A single channel queueing system with bulk service was first considered by Bailey (1954). Many types of bulk queues have since been studied in great depth by Bailey (1954), Prabhu (1955), Jaiswal (1955), Downton (1955, 1956), Bhap (1964), etc. The policy followed for batch service usually is that the service facility can offer service to at most, say, $b$ customers simultaneously and if at any time at the initiation of service, the queue length falls short of $b$, the server takes the entire queue length for service or waits till the capacity is reached (the latter being the case of service in batches of fixed size). There is a modified service policy in which the capacity of the service channel is assumed to be a random variable and it is determined at the beginning of each service. In such a modified system, if the capacity of the service channel at any instant of service is zero, then a batch of zero units is taken in for service (the service time of a batch of zero units may be identified with the time when the service channel is not available); if the capacity of the service channel is, say, $j(>0)$ and the queue length (the number of units waiting excluding those being served) is, say, $n(>0)$, then a batch of min. $(n, j)$ units is
taken in for service.

Recently, NEUTS (1967) considered a more general policy for bulk service and obtained some very interesting results which include a limit theorem for the number of customers served over a long period of time for a M/G/1 queueing system. This general policy which may be called the general rule for bulk service, is that the server starts service only when a minimum number, say, a \( b \) of customers are present in the waiting line. If the length of the waiting line is \( a \) or more, a batch of size \( b \) or the entire queue length, whichever is less would receive service simultaneously at every service epoch.

Bulk service described by this general rule seems to be very realistic for many practical situations. The operational policy of an unscheduled ferry at a river bank, of special transport service at a station, of a single ground floor station of an elevator etc., may be reasonably approximated by the above description. There are some common situations in the field of commercial and industrial activities, which may also be described by such a model. In fact, an unscheduled loading and other operations in a factory, railway marshalling yard where commodities may arrive in truck-loads and leave in wagon-loads would fit the above description.

The object of the present study is to obtain the
transient and steady state behaviours of a single counter Poisson queue with the general rule for bulk service.

For the queueing system investigated in this chapter we make the following assumptions:

(i) Customers arrive at a service facility in accordance to a Poisson process with parameter $\lambda$ and form a queue.

(ii) The queue discipline is first come first served (FCFS).

(iii) The capacity of the service facility is a fixed number $b$.

(iv) The arriving customers are served in batches according to the following policy. If immediately after a departure, there are less than 'a' customers present in the waiting line, the server remains idle until there are 'a' customers, whereupon all the 'a' customers are taken for service simultaneously; if there are 'a' or more customers waiting, a batch of size 'b' or the entire queue length, whichever is less enters service.

(v) The distribution of the service times $v_k$, $k=1,2,\ldots$ of the successive batches is exponential with distribution function,

$$\Pr \left\{ v_k < x \right\} = 1 - e^{-\mu x} \quad (\mu > 0, k=1,2,\ldots)$$
The successive service times are independent and identical and their distributions do not depend on the number of units being serviced.

(vi) Two models A and B are considered:

In model A (2.2 - 2.5) customers arrive one by one; and in model B (2.6 - 2.10) they arrive in groups of size not more than q, the size of a group being a random variable with

\[ \Pr \{ \text{size of a group is } r \} = c_r, \quad r = 1, 2, \ldots, q, \]

such that

\[ \sum_{r=1}^{q} c_r = 1. \]

The following results are obtained:

**Model A**:

(i) the Laplace Transform (L.T.) of the state probabilities of the queue length.

(ii) the probability mass function (pf) of the queue length in steady state.

(iii) the first two moments of the distribution of the queue length.
(iv) the probability density function (pdf) of the busy period distribution.

(v) the first two moments of the distribution of the busy period.

(vi) some classical results as particular cases.

Model B:

(i) the L.T. of the probability generating function (pgf) of the waiting line when the server is busy.

(ii) the explicit expression of the pgf in steady state.

(iii) the mean queue length when the server is busy.

(iv) the L.T. of the state probabilities of the queue length under a given condition and their corresponding expressions in steady state.

(v) the results in (ii) of model A as its special case.

2.1 NOTATIONS AND DEFINITIONS

Denote

\[ P_n(t) = \text{Probability that at time } t, \text{ there are } n \]
customers waiting in the queue and the server is busy \((n > 0)\),

\[ Q_r(t) = \text{Probability that at time } t, \text{ there are } r \text{ customers waiting in the queue and the server is idle } (0 \leq r \leq a - 1), \]

\[ R_n(t) = \text{Probability that at time } t, \text{ there are } n \text{ customers waiting in the queue (irrespective of whether the server is busy or idle) } (n > 0), \]

\[ P(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n, \quad |z| \leq 1, \text{ is the generating function of } P_n(t), \]

\[ C(z) = \sum_{i=1}^{\infty} c_i z^i, \quad |z| \leq 1 \text{ is the generating function of } c_i, \]

\[ f^*(s) = \int_0^{\infty} e^{-st} f(t) \, dt \text{ is the L.T. of an arbitrary function } f(t), \]

\[ a_r = \sum_{j=1}^{q} c_j j^r \text{ is the } r\text{th moment (about origin) of the distribution of the size of the arriving groups}, \]
\[ \lambda \] is the traffic intensity of the system when arrivals occur singly,

\[ \frac{\lambda a_1}{\mu b} \] is the traffic intensity of the system when arrivals occur in groups,

\[ E_b(\nu) = \text{The average queue length when the channel is busy}, \]

\[ E(\nu) = \text{The average queue length irrespective of whether the channel is busy or idle}, \]

\( P_n, Q_r, R_n \) and \( P(z) \) are the steady state expressions corresponding to \( P_n(t), Q_r(t), R_n(t) \) and \( P(z,t) \) respectively.

