CHAPTER IV

A QUEUEING SYSTEM WITH BULK SERVICE AND
GENERAL SERVICE TIME DISTRIBUTION

4.0 INTRODUCTION

Earlier studies of bulk arrival or bulk service queueing problems have been extended by many authors to cases in which both arrival and service take place in groups of variable size. A study of such a bulk queue occurred in the work of MILLER (1959), who, using the embedded Markov chain technique discussed some important properties of the queue. The study was then followed by many authors, notable among them being KEILSON (1962), Bhat (1964), LAMBOTTE et al (1969) etc.

The objective of our present study is to analyse the model M/G/1 in which customers arriving in groups are served in batches under the general rule for bulk service.

Thus this chapter deals with the transient behaviour of such a queueing system under the following assumptions:
(i) Customers arrive in groups of variable size at arrival epochs \( t_0, t_1, \ldots \) in accordance to a Poisson process with rate \( \lambda \); \( \lambda \sum_{r=1}^{\infty} c_r S_t = 1 \) is the first order probability that a group of \( r \) units arrive in time \( S_t \).

(ii) The queue discipline is FCFS.

(iii) The capacity of the server is 'b'.

(iv) Service occurs in batches of variable size under the general policy for bulk service (as discussed in Chapter II); the minimum number of customers in a batch being \( a \).

(v) The processing time of each batch is general with pdf

\[
(a) \quad D(x) = \eta(x) \exp\left\{-\int_0^x \eta(y)dy\right\}, \text{ independent of batch size} \quad \text{Model A (Section 4.2 — 4.6)}
\]

\[
(b) \quad D(x) = \eta_j(x) \exp\left\{-\int_0^x \eta_j(y)dy\right\}, \text{ depending on the batch size } j \quad \text{Model B (Section 4.7 — 4.8)}.
\]
(vi) The interarrival times of arriving groups and the service times of batches are independent of each other.

The following results are obtained:

**Model A**

(i) the L.T. of the pgf of the distribution of the queue length when the server is busy

(ii) the expression of the pgf in (i) when the service time distribution is phase-type-hyper-exponential

(iii) the steady state solution of the system

(iv) an explicit expression of the mean queue length in steady state

(v) the pdf of the busy period distribution as a solution of a set of linear equations

(vi) some known results due to JAISWAL (1960), GUPTA and GOYAL (1966) and CHAUDHRY et al (1972) as particular cases

**Model B**

(i) the L.T. of the pgf of the distribution of the queue length when the server is busy

(ii) the busy period distribution as a solution of a set of linear equations
(iii) the results in (i) of model A and that due to NEUTS (1967) as particular cases

4.1 NOTATIONS AND DEFINITIONS

Denote

\[ P_{n}(x,t)dx = \text{Probability that at time } t, \text{ the queue length is equal to } n \text{ and a batch is being served with elapsed service time lying between } x \text{ and } x+dx, \ (n \geq 0), \]

\[ P_{n,j}(x,t)dx = \text{Probability that at time } t, \text{ the queue length is equal to } n \text{ and a batch of size } j \ (a \leq j \leq b) \text{ is being served with elapsed service time lying between } x \text{ and } x+dx, \ (n \geq 0), \]

\[ P_{n}(t) = \text{Probability that at time } t, \text{ the queue length is equal to } n \text{ and the server is busy } (n \geq 0), \]

\[ P_{n,j}(t) = \text{Probability that at time } t, \text{ the queue length is equal to } n \text{ and the server is busy with a batch of size } j \ (a \leq j \leq b, \ n \geq 0) \]

\[ Q_{r}(t) = \text{Probability that at time } t, \text{ the queue length is equal to } r \text{ and the server is idle} (0 \leq r < a-1), \]
$R_n(t) = \text{Probability that at time } t, \text{ the queue length is equal to } n \ (n \geq 0).$ 

Suppose

$$P(x,t;z) = \sum_{n=0}^{\infty} z^n P_n(x,t), \ |z| \leq 1,$$

is the generating function of $P_n(x,t)$

$$P_j(x,t;z) = \sum_{n=0}^{\infty} z^n P_{n,j}(x,t), \ |z| \leq 1,$$

is the generating function of $P_{n,j}(x,t)$

$$P(t,z) = \sum_{n=0}^{\infty} z^n P_n(t) = \int_0^{\infty} P(x,t;z)dx$$

is the generating function of $P_n(t)$

$$P_j(t;z) = \sum_{n=0}^{\infty} z^n P_{n,j}(t) = \int_0^{\infty} P_j(x,t;z)dx$$

is the generating function of $P_{n,j}(t)$

$$R(t,z) = \sum_{n=0}^{\infty} z^n R_n(t)$$

is the generating function of $R_n(t)$
\[ C(z) = \sum_{i=1}^{\infty} z^i c_i \] is the generating function of \( c_i \)

\[ f^*(s) = \int_{0}^{\infty} e^{-st} f(t) dt \] is the L.T. of an arbitrary function \( f(t) \)

\[ a_r = \sum_{i=1}^{\infty} c_i i^r \] is the rth moment of the size of an arriving group

\[ p_r = \int_{0}^{\infty} x^r D(x) \] is the rth moment about origin of the service time distribution \( D(x) \)

\[ \rho = \lambda a_1 \mu_1 / b \] is the traffic intensity of the system

\[ E_B(v) = \text{The mean queue length in steady state when the server is busy} \]

\[ E(v) = \text{The mean queue length in steady state irrespective of whether the server is busy or idle.} \]

\( P_n(x), P_n, Q_r, R_n, P(x,z), P(z) \) are the corresponding steady state expressions of \( P_n(x,t), P_n(t), Q_r(t), R_n(t), P(x,t;z) \) and \( P(t,z) \).
4.2 FORMULATION OF THE EQUATIONS.

