\alpha+8)\]

\[-630 \frac{\alpha}{\lambda\sigma} \{6b^2(1-b)^2+4b^3(1-b)+b^4(1+12\alpha)-12ab^2\}\]

where \(b = \frac{\lambda\sigma}{1+\lambda\sigma}\)

and \(a = \frac{1}{12} [3b^3-8b^2+6b+12b^3\alpha]\)

### 2.4 Discussion:

The velocity in the porous medium is given by \((2.10)\). The expressions for \(F'(\eta)\) and \(G(\eta)\) given by \((2.21)\) and \((2.23)\) suggest that the depth of penetration of the flow \(\sigma\eta = \frac{z}{k}\) is inversely proportional to the square root of the permeability of the porous medium and is independent of the viscosity of the fluid, angular velocity of the rotating disk and the distance of the disk from the surface of the interface. The graphs of \(F'(\eta), G(\eta)\) and \(-F(\eta)\) against \((-\eta)\) have been plotted by taking \(\eta = 1.5, \phi = 0.5, \sigma = 2\) for \(\alpha_1 = 0, 0.1, 0.2\) in figure 2.1, the values of \(\lambda, \phi\) and \(\sigma\) are
consistant with those suggested by Williams (1978). Two values of non-
Newtonian parameter \( \alpha_1 \) are taken to study the effect of the second-order
terms in the constitutive equation and the graph for \( \alpha_1 = 0 \) is also plotted
to show the comparative effect. The graph reveals that the rotational and
radial velocity components are maximum at the interface and gradually
vanish at a large distance from it for all values of \( \alpha_1 \). The effect of \( \alpha_1 \),
i.e. non-Newtonian term is to increase the magnitude of these velocity
components. The axial velocity is present in the porous medium at a large
distance from the interface and fluid there move towards the interface to
maintain the continuity of the flow as the rotating fluid is thrown out radially
due to the centrifugal forces. The effect of non-Newtonian terms are
to increase the magnitude of this axial velocity. This fact can be used to
pump out fluid from a porous ground. The presence of mud or twigs in
the water near the banks of the rivers which make the fluid behaviour
non-Newtonian increase the flow inside the porous ground and may cause
erosion.

Shearing stresses at the rotating plate are given by:

\[
[\tau_{zr}]_{n=1} = \frac{\mu \Omega r}{d} \cdot f'(1)
\]

\[
= \frac{\mu \Omega r}{d} \cdot R_e f_1''(1)
\]

\[
= \frac{\mu \Omega r}{4d} \cdot \frac{1}{R_e} \cdot \left\{ \frac{1}{4} \left[ (b^3 - 4b^2 + 6b - 4) - b^2 \alpha_1 (2-b) \right] \right\}
\] (2.33)
\[ [\tau_{zo}]_{n=1} = \frac{\mu \Omega r}{d} g'(1) \]

\[ = \frac{\mu \Omega r}{d} \left[ b + R_c^2 \left\{ \left( \frac{12a - 270b + 246b^2 - 101b^3 + 15b^4}{180} \right) \right. \right. \]

\[ - \left. \left. \alpha \left( \frac{6b - 22b^2 + 19b^3 - 5b^4}{6} \right) \right] - 2b^3 \alpha^2 (1-b)+c \right\} \]

\( (2.34) \)

Though the expressions in (2.33) and (2.34) have been derived by taking the radius of the disk to be infinite but it is applicable to finite disk if the edge effects are negligible which is valid when the radius of the disk \( a \gg d \). These stresses can be measured experimentally and the effects of permeability of the medium and the width of the gap between the disk and the interface on these stresses can be studied. We have given in table 2.1, the values of \( f_1'(1), g_2'(1), g_2'(1) \) for \( \lambda = 1.5, \phi = 0.5, \sigma = 2 \) and \( \alpha = 0, 0.1, 0.2 \) which show that with the increase of the values of non-Newtonian parameter both the components of the shearing stresses increase. The values of \( B_0 \) and \( g_0'(1) \) are independent of \( \alpha_1 \) and their values are 0.5000 and 0.7500 respectively for the above values of the parameters. In table 2.2, we have given the values of \( f_1''(1), g_0'(1) \) and \( g_2'(1) \) for \( \lambda = 1.5, \phi = 0.5, \alpha = 0.1 \) and \( \sigma = 1, 2, 3, 4, 5 \) which show that with the increase of \( \sigma \) both components of the shearing stresses decrease.
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Table - 2.1

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<td>0.0122</td>
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Table - 2.2
FIG. 2.1
VELOCITY COMPONENTS IN POROUS MEDIUM AGAINST THE DISTANCE

FIG. 2.2