**MODEL A**

**2.2 FORMULATION OF THE EQUATIONS**

Elementary probability reasoning leads to the following set of difference - differential equations:

\[
\frac{d}{dt}P_n(t) = - (\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+b}(t) ; \quad (2.2.1)
\]

\[
\text{if } n \geq 1
\]

\[
\frac{d}{dt}P_0(t) = - (\lambda + \mu)P_0(t) + \lambda Q_{a-1}(t) + \mu \sum_{k=a}^{b} P_k(t) ; \quad (2.2.2)
\]
\[
(d/dt)Q_m(t) = -\lambda Q_m(t) + \lambda Q_{m-1}(t) + \mu P_m(t), \quad (2.2.3)
\]
\[1 \leq m \leq a-1.
\]
\[
(d/dt)Q_0(t) = -\lambda Q_0(t) + \mu P_0(t)
\]
\[(2.2.4)\]

Writing \( P_{-1}(t) = 0 \) we can include (2.2.4) in (2.2.3). It is to be noted that when \( a = 1 \), (2.2.3) will not occur.

2.3 SOLUTION OF THE EQUATIONS

Let us assume that the initial condition of the system be
\[
P_0(0) = 1
\]
\[(2.3.1)\]
i.e., at time \( t \), the system is busy and none is waiting in the queue, so that
\[
P(z,0) = 1.
\]
\[(2.3.2)\]

Multiplying (2.2.1) and (2.2.2) by the appropriate powers of \( z \) and summing over \( n \), we get
\[
(d/dt)P(z,t) = -\left[ \lambda + \mu - \lambda z - \mu z^{-b} \right] P(z,t) +
\]
\[\mu \sum_{k=a}^{b-1} \frac{1-z^{-b}}{(1-z^{-b})} P_k(t) - \mu \sum_{r=0}^{a-1} z^{-b} P_r(t) + \lambda Q_{a-1}(t).
\]
\[(2.3.3)\]
Now taking the L.T. of (2.3.3) and using (2.3.2), we find
\[ P^*(z,s) = \int z^b + \mu \sum_{k=a}^{b-1} (z^b - z^k) p_k(s) - \mu \sum_{r=0}^{a-1} z^r p^*_r(s) \]
\[ + \lambda z^b Q_{a-1}^*(s) / \left[ (s + \lambda + \mu - \lambda z)z^b - \mu \right]. \]

The zeros of the denominator of (2.3.4) can be obtained from the solution of the equation
\[ (s + \lambda + \mu - \lambda z)z^b - \mu = 0. \]
\[ (2.3.5) \]

To apply Rouche's theorem to the denominator of (2.3.4), let us consider the following two functions:

\[ f(z) = (s + \lambda + \mu)z^b \]
\[ g(z) = -\mu - \lambda z^{b+1} \]

We have

(i) both \( f(z) \) and \( g(z) \) are analytic on the unit circle \( |z| = 1 \)

(ii) on \( |z| = 1, \Re(s) > 0 \)
\[ |f(z)| = |(s + \lambda + \mu)z^b| \]
\[ = |s + \lambda + \mu| \times \lambda + \mu \times |\mu + \lambda z^{b+1}| = |g(z)| \]
Thus we find that all the conditions of Rouche's theorem are satisfied. Hence the denominator of (2.3.4) has \( b \) zeros inside \(|z| = 1\) and one zero outside it. Let the outside zero be denoted by \( z_0 = z_0(s) \), a function of \( s \).

Since the degree of the numerator in (2.3.4) is one less than that of the denominator, the ratio \( P^*(z,s) \) can be expanded in terms of the exterior zero \( z_0 \) as

\[
P^*(z,s) = \frac{V(s)}{(z_0 - z)} = \sum_{r=0}^{\infty} \left[ \frac{V(s)}{z_0^{r+1}} \right] z^r, \quad (2.3.6)
\]

where \( V(s) \) is a function of \( s \) to be evaluated later.

From (2.3.6), we find

\[
P_n^*(s) = \frac{V(s)}{z_0^{n+1}}; \quad n \geq 0. \quad (2.3.7)
\]

The L.T. of (2.2.3) is

\[
(s + \lambda)Q^*_m(s) = \lambda Q^*_{m-1}(s) + \mu P^*_m(s). \quad (2.3.8)
\]

Solving (2.3.8) recursively, we get on using (2.3.7)
\[ Q_m^*(s) = \left( \frac{\lambda}{\lambda + s} \right)^m Q_0^*(s) + \frac{\mu V(s) \{ (\lambda z_0)^m - (s + \lambda)^m \}}{z_0^{m+1} (s + \lambda)^m (\lambda z_0 - \lambda - s)} \]  

(2.3.9)

In particular,

\[ Q_{a-1}^*(s) = \left( \frac{\lambda}{\lambda + s} \right)^{a-1} Q_0^*(s) + \frac{\mu V(s) \{ (\lambda z_0)^{a-1} - (s + \lambda)^{a-1} \}}{z_0^a (s + \lambda)^{a-1} (\lambda z_0 - \lambda - s)} \]  

(2.3.10)

Rewriting (2.3.4) and (2.3.6) at \( z=1 \), we find

\[ P^*(1,s) = \frac{V(s)}{z_0 - 1} = \frac{1 - \mu \sum_{r=0}^{a-1} P_r^*(s) + \lambda Q_{a-1}^*(s)}{s} \]  

(2.3.11)

Substituting from (2.3.7) and (2.3.10) into (2.3.11), we find

\[ V^*(s) = M(s) \left[ 1 + \lambda^a (s + \lambda)^{1-a} Q_0^*(s) \right], \]  

(2.3.12)

where

\[ \frac{1}{M(s)} = \left\{ \left( s + \lambda \right)^{a-1} \left\{ \lambda \mu (z_0 - 1) z_0^a + \{ \lambda (z_0^{a-1} - z_0^a) - \mu (z_0^a - 1) \} s - z_0^a s^2 \} - \mu \lambda z_0^{a-1} (z_0 - 1) \right\} / \left[ z_0^a (z_0 - 1)(s + \lambda)^{a-1} \right] \]  

(2.3.13)
Since the L.T. of the sum of all probabilities is \( \frac{1}{s} \) at any \( t \), we have
\[
\frac{V(s)}{z_0^{-1}} + Q_0^*(s) + \sum_{m=1}^{n-1} Q_m^*(s) = \frac{1}{s}. \quad (2.3.14)
\]

Substituting from (2.3.9) and (2.3.12) into (2.3.14), we get
\[
Q_0^*(s) = \frac{(s + \lambda)^{a-1} \left[ 1 - s\in(s)R(s) \right]}{\left\{ (s + \lambda)^a - \lambda^a \right\} + s \lambda^a M(s)R(s)}, \quad (2.3.15)
\]

where
\[
R(s) = \frac{1}{z_0^{-1}} + \frac{\mu(s + \lambda)^{a-1} \left\{ \lambda(z_0^{-1})z_0^{-1} - (z_0^{-1} - 1)s \right\}^{a-1}}{sz_0^a (z_0 - 1)(\lambda z_0 - s - \lambda)(s + \lambda)^{a-1}} \cdot (2.3.16)
\]

Similarly, (2.3.7) and (2.3.9), on using (2.3.12), yield
Equations (2.3.15), (2.3.17) and (2.3.18) give the L.T. of the state probabilities.