It is obvious that

\[ R_n(t) = P_n(t) + Q_n(t) , \quad 0 \leq n \leq a - 1 \]  

(4.2.1)

\[ = P_n(t) , \quad n \geq a \]

Following KEILSON and KOCHARIAN (1960) we obtain the following difference — differential equations governing the system:

\[ (d/dt)Q_0(t) = -\lambda Q_0(t) + \int_0^\infty P_0(x,t)\eta(x)dx \]  

(4.2.2)

\[ (d/dt)Q_r(t) = -\lambda Q_r(t) + \sum_{i=1}^{r} \sigma_i Q_{r-i}(t) \]  

(4.2.3)

\[ + \int_0^\infty P_r(x,t)\eta(x)dx , \quad 1 \leq r \leq a - 1 \]

\[ (\partial/\partial t)P_0(x,t) + (\partial/\partial x)P_0(x,t) = -[\lambda + \eta(x)]P_0(x,t) \]  

(4.2.4)

\[ (\partial/\partial t)P_n(x,t) + (\partial/\partial x)P_n(x,t) = -[\lambda + \eta(x)]P_n(x,t) \]

\[ + \lambda \sum_{i=1}^{n} \sigma_i P_{n-i}(x,t) , \quad n > 0 \]  

(4.2.5)
where \( \eta(x)dx \) occurring in the above equations is the first order conditional probability that the service will be completed in the interval \((x, x+dx)\) given that it has not been completed till time \(x\): further the pdf of the overall service time distribution \(D(x)\) of a batch is given by

\[
D(x) = \eta(x)\exp\left\{- \int_0^x \eta(y)dy\right\}.
\] (4.2.6)

### 4.3 SOLUTION OF THE EQUATIONS

The equations (4.2.2) through (4.2.5) are to be solved subject to the following boundary conditions:

\[
P_0(0, t) = \sum_{r=a}^{b} \int_0^\infty P_r(x, t) \eta(x)dx
\] (4.3.1)

\[
P_n(0, t) = \sum_{r=a}^{b} \int_0^\infty P_{n+r}(x, t) \eta(x)dx
\] (4.3.2)

\[
+ \lambda \sum_{j=0}^{a-1} \sum_{i=a-j}^{b-j} e_i Q_j(t)
\]

\[
+ \lambda \sum_{j=0}^{a-1} e_{n+b-j} Q_j(t), \quad n > 0
\]
Let the initial condition be

\begin{align*}
Q_0(0) &= \delta_{p,0} \quad (4.3.3a) \\
\mathbb{P}_n(x,0) &= \delta(x) \delta_{p,n+1} \quad n \geq 0 \quad (4.3.3b)
\end{align*}

where \(\delta(x)\) is Dirac's delta function and \(\delta_{p,r}\) Kronecker delta, \(p\) being any non-negative integer.

This condition implies that

(a) when \(p=0\), time is reckoned from the instant when the server is idle and there is none waiting in the queue

and (b) when \(p>0\), time is reckoned from the instant when the server has just taken in a batch for service leaving \(p-1\) units or customers in the waiting line.

Thus, with the help of (4.3.3b), we can write

\[
P(x,0;z) = \sum_{n=0}^{\infty} z^n \mathbb{P}_n(x,0) \\
= \sum_{n=0}^{\infty} z^n \delta(x) \delta_{p,n+1} = \delta(x)K(z), \]
where

$$K(z) = \sum_{n=0}^{\infty} z^n \delta_{p,n+1}$$

$$= \begin{cases} 
0 & \text{for } p=0 \\
\frac{z}{p-1} & \text{for } p \geq 1
\end{cases}$$

Now multiplying (4.2.4) and (4.2.5) by the appropriate powers of $z$ and then summing over all $n$, we get

$$\frac{\partial}{\partial t} P(x,t;z) + \frac{\partial}{\partial x} P(x,t;z) + \left[\lambda - \lambda C(z) + \eta(x)\right] P(x,t;z) = 0$$

(4.3.5)

Similarly from (4.3.1) and (4.3.2), we find

$$z^b P(0,t;z) = \int_0^\infty P(x,t;z) \eta(x) dx - \sum_{r=0}^{\infty} z^r P_r(x,t) \eta(x) dx$$

$$+ \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r(x,t) \eta(x) dx$$

$$+ \lambda \sum_{j=0}^{a-1} \left[ z^b \sum_{i=a-j}^{b-j-1} c_i \right]$$
The equation (4.3.5) is of Lagrangian type. Its general solution is

\[ P(x, t; z) = H_0(t-x, z) \exp \left\{ -N(x) - \lambda \left[ 1 - C(z) \right] x \right\}, \quad (4.3.7) \]

where

\[ N(x) = \int_0^x \gamma(y) \, dy \] and the function \( H_0(\cdot, \cdot, z) \) is given by

\[ H_0(-y, z) = \delta(y) K(z) \quad \text{for} \quad y > 0 \quad (\text{using (4.3.4)}) \]

\[ H_0(t, z) = P(0, t; z) \quad \text{for} \quad t > 0 \]

On taking the L.T.'s of (4.2.1), (4.2.2), (4.2.3), (4.3.6), (4.3.7) and using (4.3.3*), (4.3.4), we have respectively

\[ R_n^*(s) = P_n^*(s) + Q_n^*(s), \quad 0 \leq n \leq a - 1 \quad (4.3.8) \]

\[ = P_n^*(s), \quad n > a \]
\[(s + \lambda)Q_0^*(s) = \delta_{p,0} + \int_0^\infty p_0^*(x,s)\eta(x)dx \quad (4.3.9)\]

\[(s + \lambda)Q_r^*(s) = \sum_{i=1}^{r} a_i Q_{r-i}^*(s) + \int_0^\infty p_r^*(x,s)\eta(x)dx \quad (4.3.10)\]

\[1 \leq r \leq a - 1\]

\[z^b P^*(0,s;z) = \int_0^\infty P^*(x,s;z)\eta(x)dx - \sum_{r=0}^{a-1} z^r \int_0^\infty p_r^*(x,s)\eta(x)dx\]

\[+ \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty p_r^*(x,s)\eta(x)dx\]

\[+ \sum_{j=0}^{a-1} \left[ z^b \sum_{i=a-j}^{b-j-1} a_i \right] Q_j(t) \]

\[+ \left\{ C(z) - \sum_{r=1}^{b-j-1} a_r z^r \right\} z^j \]

and

\[P^*(x,s;z) = \left[ K(z) + P^*(0,s;z) \right] \exp \left\{ -N(x) - nh(s,z) \right\} \]

\[\quad (4.3.12a)\]
where

\[ h(s, z) = s + \lambda - \lambda C(z) \quad (4.3.12b) \]

Again, by making use of (4.3.12) we can write that

\[
\int_0^\infty P_r^*(x,s;z) \eta(x) dx = [K(z) + P_0^*(0,s;z)] D^* \{ h(s, z) \} \quad (4.3.13)
\]

Now, substituting for \( \int_0^{\infty} P_r^*(x,s) \eta(x) dx \), \( r=0,1, \ldots, \)

\( a-1 \); from (4.3.9) and (4.3.10) and using (4.3.13) in (4.3.11),

we get on simplification

\[
K(z) + P_0^*(0,s;z) = \frac{A(s, z)}{z^b - D^* \{ h(s, z) \}} \quad , \quad (4.3.14a)
\]

where

\[
A(s, z) = \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r^*(x,s) \eta(x) dx + z^b K(z)
\]

\[
+ \delta_{p,0} + \sum_{j=0}^{a-1} \sum_{i=a-j}^{b-j-1} c_i (z^b - z^{j+i})
\]

\[- z^j h(s, z) Q_j^*(s) \quad (4.3.14b)\]
Therefore, on integrating (4.3.12a) from 0 to \( \infty \),
we get

\[
P^*(s,z) = \int_0^\infty P^*(x,s;z) \, dx
\]

\[
= \left[ K(z) + P^*(0,s;z) \right] \left[ \frac{1 - D^*\{h(s,z)\}}{h(s,z)} \right],
\]  (4.3.15)

where \( K(z) + P^*(0,s;z) \) is given by (4.3.14a).

It is to be noted that (4.3.14) gives a relation
between \( P^*(0,s;z) \) and the \( b \) unknowns \( Q^*_r(s), r=0,1, \ldots, a-1; \)

\[
\begin{align*}
\int_0^\infty & P^*_r(x,s) \eta(x) \, dx, \quad r=a, a+1, \ldots, b-1.
\end{align*}
\]

To obtain these \( b \) unknowns we apply Rouche's theorem to the denominator of (4.3.14):

Let

\[
f(z) = z^b
\]

and

\[
g(z) = D^*\{h(s,z)\}
\]
(a) both \( f(z) \) and \( g(z) \) are analytic inside and on the unit circle \( C \) given by \( |z| = 1 \).

(b) on \( |z| = 1 \), we have

\[
|g(z)| \leq |D^*\{h(s,z)\}|
\]

\[
= |D^*\{s + \lambda \cdot \lambda C(z)\}|
\]

\[
= \left| \int_0^\infty e^{-(s + \lambda \cdot \lambda C(z))} D(x) dx \right|
\]

\[
\leq \int_0^\infty \frac{|D(x)|}{s + \lambda \cdot \lambda C(z)} \frac{dx}{|e|}
\]

\[
< \int_0^\infty |D(x)| \; dx \quad \text{for} \quad \text{Re}(s) > 0
\]

\[
= 1
\]

\[
= |f(z)|
\]

Also \( f(z) \) has only \( b \) zeros inside \( C \).

Thus we find that all the conditions of Rouche's theorem are satisfied. Hence the denominator of (4.3.14) has \( b \) zeros \( z_i (\neq z_i(s), \text{a function of } s), i=1, 2, \ldots b \) (say) inside \( |z| = 1 \). For each of these \( b \) zeros the
numerator vanishes, giving rise to $b$ equations

$$A(s,z_i) = 0, \quad i = 1,2, \ldots, b \quad (4.3.16)$$

Solving this set of equations, one can obtain the $b$ unknowns uniquely, viz.,

$$Q^*_r(s), \quad r = 0, 1, \ldots, a-1$$

$$\int_0^\infty P^*_r(x,s)\eta(x)dx, \quad r = a, a+1, \ldots, b-1$$

Thus $K(z) + P^*(0,s;z)$ is completely determined and hence $(4.3.15)$ is completely known.

The expression in $(4.3.15)$ gives the L.T. of the pgf of the queue length when the channel is busy. This enables us to determine theoretically all $P^*_n(s), \ n \geq 0$. Utilising these values of $P^*_n(s)$ and $Q^*_n(s)$ one can obtain all $R^*_n(s), \ n \geq 0$, by $(4.3.8)$. These on inversion determine $R_n(t), \ n \geq 0$. 
4.4 AN EXAMPLE

Suppose the service system consists of \( r \) branches, having \( s \) phases in each branch. Assume that a batch taken in for service in the \( i \)th branch \( (1 \leq i \leq r) \) demands \( j \) phases \( (1 \leq j \leq k) \) with probability, say, \( e_{ij} \) and the service time distribution in each phase of the \( i \)th branch is negative exponential with mean service time \( 1/\mu_i \). The overall service time distribution which may be called phase-type hyper-exponential is one from which many service time distributions can be either derived or approximated. Further, there may be situations where this type of service time distribution may be considered realistic. Consider, for example, the service time distribution of outdoor patients in a hospital. Patients may need care in one of the several branches (general medicine, cardiology etc.) with a certain relative frequency and in a particular branch a patient may need \( j \) phases (general examination, X-ray, blood test etc.) for service.