2.4. STEADY STATE SOLUTION

The steady state solution can be obtained by the well known property of L.T. viz.,

\[
\text{Lt } \frac{f^*(s)}{s} = \text{Lt } f(t) = f \text{ (say) } \quad (2.4.1)
\]

\[
s \to 0 \quad t \to \infty
\]

if the limit on the right exists.
Thus, if \( \lim_{t \to \infty} P_n(t) = P_n \) and \( \lim_{t \to \infty} Q_m(t) = Q_m \), etc., then in steady state the following expressions are obtained:

\[
\begin{align*}
\text{Lt } s V(s) &= dbQ_0 \\
\text{Lt } 1/M(s) &= \mu/d \\
\text{Lt } R(s) &= \frac{1}{d-1} + \frac{(a-1)d^a - ad^{a-1} + 1}{b \cdot d^a(d-1)^2}, \\
\end{align*}
\]

where \( d = \lim_{s \to 0} z_0 > 1 \) satisfies the equation

\[
b \cdot Q = x^{-b} \frac{(x^b-1)}{(x-1)}. \]

By virtue of (2.4.1) and (2.4.2), we obtain from (2.3.17), (2.3.18) and (2.3.15), respectively

\[
\begin{align*}
P_n &= d^{-n} \cdot b \cdot Q_0, \quad n > 0 \\
Q_m &= \left[ \frac{(d^{m+1} - 1)}{d^m(d-1)} \right] Q_0, \quad 1 \leq m \leq a-1 \\
Q_0 &= d^{a-1}(a-1)^2 / \left[ d^{a+1}(a + b \cdot Q) - d^a(a+b \cdot Q+1) + 1 \right].
\end{align*}
\]
Moments

The first and the second moments of the distribution of the number of customers are given by

\[ E(v) = \sum_{n=0}^{\infty} np_n + \sum_{r=1}^{a-1} rQ_r \]  \hspace{1cm} (2.4.4)

\[ = \frac{a(a-1)d^{a+2} + 2(b\rho - a^2 + a)d^{a+1} + (a^2 - 2b\rho - 2)d^{a+2} + 2a - 2(b\rho + 2d - 2(d-1))}{2(d-1) \left[ d^{a+1}(a+b\rho) - d^a(a+b\rho+1) + 1 \right]} \]

and

\[ E(v^2) = \sum_{n=0}^{\infty} n^2p_n \]  \hspace{1cm} (2.4.5)

\[ = \frac{6(b\rho - 2)d^{a+1} + \left\{ a(a-1)(2a-1) + 6b\rho \right\} d^a}{6(d-1) \left[ d^{a+1}(a+b\rho) - d^a(a+b\rho+1)+1 \right]} \]
Particular cases

I. The system $M/M_1;b/1$:

When $a=1$, i.e., an idle server starts service on the arrival of a customer, then

\[ P_n = d^{-n}bQ_0, \quad n \geq 0 \]
\[ Q_0 = \frac{(d-1)/(bQ_0 d + d - 1)}{1} \]
\[ E(v) = bQ_0 d/(d - 1)(bQ_0 d + d - 1) \]
\[ E(v^2) = bQ_0 d(d + 1)/[(d - 1)[d^2(1+bQ_0 ) - d(2+bQ_0 ) + 1]]. \]

The above results are in conformity with the corresponding results due to GHAfiE (1968) for $a=1$ in his paper.

II. The system $M/M_b;b/1$:

When $a=b$, i.e., the service takes place in batches of fixed size, say, $b$, then

\[ P_n = d^{-n}bQ_0, \quad n \geq 0 \]
\[ Q_m = \left[ \frac{(d^{m+1} - 1)/d^m(d - 1)}{1} \right] Q_0, \quad 1 \leq m \leq b - 1 \]
\[ Q_0 = d^{-b}b^{d-1}(1+bQ_0 ) - d^b(b+bQ_0 + 1) + 1] \]
\[ E(v) = (b-1)/2 + d^{-b}(d-1)^{-1} \]
E(v^2) = \frac{1}{6(d-1)[bd^{b+1}(1+\varphi) - d^b(b+b\varphi +1) + 1]}
\times
\left[6(bQ-2)d^{b+1} + \{b(b-1)(2b-1) + 6b\varphi\} d^b - 2b(b-1)(2b-1)d^{b-1} + b(b-1)(2b-1)d^{b-2} + 6b(b+1)d^2 - 12(b+1)(b-1)d + 6b(b-1)\right].

In the special case when the service occurs one by one (i.e. \(a = b = 1\)) the above expressions lead to the classical results of the simple queue, viz.,

\[P_n = (1 - \varphi_0) \varphi_0^{n+1}, \quad n \geq 0\]

\[Q_0 = (1 - \varphi_0)\]

\[E(v) = \varphi_0^2/(1 - \varphi_0)\]

\[E(v^2) = \varphi_0^2(1 + \varphi_0)/(1 - \varphi_0)^2,\]

where

\[\varphi_0 = \lambda/\mu.\]
2.5 BUSY PERIOD DISTRIBUTION

By a busy period we mean the time interval between the instant at which the idle server finds a batch of 'a' customers or more in the queue and the time when the number of customers in the queue drops below a for the first time thereafter just after the completion of the service of a batch. The initial busy period depends on the initial condition of the system.

Let us now consider the first busy period and assume the initial condition $P_0(0) = 1$, i.e., at time $t = 0$, the server has just taken in a batch of size $a$ leaving zero customers in the waiting line.