Here the probability density function is

\[
D(x) = \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} x^{j-1} \frac{1}{\mu_i} \exp\left(-\frac{1}{\mu_i}x\right)/(j-1)!
\]

\[(4.4.1)\]
and therefore

\[ D^*\{h(s,z)\} = \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{h(s,z) + \mu_i} \right]^j \]

(4.4.2)

Substituting for \( D^*\{h(s,z)\} \) in (4.3.14), we get

\[ p^*(s,z) = \frac{1 - \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{h(s,z) + \mu_i} \right]^j}{h(s,z)} \]

\[ x_{\frac{1}{x}} \]

\[ s^b - \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{h(s,z) + \mu_i} \right]^j \]

\[ \left[ \sum_{r=a}^{b-1} \left( s^b - z^r \right) \right] \left[ \sum_{r=0}^{\infty} \frac{p^*(x,s) \gamma(x) dx}{\left( s^b - z^r \right)} \right] \]

\[ + s^b K(z) + \delta_{p,0} \]
Evaluating the unknown constants as before $P^*(s, z)$ can be completely determined whence $R_n^*(s)$ and so $R_n(t)$ can be derived.

**Particular cases**

I. When there is only one phase in each branch so that the service time distribution is hyper-exponential.

Here

$$k = 1, e_{ij} = e_{i1} = \nu_i \text{ (say)}$$

We have then

$$D^*\{h(s, z)\} = \sum_{i=1}^{r} \frac{\nu_i \mu_i}{h(s, z) + \mu_i}$$

and therefore
Also, if $a=1$, i.e., service takes place in batches of variable size up to a maximum of $b$ units, then

$$P^*(s,z) = \frac{1 - \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s,z) + \mu_i}}{h(s,z)} \times$$

$$x \frac{1}{z^b - \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s,z) + \mu_i}}$$

$$\left[ \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r(x,s) \gamma(x) dx + z^b K(z) \right.$$  

$$+ \delta_{p,0} + \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} c_i (z^b - z^{j+1}) \right\} Q_j^*(s) \right]$$

Also, if $a=1$, i.e., service takes place in batches of variable size up to a maximum of $b$ units, then

$$P^*(s,z) = \frac{1 - \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s,z) + \mu_i}}{h(s,z)} \times$$

$$h(s,z)$$
and therefore

\[ R^*(s,z) = P^*(s,z) + Q_0^*(s) \]

Further, if \( a = b = 1, i.e., \), service occurs one by one, then

\[ R^*(s,z) = \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s,z) + \mu_i} \times \]

\[ zK(z) + \delta_{p,0} - h(s,z)Q_0^*(s) \]

\[ x \]
For $p > 0$,

$$R^*(s, z) = \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s, z) + \mu_i} \times$$

$$\frac{z^p - h(s, z) Q_0^*(s)}{z - \sum_{i=1}^{r} \frac{v_i \mu_i}{h(s, z) + \mu_i}} + Q_0^*(s)$$

which agrees with the corresponding results due to GUPTA and GOYAL (1966) (except for notation).

II. When there is only one branch, so that the service time distribution is simple phase type:

Here,

$$r = 1, \; e_{ij} = e_{_1} = w_j \text{ (say)}, \; \mu_1 = \mu_1 = \mu \text{ (say)}$$

We have then

$$D^*\{h(s, z)\} = \sum_{j=1}^{k} \frac{\mu}{w_j \left[ \frac{\mu}{h(s, z) + \mu} \right]^j}$$

and therefore
\[ P^*(s,z) = \frac{1 - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{h(s,z) + \mu} \right]^j}{h(s,z)} \times \]

\[ z^b - \sum_{j=1}^{r} w_j \left[ \frac{\mu}{h(s,z) + \mu} \right]^j \times \]

\[ \sum_{r=a}^{b-1} (z^b - z^r) \left\{ P^*_t(x,s)\eta(x)dx + z^b K(s) \right\} \]

\[ + \delta_p,0 + \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} c_i (z^b - z^{j+1}) \right\} Q_j^*(s) \]

Also, if arrivals occur singly, i.e., \( c_r = \delta_{1,r} \), then

\[ P^*(s,z) = \frac{1 - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{s + \lambda + \mu - \lambda z} \right]^j}{s + \lambda - \lambda z} \times \]

\[ z^b - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{s + \lambda + \mu - \lambda z} \right]^j \]
Further, two special cases are considered: (i) case for $a=1$ and (ii) case for $k=1$.

(i) Here,

$$R^*(s,z) = \frac{1 - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{s + \lambda + \mu - \lambda z} \right]^j}{s + \lambda - \lambda z}$$

$$x \left[ \sum_{r=1}^{b-1} (z^b - z^r) \int_0^{\infty} p_r^*(x,s) \eta(x) dx + z^b k(z) \right]$$

$$+ \delta_{p,0} + (\lambda z^b - s - \lambda) Q_0^*(s) \right] + Q_0^*(s)$$
The results (except for notation) obtained by JAISWAL (1960) correspond to the case $p=0$.

(ii) Here,

\[ w_i = 1, \quad \eta(x) = \mu \quad \text{and} \quad \int_0^\infty P_r^*(x,s)\eta(x)dx = \mu P_r^*(s) \text{(say)}\]

and therefore, after some readjustment, we get

\[ P^*(s,z) = \frac{1}{(s + \lambda + \mu - \lambda z)z^b - \mu} \times \]

\[ \times \left[ \mu \sum_{r=a}^{b-1} (z^b - z^r) P_r^*(s) + z^b K(z) + \delta_{p,0} \right. \]

\[ \left. \quad - \mu \sum_{r=0}^{a-1} z^r P_r^*(s) + \lambda z^b Q_{a-1}^*(s) \right] \]

For $p=1$, this result agrees with that of (2.3.4).