Now, if $Q_r(t)$, $r = 0, 1, \ldots, a-1$, are the probabilities that at time $t$, the system attains the idle state for the first time, the pdfs for the busy period ending with $r$ customers ($r = 0, 1, \ldots, a-1$) are

$$f_r(t) = \mu P_r(t) = (d/dt)Q_r(t), \quad r = 0, 1, \ldots, a-1, \quad (2.5.1)$$

where $f_r(t)$ are the solutions of the following equations:

$$(d/dt)P_n(t) = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+b}(t), \quad (2.5.2)$$

$$n \geq 1$$
\[ \frac{d}{dt} P_0(t) = - (\lambda + \mu) P_0(t) + \mu \sum_{k=a}^{b} P_k(t) \quad (2.5.3) \]

\[ \frac{d}{dt} Q_r(t) = \mu P_r(t) , \quad 0 \leq r \leq a-1 . \quad (2.5.4) \]

Clearly, \( P_n(t) , n > 0 \) are the probabilities defined for the busy period process.

Multiplying (2.5.2) and (2.5.3) by the appropriate powers of \( z \) and adding over all \( n \), we get

\[ \frac{d}{dt} P(z,t) = - [\lambda + \mu - \lambda z - \mu z^{-b}] P(z,t) + \]

\[ \mu \sum_{k=a}^{b-1} (1-z^{-k-b}) P_k(t) - \mu \sum_{r=0}^{a-1} z^{-r-b} P_r(t) . \quad (2.5.5) \]

The L.T. of (2.5.5) yields

\[ P^*(z,s) = \frac{z^b + \mu \sum_{k=a}^{b-1} (z^b - z^k) P_k^*(s) - \mu \sum_{r=0}^{a-1} z^{-r-b} P_r^*(s)}{z^b(s + \lambda + \mu - \lambda z) - \mu} . \quad (2.5.6) \]

Since the denominator of (2.5.6) is the same as that of (2.3.4), the expression \( P^*(z,s) \) can be readily expanded in
terms of the exterior zero $z_0$ as

$$P^*(z,s) = \frac{B(s)}{(z_0 - z)}. \quad (2.5.7)$$

and so

$$P_n^*(s) = \frac{B(s)}{z_0^{n+1}}, \quad n > 0 \quad (2.5.8)$$

where $B(s)$ is to be determined.

and (2.5.7)

Rewriting (2.5.6) at $z=1$ and substituting from (2.5.8), we find after simplification

$$B(s) = z_0^a (z_0 - 1) / [(s+\mu)z_0^a - \mu]. \quad (2.5.9)$$

Hence, (2.5.8), on using (2.5.9), yields

$$P_n^*(s) = z_0^{a-n} (z_0 - 1) / [(s+\mu)z_0^a - \mu], \quad (2.5.10)$$

$$n > 0.$$

By definition, the L.T. of the pdf of the busy period becomes
\[ f^*_r(s) = \frac{\mu z_0^{a-r} - 1}{(s+\mu)z_0^a - \mu} \quad , \quad 0 < r < a-1 \quad (2.5.11) \]

which can be expressed in the form

\[ f^*_r(s) = \sum_{i=0}^{\infty} \frac{\mu}{s+\mu} \left[ z_0^{-(ia+r)} - z_0^{-(ia+r+1)} \right] \quad (2.5.12) \]

To determine the Laplace inverse of \( z_0 \) let us first consider the well known Lagrange's formula given below:

The equation \( z = h + w \phi(z) \) has one and only one root that approaches \( h \) as \( w \) approaches zero and if this root is \( y_0 \), then

\[ f(y_0) = f(h) + \sum_{j=1}^{\infty} \frac{1}{j!} (w/j!) \frac{d^{j-1}}{dh^{j-1}} \left[ \phi^j(h)f'(h) \right], \]

where \( \phi(\cdot) \) and \( f(\cdot) \) are functions of \( z \).

Let us now write \((2.3.5)\) in the form

\[ z = \frac{(s+\lambda+\mu)/\lambda - (\mu/\lambda)z^{-b}}{(s+\mu + \lambda/\lambda)} \quad (2.5.13) \]
By the theorem given above, we observe that (2.5.13) has one and only root which approaches \((1 + \frac{s}{a})\) as \(\mu/a \to 0\), the modulus of which is greater than unity, since \(\text{Re}(s) > 0\). The denominator of the above equation is:

\[
\text{denominator of (2.5.13)} = \frac{1}{(1+s/a)^n}
\]

Also we know, by Rouché's theorem, that (2.5.6) has only one root \(z_0\), whose modulus is greater than one. So \(z_0\) is the same root as obtained by Lagrange's formula.

Hence, writing \(f(z) = z^{-\sigma}, h = (s+\lambda+\mu)/\lambda\), \(w = -\mu/\lambda\) and \(\Phi(z) = z^{-b}\) where \(\sigma\) is any positive integer, we get (see LUCHAK (1956) and JAISWAL (1960))

\[
z_0^{-\sigma} = \left\{ h^{-\sigma} + \sigma \sum_{j=1}^{\infty} \frac{(-1)^{j+1}/j!}{(\mu/\lambda)^j} \cdot (\mu/\lambda)^j \cdot \frac{d^{j-1}}{dh^{j-1}} \left[ \frac{1}{h-(1+\sigma+bj)} \right] \right\}
\]

\[
= \sigma \sum_{j=0}^{\infty} \frac{(\mu/\lambda)^j/j!}{(bj+\sigma)} \cdot \left[ \frac{(b+1)+\sigma-1}{(s+\lambda+\mu) \lambda} \right] \cdot \left[ \frac{1}{(bj+\sigma)!} \right]
\]

(2.5.14)

The Laplace inverse of (2.5.14) is given by (see ERDÉLYI, et al. (1954), page 144, formula (3))
\[ L^{-1}(z_0^{-\sigma}) = \left( \frac{\sigma}{t} \right) \sum_{j=0}^{\infty} \left( \frac{\mu}{\lambda} \right)^j e^{-\left( \lambda + \mu \right) t} \]

\[ \left( \lambda t \right)^j (b+1) + \sigma \left( 1/j! (b j + \sigma)! \right) \]

\[ = \left( \frac{\sigma}{t} \right) e^{-\sigma} e^{-(\lambda + \mu)t} I^b_\sigma(2\lambda pt), \quad (2.5.15) \]

where

\[ p^{b+1} = \frac{\mu}{\lambda} \]

and

\[ I^b_\sigma(x) = \sum_{n=0}^{\infty} \left( \frac{x}{n!} \right)^{\sigma+n(b+1)} x \frac{1}{n!(bn+\sigma)!} \]

\[ I^1_\sigma(x) \text{ being the modified BESSELS function of order } \sigma \text{ (FELLER Vol. II, page 57).} \]

To invert (2.5.12) we use (2.5.15) and the convolution property of L.T., viz.,

\[ L^{-1} [f_1^*(s) \cdot f_2^*(s)] = \int_0^t f_1(t-u) \cdot f_2(u) du , \]

to get
The moments of the distribution of the busy period can be obtained by differentiating \( f_r(s) \) and setting \( s \to 0 \). Thus, the first two moments of the distribution of the busy period ending with \( r \) customers are found to be

\[
\begin{align*}
E(\tau) &= \left[ -\frac{df_r(s)}{ds} \right]_{s=0} = \frac{k^{a-r-1}}{\pi L(k-1)^2} x \times \\
&= \left[ b Q (b+1)k^{a+2} + (r - 2b^2 Q - b Q + b)k^{a+1} \\
&+ (b^2 Q + b - r - 1)k^a + (a - r)k + r - a + 1 \right] \times \\
&0 \leq r \leq a - 1
\end{align*}
\]
\[ E(T^2) = \left[ \frac{d^2 f_T(s)}{ds^2} \right]_{s=0} = \frac{k^{a-r-1}}{\mu^2 \chi^3 (k-1)^3} \times \]

\[
\begin{align*}
\left[ 2k^{2a}(k-1)L^3 + \left\{ 4a(k-1)k^2 - k^a(k^a - 1) \right\} \times \\
(2M_1 + 2a(k-1)) \right] L^2 + \left\{ 2a^2(k-1)k^2 - k^a(k^a - 1) \times \\
(aM_1 + a(k-1)(a+b-1)) + (k^a-1)^2(bM_1 + (a-r-1)M_2) \\
- ak^a(k^a - 1)M_1 \right] L + ab(k-1)k^a(k^a - 1)(b \varphi + 1) \\
- b(k^a - 1)^2(b \varphi + 1)M_1 \right] \\
\end{align*}
\]

(2.5.17)

\[ 0 \leq r \leq a-1 \]

where

\[ M_i = (a-r-1)(k-1) + ik \]

\[ L = b \varphi (b+1)k - b(b \varphi + 1) \]

and \( k \) is the root of the equation

\[ b \varphi = (d^b - 1)/d^b(d-1) \]
Particular cases

In the particular case when \( a = 1 \), the busy period distribution and the moments can easily be obtained from (2.5.11), (2.5.16) and (2.5.17) respectively in the form

\[
f^*(s) = f_0(s) = \frac{\mu(s_0 - 1)}{[(s+\mu)s_0 - \mu]} \]

\[
f(t) = f_0(t) = \sum_{i=0}^{\infty} (\mu)^{i+1} \int_{0}^{t} e^{-(\mu t + \lambda u)} \frac{(t-u)^i}{i!} \, du
\]

\[
\begin{bmatrix}
i \\
p^i \\
i + 1 \\
p^{i+1} \\
i + 1 \\
p^{i+1}
\end{bmatrix} 
\left(2 \lambda pu\right) - \left[ I_i(2 \lambda pu) - \frac{i+1}{p^{i+1}} I_{i+1}(2 \lambda pu) \right]
\] 

\[
E(T) = E_0(T) = \frac{k}{\mu(k-1)}
\]

\[
E(T^2) = E_0(T^2) = \frac{(2k^2L + 2k)}{\mu^2L(k-1)^2}
\]

and therefore

\[
\text{Var}(T) = E(T^2) - \left[ E(T) \right]^2
\]

\[
= \frac{(k^2L + 2k)}{\mu^2L(k-1)^2}.
\]
Further, for $b=1$, $f^*(s)$ takes the form

$$f^*(s) = (\mu/\kappa)z_1^{-1},$$

where $z_1$ is the root of the equation $\kappa z^2 - (s + \kappa + \mu)z + \mu = 0$, and it tends to $\mu/\kappa$ as $s \to 0$.

Inverting it with the help of (2.5.15), we get

$$f(t) = (1/\sqrt{\kappa/\mu})(1/t) e^{-((\kappa + \mu)/t)} t 1 I_1 \left(2\sqrt{\kappa/\mu} t\right)$$

Also,

$$E(T) = 1/(\mu - \kappa)$$

$$E(T^2) = 2\mu/(\mu - \kappa)^3$$

$$\text{Var}(T) = (\kappa + \mu)/(\mu - \kappa)^3$$


**MODEL B**

In this model, it is assumed that arrivals occur in groups of size $r (\leq q)$ with probability $q_r$. We now discuss...
below the queueing system $M^r/M_{a,b}/1$ : $(r \leq q)$ with the
general rule for bulk service.

2.6 FORMULATION OF THE EQUATIONS

It is obvious that

$$R_n(t) = P_n(t) + Q_n(t) , \quad 0 \leq n \leq a-1$$

$$= P_n(t) , \quad n > a \quad (2.6.1)$$

From the probability consideration we see that the
system is governed by the following difference - differential
equations:

$$(d/dt)P_n(t) = - (\lambda + \mu)P_n(t) + \sum_{j=0}^{a-1} \sigma_{n+b-j}Q_j(t)$$

$$+ \sum_{r=1}^{k} e_rP_{n-r}(t) + \mu P_{n+b}(t) , \quad n > 0 \quad (2.6.2)$$

$$(d/dt)P_0(t) = - (\lambda + \mu)P_0(t) + \sum_{r=0}^{a-1} \sum_{j=a-r}^{b-r} e_jQ_r(t)$$

$$+ \mu \sum_{j=a}^{b} P_j(t) \quad (2.6.3)$$
\[
(d/dt)Q_0(t) = -\lambda Q_0(t) + \mu P_0(t) \quad (2.6.4)
\]
\[
(d/dt)Q_n(t) = -\lambda Q_n(t) + \gamma \sum_{m=1}^{k} e^{m} Q_{n-m}(t) + \mu P_n(t), \quad 1 \leq n \leq a - 1
\]

where
\[
k = n \text{ if } n \leq q
\]
\[
= q \text{ if } n > q
\]

and the terms involving $e_i$ in the above summations will not occur when $i > q$.