III. When $a=b$, i.e., service occurs in batches of fixed size $b$;
\[ P^*(s, z) = 1 - \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{h(s, z) + \mu_i} \right] \]
\[ \times \frac{1}{h(s, z)} \]
\[ z^b K(z) + \delta_{p, 0} - \sum_{j=0}^{b-1} z^j h(s, z) Q^*_j(s) \]
\[ \times \frac{1}{z^b - \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{h(s, z) + \mu_i} \right]} \]

4.5 STEADY STATE SOLUTIONS

To obtain \( P(z) \), the pgf of the sequence \( \{ P_n \} \), we apply the property given by (2.4.1) to (4.3.14) and (4.3.15). Thus we get,

\[ P(z) = \frac{1 - D^* \{ \lambda - \lambda C(z) \}}{\lambda - \lambda C(z)} \times \frac{1}{z^b - D^* \{ \lambda - \lambda C(z) \}} \times \]
\[ \left[ \sum_{r=a}^{b-1} (z^b - z^r) \int_0^{\infty} P_r(x) \eta(x) \, dx \right] \]
\[ + \sum_{j=0}^{a-1} \lambda \left( \sum_{i=a-j}^{b-j-1} a_i (z^b - z^{j+i}) - z^j (1 - C(z)) \right) Q^*_j \]
Evaluating the unknown constants as before, $P(z)$ can be completely determined, whence all $R_n$ can be derived.

For the phase-type hyper-exponential service time distribution discussed in section 4.4, the following expression is observed:

$$D^*\{\lambda - \lambda C(z)\} = \lim_{s \to 0} D^*\{h(s,z)\}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{k} e_{ij} \left[ \frac{\mu_i}{\lambda - \lambda C(z) + \mu_i} \right]^j$$

**The Mean Queue Length:**

To obtain $E_B(v)$ we rewrite (4.5.1) as

$$P(z) = \frac{V(z) W(z)}{\lambda U(z)}$$

which, on being differentiated with respect to $z$ at $z = 1$, gives
$$E_B(v) = \frac{U(1)^{(1)}(V(1)W(1) + V(1)W(1)) - U(1)V(1)W(1)}{\lambda [U(1)]^2}$$

(4.5.2)

where

$$U(z) = \frac{1}{z-1} \cdot \left[ 1 - z^{-b} D^\ast \{\lambda - \lambda C(z)\} \right]$$

$$V(z) = \frac{1}{1 - C(z)} \cdot \left[ 1 - D^\ast \{\lambda - \lambda C(z)\} \right]$$

(4.5.3)

$$W(z) = \frac{1}{z-1} \cdot \left[ \sum_{r=a}^{b-1} \int_{0}^{\infty} p^r(x) \eta(x) dx \right]$$

$$+ \sum_{j=0}^{a-1} \sum_{i=a-j}^{b-j-1} a_i (1 - z^{i+j-b})$$

$$- z^{j-b}(1 - C(z)) Q_j$$

Note that

$$\frac{d^r}{dz^r} D^\ast \{\lambda - \lambda C(z)\} \bigg|_{z=1} = \begin{cases} \lambda \mu_1 a_1 & \text{for } r=1 \\ -\lambda \mu_1 a_1 + \lambda \mu_1 a_2 + \lambda^2 u_2 a_1^2 & \text{for } r=2 \end{cases}$$

(4.5.4)
\[
\frac{d^r}{dz^r} C(z) \bigg|_{z=1} = \begin{cases} 
1 & \text{for } r=0 \\
 a_1 & \text{for } r=1 \\
 a_2 - a_1 & \text{for } r=2 
\end{cases}
\]

Using (4.5.4) in evaluating the necessary limiting values of (4.5.3), we get

\[U(1) = b(1 - \ell)\]

\[V(1) = \lambda \mu_1\]

\[W(1) = \sum_{r=a}^{b-1} (b - r) \int_0^\infty P_r(x) \eta(x) dx\]

\[(4.5.5)\]

\[+ \lambda \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} x_i (b-i-j) + a_1 \right\} Q_j\]

\[U(1) = \frac{1}{2} \left[ \frac{b}{a_1} (a_1 - a_2) - b^2 - b + 2b \lambda \mu_1 a_1 - \lambda^2 \mu_2 a_1^2 \right]\]

\[V(1) = \frac{1}{2} \lambda^2 \mu_2 a_1\]
\[ W(1) = \frac{1}{2} \left[ \sum_{r=0}^{b-1} (b-r)(r-b-1) \int_{0}^{\infty} p_{r}(x) \eta(x) dx \right. \\
+ \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} (b-i-j)(i+j-b-1) \right\} Q_{j} \right] \\
+ \lambda \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} (b-i-j)(i+j-b-1) \right\} Q_{j} \\
+ \lambda \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} (b-i-j)(i+j-b-1) \right\} Q_{j} \\
+ a_{2} + a_{1}(2j-2b-1) \right] Q_{j} \]

Thus, on substituting (4.5.5) in (4.5.2), we get

\[ E_{b}(\nu) = \frac{1}{2b^{2}(1-\nu)^{2}} \left\{ \lambda b(1-\nu)a_{1} - \mu_{1}\left\{ \frac{b}{a_{1}}(a_{1}-a_{2})\right\} \right. \\
- b^{2} - b + 2\lambda b\mu_{1}a_{1} - \lambda^{2}\mu_{2}a^{2} \left[ \sum_{r=0}^{b-1} (b-r) \int_{0}^{\infty} p_{r}(x) \eta(x) dx \right. \\
+ \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} a_{1}(b-i-j) + a_{1} \right\} Q_{j} \left. \right\} \]

\[ + b(1-\nu)\mu_{1} \left[ \sum_{r=0}^{b-1} (b-r)(r-b-1) \int_{0}^{\infty} p_{r}(x) \eta(x) dx \right. \\
+ \sum_{j=0}^{a-1} \left\{ \sum_{i=a-j}^{b-j-1} (b-i-j) + a_{1} \right\} Q_{j} \left. \right\} \]
\[
\sum_{j=0}^{a-1} \sum_{i=a-j}^{b-j-1} c_i (b-i-j)(i+j-b-1) + a_2 + a_1 (2j - 2b - 1) Q_j \]

(4.5.6)

**Particular cases:**

As before, by varying \( a_i, a, r, k \) and \( e_{ij} \), many special cases can be studied. We have considered here some of them and have obtained the expressions for \( P(z) \) only.