It is also to be noted that when $a=1$, (2.6.5) will not occur.

2.7 SOLUTION OF THE EQUATIONS

Let the initial condition of the system be $P_0(0)=1$ so that (2.3.2) holds. Now multiplying (2.6.2) and (2.6.3) by the appropriate powers of $z$ and summing over all $n$, we get.
\[ (d/dt)P(z,t) = -\left[ \lambda + \mu - \lambda c(z) - \mu z^{-b} \right] \]
\[ + \mu \sum_{j=a}^{b-1} (1-z^{-j-b})P_j(t) - \mu \sum_{r=0}^{a-1} z^{-r-b}P_r(t) \]
\[ + \lambda \sum_{r=0}^{a-1} \left[ \sum_{n=1}^{\infty} z^n c_{n+a-r} \right] Q_r(t). \]

As usual, we get from (2.6.1), (2.6.4), (2.6.5) and (2.7.1), respectively,

\[ R_n^*(s) = P_n^*(s) + Q_n^*(s), \quad 0 \leq n \leq a - 1 \]
\[ = P_n^*(s), \quad n > a \]
\[ (s+\lambda)Q_0^*(s) = \mu P_0^*(s) \]
\[ (s+\lambda)Q_n^*(s) = + \lambda Q_{n-1}^*(s) + \mu P_n^*(s) \]

and
\[ P^*(z,s) = K(s,z)/\left[ z^b \{ s + \lambda + \mu - \lambda c(z) \} - \mu \right], \]

where
\[ K(s, z) = z^b + \mu \sum_{j=a}^{b-1} (z^b - z^j) P_j^*(s) - \mu \sum_{r=0}^{a-1} z^r P_r^*(s) \]

\[ + \lambda \sum_{r=0}^{a-1} \left[ z^b \sum_{k=a-r}^{b-r-1} e_k + \left\{ C(z) \right\} \right] Q_r^*(s). \]

On using (2.7.3) and (2.7.4) in (2.7.5b), \( K(s, z) \) reduces to

\[ K(s, z) = z^b + \mu \sum_{j=a}^{b-1} (z^b - z^j) P_j^*(s) + \]

\[ + \sum_{r=0}^{a-1} \left[ \sum_{k=a-r}^{b-r-1} \lambda e_k (z^b - z^{r+k}) \right] z^r \]

\[ - \left\{ s + \lambda - \lambda C(z) \right\} Q_r^*(s). \]

To obtain the \( b \) unknown constants \( Q_r^*(s) \), \( r=0, 1, \ldots, a-1; \)

\( P_j^*(s), j=a, a+1, \ldots, b-1, \) occurring in (2.7.5a) we apply

Rouché's theorem to the denominator of (2.7.5a).
Now let,

\[
\begin{align*}
    f(z) &= z^b(s+\lambda+\mu) \\
    g(z) &= -\mu - \lambda C(z)z^b
\end{align*}
\]

Here both \(f(z)\) and \(g(z)\) are analytic inside and on the unit circle \(|z| = 1\). Also at \(|z| = 1\) and \(\text{Re}(s) > 0\), we have

\[
|f(z)| = |z^b(s+\lambda+\mu)| = |s+\lambda+\mu| > \lambda + \mu
\]

\[
> |\lambda C(z)z^b + \mu| = |g(z)|
\]

Thus all the conditions of Rouché's theorem are satisfied. So

\[
z^b\{s + \lambda + \mu - \lambda C(z)\} - \mu = 0 \quad (2.7.6)
\]

has \(b\) solutions on and \(q\) solutions outside the unit disc \(|z| = 1\) as there are \(b+q\) zeros in all. For each of these \(b\) zeros \(z_i (\equiv z_i(s), a\ function\ of\ s), i=1,2, \ldots, b\) lying on the unit disc, the numerator vanishes giving rise to \(b\) equations

\[
X(s,z_i) = 0, \quad i=1,2, \ldots, b
\]
which when solved give the b unknowns

\[ Q_r(s) \quad , \quad r = 0, 1, \ldots, a-1 \]

\[ P^*_j(s) \quad , \quad j = a, a+1, \ldots, b-1. \]

Thus \( P^*(z,s) \), the L.T. of the pgf of the queue length when the server is busy, is completely known. This when expanded in power series of \( z \), generates \( P^*_n(s) \), \( n > 0 \).

Putting these \( P^*_n(s) \) and \( Q^*_r(s) \) in (2.7.2) all \( R^*_n(s); n > 0 \), can be obtained and these when inverted lead to the determination of \( R_n(z), n > 0 \).

\[ 2.8 \text{ STEADY STATE SOLUTIONS} \]

To find \( P(z) \), the generating function of \( \{ P_n \} \), we apply to (2.7.5) the property (2.4.1). Thus

\[
P(z) = \frac{1}{z^b \{ \lambda + \mu - \lambda C(z) \} - \mu}
\]

\[
\left\{ \begin{array}{l}
\mu \sum_{j=a}^{b-1} (z^b - z^j) P_j + \lambda \sum_{r=0}^{a-1} \left[ \sum_{k=a-r}^{b-r-1} e_k \right] Q_r \\
(z^b - z^{r+k}) - \left\{ 1 - C(z) \right\} z^r Q_r
\end{array} \right\}
\]

(2.8.1)
The unknowns can be evaluated as before.

**Average Queue Length**

The average queue length $B(v)$ is given by

$$E(v) = E_B(v) + \sum_{r=1}^{a-1} rQ_r \quad (2.8.2)$$

To obtain $E_B(v)$, we rewrite (2.8.1) as

$$P(z) = B(z)/A(z)$$

which on differentiation with respect to $z$ at $z=1$ yields

$$E_B(v) = \frac{A(1)B(1) - A(1)B(1)}{[A(1)]^2}, \quad (2.8.3)$$

where

$$A(z) = \frac{\lambda + \mu - \lambda C(z) - \mu z^{-b}}{z - 1}$$

$$B(z) = \frac{\left\{ \sum_{j=a}^{b-1} (1 - z^{-j-b}) P_j + \lambda \sum_{r=0}^{a-1} \left[ \sum_{k=a-r}^{b-r-1} c_k \right] \right\} Q_r}{z - 1}$$
From the above, we get

$$A(1) = b\mu[1 - \rho_1]$$

$$B(1) = \mu \sum_{j=a}^{b-1} (b-j)P_j + \sum_{r=0}^{a-1} \left[ \sum_{k=a-r}^{b-r-1} c_k \times (b-r-k) + a_1 \right] Q_r$$

$$(1) A(1) = i \left[ \nu(a_1 - a_2) - b(b + 1)\mu \right]$$

$$(1) B(1) = i \left[ \mu \sum_{j=a}^{b-1} (b-j)(j-b-1)P_j + \sum_{r=0}^{a-1} \left\{ \sum_{k=a-r}^{b-r-1} c_k (b-r-k)(r+k-b-1) + a_2 + a_1(2r - 2b - 1) \right\} Q_r \right]$$

On making the above substitution in (2.8.3), we get

$$E_B(v) = \frac{M}{2b^2\mu^2(1-\rho_1)^2}$$

where
\[ M = b_0(1 - \phi_1) \left[ \mu \sum_{j=a}^{b-1} (b-j)(j-b-1)P_j + \right. \]

\[ \left. \sum_{r=0}^{r-1} \sum_{k=a-r}^{\lambda} c_k (b-r-k)(r+k-b-1) + a_2 + \right. \]

\[ a_1(2r - 2b - 1)Q_r \right] + \left[ \gamma(a_2 - a_1) + b(b+1)/\mu \right]^x \]

\[ \left[ \mu \sum_{j=0}^{b-1} (b-j)P_j + \gamma \sum_{r=0}^{r-1} \sum_{k=a-r}^{\lambda} c_k x \right. \]

\[ (b-r-k) + a_1 \} Q_r \right] \]

(2.8.4)

2.9 EXACT EXPRESSION OF \( P_n^*(s) \)

The analysis discussed in Section 2.7 leads to the theoretical determination of \( R_n^*(s) \). But from the practical point of view, it is necessary to investigate the explicit form of the probability \( P_n(t) \). We proceed to show below that under a certain condition, the probability \( P_n^*(s) \) and hence \( P_n^*(s) \) can be explicitly derived.
Let us now refer to (2.7.5a) and (2.7.5c). If $b \geq q+a-1$, the numerator of (2.7.5a) will be of degree $b$. Consequently, since the degree of the denominator is $b+q$, the ratio $P^*(z,s)$ can be readily expanded in terms of the $q$ external solutions $z_i = z_i(s)$, a function of $s$, $i=1,2,...,q$ as

$$P^*(z,s) = \frac{W(s)}{F(z)} \quad , \quad (2.9.1)$$

where $W(s)$ is the proportionality constant (to be evaluated). and

$$F(z) = \prod_{j=1}^{q} (z_j - z) \quad (2.9.2)$$

From the expanded form of (2.9.1), we get

$$P^*_n(s) = W(s) \phi_{q,n}(s) \quad , \quad n \geq 0 \quad , \quad (2.9.3)$$

where

$$\phi_{q,n}(s) = - \sum_{i=1}^{q} \left[ \frac{z_i^{-(n+1)}}{F'(z_i)} \right]$$

and

$$F'(z_i) = \left. (d/dt)F(z) \right|_{z=z_i}$$
Let us define a function $U_j(s)$ by means of the recurrence relation

$$U_j(s) = \left[ \frac{\gamma}{(s+\gamma)} \right] \sum_{r=1}^{j} c_r U_{j-r}(s); j \geq 1 \quad (2.9.4)$$

with

$$U_0(s) = U_0 = 1$$

From (2.7.4), by subsequent reduction and using (2.9.4), we get

$$Q^*_j(s) = U_j(s)Q_0^*(s) + \left( \frac{\mu}{s+\lambda} \right) \sum_{i=1}^{j} U_{j-i}(s)F_i^*(s), \quad 1 \leq j \leq a - 1 \quad (2.9.5)$$

Rewriting (2.7.5) and (2.9.1) at $z=1$, we find

$$P^*_1(s) = \frac{W(s)}{F(1)} \quad (2.9.6)$$
where

\[ M_r = \begin{cases} 1 & \text{for } r = a - 1 \\ \frac{a-r-1}{a-1} & \text{otherwise} \end{cases} \]

Substituting from (2.9.3) and (2.9.5) into (2.9.6) and then simplifying, we find

\[ W(s) = R(s) \left[ 1 + \lambda L(s) Q_\ast^*(s) \right] \] (2.9.7)

where

\[ \frac{1}{R(s)} = \frac{s}{F(1)} + \mu \sum_{r=0}^{a-1} \phi_{q,r}(s) - \left( \frac{\mu}{s + \lambda} \right) \sum_{i=1}^{a-1} \sum_{j=1}^{i} M_{i,j}(s) \phi_{q,j}(s) \]

and

\[ L(s) = \sum_{r=0}^{a-1} M_r U_r(s) \]

On using (2.9.7) in (2.9.3) and (2.9.5), we get

\[ P_m^*(s) = R(s) \phi_{q,m}(s) \left[ 1 + \lambda L(s) Q_\ast^*(s) \right] \] (2.9.8)
\[ Q_j^*(s) = \left[ U_j(s) + \left( \frac{\mu}{s+\lambda} \right) R(s) L(s) \right] \]

\[ \sum_{i=1}^{j} U_{j-i}(s) \phi_{q,i}(s)\right] Q_0^*(s) + \]

\[ \left( \frac{\mu}{s+\lambda} \right) R(s) \sum_{i=1}^{j} U_{j-i}(s) \phi_{q,i}(s). \]

(2.9.9)

Since the L.T. of the sum of all probabilities, at any \( t \) is \( 1/s \),

\[ \frac{W(s)}{P(1)} + \sum_{j=1}^{a-1} Q_j^*(s) + Q_0^*(s) = 1/s \]

(2.9.10)

Substituting from (2.9.8) and (2.9.9) into (2.9.10), we get

\[ Q_0^*(s) = \left[ (1 - sR(s) \left\{ \sum_{i=1}^{q} \left( (1-z_i) F(z_i) \right)^{-1} \right\} + \]

\[ \left( \frac{\mu}{s+\lambda} \right) \sum_{j=1}^{a-1} \sum_{i=1}^{j} U_{j-i}(s) \phi_{q,i}(s) \right)/s \right] \]
Equations (2.9.9) and (2.9.10) give the L.T. of all the transient state probabilities.