I. When there is only one branch, we get from (4.5.1)

\[
P(z) = \frac{1 - \sum_{j=1}^{k} \frac{\mu}{\lambda - \lambda C(z) + \mu}^j w_j}{\lambda - \lambda C(z)}
\]

\[
x z^b - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{\lambda - \lambda C(z) + \mu} \right]^j
\]

\[
\sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r(x)\eta(x)dx +
\]
Also (i) if $a = 1$, then

$$
P(z) = \frac{1 - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{\lambda + \mu - \lambda C(z)} \right]^j}{\gamma - \lambda C(z)} x.\]

$$

also (i) if $a = 1$, then

\[
\sum_{r=1}^{b-1} (z^b - z^r) \int_0^\infty P_r(x) \eta(x) dx + \lambda c_r Q_0 \]

\[
- \lambda \{1 - C(z)\} Q_0 \]

which corresponds to the result due to CHAUDHRY and TEMPLETON (1972) (except for notation).
Further, if $a_i = \delta_{1,i}$, then

$$P(z) = \frac{1 - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{\gamma + \mu - \gamma z} \right]^j}{\gamma(1 - z)}$$

$$= \sum_{r=0}^{b-1} (z^b - z^r) \int_0^\infty P_r(x) \eta(x) dx$$

$$= z^b - \sum_{j=1}^{k} w_j \left[ \frac{\mu}{\gamma + \mu - \gamma z} \right]^j$$

a result due to JAISWAL (1960) (except for notation).

It may be noted here that in the derivation of $P(z)$, we have used the equation $\gamma Q_0 = \int_0^\infty P_0(x) \eta(x) dx$

being the equation obtained from (4.2.2) in steady state.

(ii) if $w_k = 1$, $w_j = 0$, $j \neq k$, then

$$P(z) = \frac{1 - \left[ \frac{\mu}{\gamma + \mu - \gamma C(z)} \right]^k}{\gamma - \gamma C(z)}$$
which gives the pgf of the distribution of the queue length when the service time distribution is Erlang-K.

(iii) if $k \to \infty$ and $\mu \to \infty$ so that their ratio is equal to unity, then following BAILEY (1954) we get the result for the constant service time. Clearly

$$\rho = \frac{\gamma a_1 \mu_1}{b} \to \frac{\gamma a_1 k}{b \mu} \to \frac{\gamma a_1}{b}$$

and therefore we get from (ii) above
P(z) = \frac{\frac{b}{a_1} [1 - C(z)]}{e [1 - C(z)]} \cdot \frac{1}{z^b e} - 1

\times \left[ \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r(x) \eta(x) dx \right]

+ \sum_{j=0}^{a-1} \sum_{i=a-j}^{b-j-1} \sum_{i=a-j}^{b-j+1} \left[ \sum_{i=a-j}^{b-j-1} c_i (z^b - z^{j+i}) - z^j (1 - C(z)) \right] Q_j

II. When there is only one phase in each branch, we get

P(z) = \frac{1 - \sum_{i=1}^{r} v_i \left[ \frac{\mu_i}{\lambda + \mu_i - \lambda C(z)} \right]}{\lambda - \lambda C(z)} \times \frac{1}{z^b \sum_{i=1}^{r} v_i \left[ \frac{\mu_i}{\lambda + \mu_i - \lambda C(z)} \right]}
Further, if $v_i = \frac{1}{r}$, i.e., there is equal chance of selecting any one of the $r$ branches, then for $\mu_i = \mu$, $i = 1, 2, ..., r$, we get

$$P(z) = \frac{1}{z^b \{ \wedge + \mu - \wedge C(z) \} - \mu}$$
Referring to chapter II for the definition of the busy period, we write that the pdfs of the busy period ending with \( r \) \((r \leq a - 1)\) customers are

\[
 f_r(t) = \frac{d}{dt} Q_r(t) , \quad r=0,1,\ldots,a-1 \quad (4.6.1)
\]

where \( f_r(t) \) are obtained as the solution of the following equations:

\[
 \left( \frac{d}{dt} \right) Q_r(t) = \int_0^\infty P_r(x,t) \eta(x) dx , \quad (4.6.2)
\]

\[
 0 \leq r \leq a - 1
\]

\[
 \left( \frac{\partial}{\partial t} \right) P_0(x,t) + \left( \frac{\partial}{\partial x} \right) P_0(x,t) = -[\lambda + \eta(x)] P_0(x,t) \quad (4.6.3)
\]

\[
 \left( \frac{\partial}{\partial t} \right) P_n(x,t) + \left( \frac{\partial}{\partial x} \right) P_n(x,t) = -[\lambda + \eta(x)] P_n(x,t) \quad (4.6.4)
\]

\[
 + \lambda \sum_{i=1}^{n} e_i P_{n-i}(x,t) ,
\]

\( n > 0 \)
Here the $P_n(t), n \geq 0$ are the probabilities defined for the busy period process.

To solve the above equations let us consider the initial condition

$$P_n(x,0) = \delta(x) \delta_{q,n}$$  \hspace{1cm} (4.6.7)

which implies that the time is reckoned from the instant when the server has just taken in a batch of size $k(a \leq k \leq b)$ leaving $q$ units in the waiting line ($k < b$ necessarily implies $q=0$ and $k=b$ implies $q \geq 0$).

By virtue of (4.6.7) we can write that at $t=0$,

$$P(x,0;z) = \delta(x)z^q$$  \hspace{1cm} (4.6.8)
From (4.6.3) and (4.6.4), we find

\[
\frac{\partial}{\partial t} P(x,t;z) + \frac{\partial}{\partial x} P(x,t;z) + \left[ \lambda - \lambda G(z) + \eta(x) \right] P(x,t;z) = 0
\]

(4.6.9)

whose general solution is

\[
P(x,t;z) = H(t-x,z) \exp \left\{ -N(x) - \lambda [1 - G(z)] x \right\},
\]

(4.6.10)

where

\[
H(-y,z) = \delta(y) z^q \quad \text{for } y > 0
\]

\[
H(\cdot, z) = P(0,t;z) \quad \text{for } t > 0
\]

and

\[
N(x) = \int_0^x \eta(y) dy
\]

Also, from (4.6.5) and (4.6.6), we get

\[
z^b P(0,t;z) = \int_0^\infty P(x,t;z) \eta(x) dx + \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P_r(x,t) \eta(x) dx
\]
\[ - \sum_{r=0}^{a-1} z^r \int_0^\infty P_r(x,t) \eta(x) dx \quad (4.6.11) \]

Now, taking the L.T. of (4.6.2), (4.6.10) and (4.6.11), we have (on using (4.6.8)) respectively

\[ f^*_r(s) = sQ^*_r(s) = \int_0^\infty P^*_r(x,s) \eta(x) dx , \quad (4.6.12) \]

\[ 0 \leq r \leq a - 1 \]

\[ P^*(x,s;z) = [z^a + P^*(0,s;z)] \exp\{-N(x) - xh(s,z)\} , \quad (4.6.13) \]

\[ z^b P^*(0,s;z) = \int_0^\infty P^*(x,s;z) \eta(x) dx \]

\[ + \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P^*_r(x,s) \eta(x) dx \quad (4.6.14) \]

\[ - \sum_{r=0}^{a-1} z^r \int_0^\infty P^*_r(x,s) \eta(x) dx , \]
where

\[ h(s,z) = s + \gamma - \gamma C(z). \]

Substituting \( P^*(x,s;z) \) from (4.6.13) into (4.6.14), we find on simplification

\begin{equation}
\begin{aligned}
\sqrt{z} + P^*(0,s;z) &= \frac{G(s,z)}{z^b - D\{h(s,z)^2\}}, \quad (4.6.15a)
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
G(s,z) &= \sum_{r=a}^{b-1} (z^b - z^r) \int_0^\infty P^*_r(x,s) \eta(x) dx \\
&\quad - \sum_{r=0}^{a-1} z^r \int_0^\infty P^*_r(x,s) \eta(x) dx + z^{b+q} \quad (4.6.15b)
\end{aligned}
\end{equation}

Now, from (4.6.13) and (4.6.15), we get

\[ P^*(s,z) = \int_0^\infty P^*(x,s;z) dx \]
\[
\frac{1 - D\{h(s,z)\}}{h(s,z)} \cdot \frac{G(s,z)}{z^b - D\{h(s,z)\}} \quad (4.6.16)
\]

Since the denominator of (4.6.15a) is the same as (4.3.14a), application of Rouche's theorem to it gives that there are \(b\) zeros, viz., \(z_i = z_i(s)\), a function of \(s\), \(i = 1, 2, \ldots, b\) lying inside the unit circle \(|z| = 1\). For each of these zeros the numerator of (4.6.15a) vanishes giving rise to \(b\) equations:

\[
z_i^{q+b} = \sum_{r=a}^{b-1} (z_i^r - z_i^b) \int_0^\infty p_r(x,s)\eta(x)dx
\]

\[
+ \sum_{r=0}^{a-1} z_i^r \int_0^\infty p_r(x,s)\eta(x)dx, \quad i = 1, 2, \ldots, b
\]

(4.6.17)

Solving the above set of equations, the unknowns \(\int_0^\infty p_r(x,s)\eta(x)dx, \quad r = 0, 1, \ldots, b-1\) can be obtained.

Thus \(P^*(s,z)\) is completely determined.

The L.T. of the pdfs of the busy period viz., \(f_r^*(s)\) are the constant \(\int_0^\infty p_r(x,s)\eta(x)dx, (0 \leq r \leq a - 1)\)
obtained as the solutions of (4.6.17) and (4.6.16) determines the L.T. of the pgf of the busy period process.

Particular cases:

I. Suppose the service time distribution of a batch is modified Erlang with the pdf

\[ D(x) = \sum_{j=1}^{k} l_j \mu (\mu x)^{j-1} e^{-\mu x} \frac{1}{(j-1)!}, \quad x > 0 \]

where \( l_j \) (\( j = 1, 2, \ldots, k \)) being the probability that a batch of size \( m(a \leq m \leq b) \) is taken in the \( j \)th phase \( (j = 1, 2, \ldots, k) \) each phase having exponential distribution with mean \( 1/\mu \).

We have then

\[ D^*\{h(s,z)\} = \sum_{j=1}^{k} l_j \left[ \frac{\mu}{h(s,z) + \mu} \right]^j \]

Making use of the above expression in (4.6.16) and (4.5.17), we get respectively,
\[
P^*(s, z) = \frac{1 - \sum_{j=1}^{k} \frac{1}{h(s, z) + \mu} \left[ \frac{\mu}{h(s, z) + \mu} \right]^j}{h(s, z)}
\]

\[
x \frac{1}{z^b - \sum_{j=1}^{k} \frac{1}{h(s, z) + \mu} \left[ \frac{\mu}{h(s, z) + \mu} \right]^j}
\]

\[
x \left[ \sum_{r=a}^{b-1} (z^b - z^r) \int_{0}^{\infty} p^*_r(x, s) \eta(x) dx \right.
\]

\[
- \sum_{r=0}^{a-1} z^r \int_{0}^{\infty} p^*_r(x, s) \eta(x) dx + z^{q+b} \right]
\]

and

\[
d_{i}^{q+b} = \sum_{r=a}^{b-1} (d_{i}^r - d_{i}^b) \int_{0}^{\infty} p^*_r(x, s) \eta(x) dx
\]

\[
+ \sum_{r=0}^{a-1} d_{i}^r \int_{0}^{\infty} p^*_r(x, s) \eta(x) dx
\]

\[i = 1, 2, \ldots, b\]
where \( d_i = d_i(s) \), a function of \( s \), \( i = 1, 2, \ldots, b \) are the roots with modulus less than unity of the equation

\[
z^b - \sum_{j=1}^{k} l_j \left[ \frac{\mu}{h(s,z) + \mu} \right]^j = 0.
\]