2.10 STEADY STATE SOLUTIONS UNDER THE CONDITION \( \beta > q + a - 1 \)

For the steady state the following expressions are considered:

\[
\phi_{q,r} = \lim_{s \to \infty} \phi_{q,r}(s) = \sum_{i=1}^{q} \left[ d_i \frac{-(r+1)}{F(d_i)} \right]
\]

\[
L = \lim_{s \to \infty} L(s) = \sum_{r=0}^{a-1} M_r U_r
\]
\[
\frac{1}{R} = \lim_{s \to 0} \frac{1}{R(s)}
\]

\[
= \mu \sum_{r=0}^{a-1} \phi_{q,r} - \mu \sum_{i=1}^{a-1} \sum_{j=1}^{i} M_{1} u_{i-j} \phi_{q,j},
\]

(2.10.1)

where \( U_j = \lim_{s \to 0} U_j(s) \) and \( d_1 = \lim_{s \to 0} z_1 (> 1) \) satisfies the equation

\[
\lambda / \mu = (d^b - 1) / d^b [0(d) - 1]
\]

Using (2.10.1) and the property given by (2.4.1), we get from (2.9.8), (2.9.9) and (2.9.11), respectively

\[
P_n = \lambda RL \phi_{q,n} q_0^{(0)}, \quad n > 0
\]

\[
Q_j = (U_j + \mu RL \sum_{i=1}^{j} U_{j-i} \phi_{q,i}) q_0^{(0)} , \quad (2.10.2)
\]

\[
Q_0 = 1 /[\lambda RL \sum_{i=1}^{a} ((1-d_1)^{(d_1)} + \sum_{j=0}^{-1} u_j +
\mu RL \sum_{j=1}^{a-1} \sum_{i=1}^{j} U_{j-i} \phi_{q,i}]}
\]
Particular cases

The system $M^q_{\text{a}, b}/1$:

Suppose $a_q = 1$, $c_r = 0$ for $r \neq q$ i.e., arrivals occur in groups of fixed size $q$. Here, $d_i$ occurring in $\phi_{q, n}$ is the root of the equation $\lambda/\mu = d^{-b}(d^b - 1)/(d^q - 1)$.

Now for

(a) $q > a$,

$U_j = 0$, $1 \leq j \leq a - 1$

$M_r = 1$, $1 \leq r \leq a - 1$

$1/R = \mu \phi_{q, 0}$

$L = 1$

and therefore

$$P_n = b \phi \left[ \phi_{q, n} \phi_{q, 0} \right] Q_0, \quad n > 0$$

$$Q_j = \left[ \phi_{q, j} \phi_{q, 0} \right] Q_0, \quad 1 \leq j \leq a - 1$$

$$Q_0 = \phi_{q, 0} \left[ b \phi \sum_{i=1}^{q} ((1-d_i) P(d_i))^{-1} + 1 + \sum_{j=1}^{a-1} \phi_{q, j} \right]$$

and

(b) \( q < a, \)

\[
U_j = \begin{cases} 
1 & \text{for } j = mq, m \text{ being an integer including zero.} \\
0, & \text{otherwise.}
\end{cases}
\]

\[
M_r = \begin{cases} 
1 & , a-q \leq r \leq a-1 \\
0 & , 0 \leq r < a-q
\end{cases}
\]

\[
1/R = \mu \sum_{r=0}^{a-1} \phi(q,r) - \mu \sum_{j=a-q}^{a-1} \sum_{j=1}^{i} U_{i-j} \phi(q,j)
\]

\[
L = \sum_{r=a-q}^{a-1} U_r
\]

Using these expressions in (2.10.2), we get the required results.

Further, if \( c_i = \delta_{1,i} \) i.e., arrivals occur singly, then

\[
U_j = 1 \quad \text{for } j > 0
\]

\[
\phi(q,r) = \phi(1,r) = d_1^{-(r+1)}
\]

\[
M_r = \begin{cases} 
1 & , r = a-1 \\
0, & \text{otherwise.}
\end{cases}
\]
\[ R = \frac{d_1}{\mu} \]
\[ L = 1 \]

and therefore

\[ P_n = b^n d_1^{-n} Q_0 \quad , \quad n \geq 0 \]
\[ Q_j = \left[ \frac{(d_1^{j+1} - 1)}{d_1^j (d_1 - 1)} \right] Q_0 \quad , \quad 1 \leq j \leq a - 1 \]
\[ Q_0 = \frac{a^{-1}}{d_1} \left( d_1 - 1 \right)^2 / \left[ (a + b^\rho) d_1 \right]^{a+1} - \frac{a}{(a + b^\rho + 1) d_1 + 1} \]

where \( d_1 (>1) \) is the root of the equation

\[ \frac{\lambda}{\mu} = -b b \quad d_1 (d - 1) / (d - 1) \]

This result agrees with that of (2.4.3) of the previous model.

### 2.11 CONCLUDING REMARKS

In this chapter we have assumed, for the sake of simplicity, that the service time distribution is independent.
of batch-size. In a subsequent chapter (chapter IV) we have considered the system wherein we have taken the service time distribution to be a function of the batch-size. In the case of group arrivals, it is also assumed that the size of each group is not more than a finite number q. This makes the analysis tractable. Further, this assumption seems to be quite realistic as groups of very large size are not likely to occur much in practice.

The means and variances of the number in the queue as well as those of the busy period distribution can easily be numerically computed. The numerical results could be used to compare the effectiveness of the model $M/M_a,b/1$ with the standard bulk service queuing models viz., $M/M_1,b/1$ and $M/M_b,b/1$. 

***