Also, if \( a = 1 \),

\[
\begin{align*}
P^*(s,z) &= \frac{1 - \sum_{j=1}^{k} l_j \left[ \frac{\mu}{h(s,z) + \mu} \right]^j}{h(s,z)} \\
&\quad \times \left\{ \sum_{r=1}^{b-1} s^b - z^r \right\} \int_{0}^{\infty} p^*_r(x,s) \eta(x) dx \\
&\quad - \int_{0}^{\infty} p^*_0(x,s) \eta(x) dx + z^{q+b} \right\}
\end{align*}
\]
\[ d_i^{q+b} = \sum_{r=1}^{b-1} (d_i^r - d_i^b) \int_0^\infty p_T^*(x,s) \eta(x) \, dx \]

Further, if \( q=0 \), i.e., \( \partial_{\mu} \delta(x) \delta_0, \eta \), then

\[ p^*(s,z) = \frac{1 - \sum_{j=1}^{k} \left[ \frac{\mu}{h(s,z) + \mu} \right]^j}{h(s,z)} \]

\[ \times \frac{1}{z^b - \sum_{j=1}^{k} \left[ \frac{\mu}{h(s,z) + \mu} \right]^j} \]

\[ \times \left[ \sum_{r=1}^{b-1} (z^b - z^r) \int_0^\infty p_T^*(x,s) \eta(x) \, dx \right. \]

\[ - \left. \int_0^\infty p_0^*(x,s) \eta(x) \, dx + z^b \right] \]
Following the procedure adopted by JAISWAL (1962) we see that the L.T. of the pdf of the busy period (the solution of the second equation given above) can be written as

\[
\varphi_0(s) = \int_0^\infty P_0(x,s)\eta(x)dx = \frac{1}{1 - \sum_{i=1}^b \frac{d_i-1}{d_i}}.
\]

This is in agreement with the result due to CHAUDHRY and TEMPLETON (1972) (except for notation).

II. Suppose that the customers arrive singly, i.e.,

\[c_i = \delta_{i,i}, \quad C(z) = z,\]

then the initial condition will be that a batch of a customers is taken for service at \(t=0\) leaving \(q=0\) units in the queue.

We have then
\[ P^*(s, z) = \frac{1 - D^*(s + \lambda - \lambda z)}{s + \lambda - \lambda z} \times \frac{1}{z^b - D^*(s + \lambda - \lambda z)} \times \]
\[ \sum_{r=0}^{a-1} z^r \left[ \int_0^{\infty} P^*_r(x, s) \eta(x) dx + z^b \right] \]
\[ b \theta_i = \sum_{r=a}^{b-1} (\theta_i^r - \theta_i^b) \int_0^{\infty} P^*_r(x, s) \eta(x) dx \]
\[ + \sum_{r=0}^{a-1} \theta_i^r \int_0^{\infty} P^*_r(x, s) \eta(x) dx , \]

where \( \theta_i = \theta_i(s) \), a function of \( s \), \( i=1, 2, \ldots, b \) are the roots with modulus less than unity of the equation
\[ z^b - D^*(s + \lambda - \lambda z) = 0. \]

Also, for \( D(x) = \mu e^{-\mu x} \), i.e., the service time distribution is exponential, we get
\[ P^*(s, z) = \frac{1}{z^b(s + \lambda + \mu - \gamma z) - \mu} \]

\[
= \mu \left[ \sum_{r=a}^{b-1} (z^b - z^r) P_r^*(s) \right] - \mu \left[ \sum_{r=0}^{a-1} z^r P_r^*(s) + z^b \right]
\]

\[ \Theta_i = \mu \sum_{r=a}^{b-1} (\Theta_i - \Theta_i) P_r^*(s) \]

\[ + \sum_{r=0}^{a-1} \mu \Theta_i P_r^*(s) \]

where

\[ \mu P_r^*(s) = \int_0^\infty P_r(x, s) \eta(x) dx \]

The above expression of \( P^*(s, z) \) agrees with that of (2.5.6)
In this model, it is assumed that the service time distribution of a batch depends on its size. Thus, the service time distribution of the batch is assumed to be general with pdf $D_j(x)$ ($j=a, a+1, ..., b$), where $j$ is the batch size. We now present below the study of the queuing system $M(x)/G_{a,b}/1$ having general rule for bulk service and some particular cases.

4.7 EQUATIONS AND THEIR SOLUTION

The equations are given by

$$P_n(t) = \sum_{j=a}^{b} P_{j,n}(t) + Q_n(t), \quad 0 \leq n \leq a - 1$$

$$= \sum_{j=a}^{b} P_{j,n}(t), \quad n \geq a$$

$$(d/dt) Q_0(t) = -Q_0(t) + \sum_{j=a}^{b} \int_{0}^{\infty} P_{j,0}(x,t) \eta(x) dx$$
\begin{align*}
\frac{d}{dt} Q_r(t) &= -\lambda Q_r(t) + \lambda \sum_{i=1}^{r} c_i Q_{r-i}(t) \\
&\quad + \sum_{j=a}^{b} \int_{a}^{\infty} P_{j,r}(x,t) \eta_j(x) dx , \\
&\quad 1 \leq r \leq a - 1
\end{align*}

\begin{align*}
\frac{\partial}{\partial t} P_{j,0}(x,t) + \frac{\partial}{\partial x} P_{j,0}(x,t) &= -\left[ \lambda + \eta_j(x) \right] P_{j,0}(x,t) , \\
&\quad a \leq j \leq b
\end{align*}

\begin{align*}
\frac{\partial}{\partial t} P_{j,n}(x,t) + \frac{\partial}{\partial x} P_{j,n}(x,t) &= -\left[ \lambda + \eta_j(x) \right] P_{j,n}(x,t) \\
&\quad + \lambda \sum_{i=1}^{n} c_i P_{j,n-i}(x,t) , \\
&\quad a \leq j \leq b , n \geq 1
\end{align*}

\begin{align*}
P_{j,0}(0,t) &= \sum_{j=a}^{b} \int_{a}^{\infty} P_{r,j}(x,t) \eta_r(x) dx \\
&\quad + \lambda \sum_{r=0}^{a-1} c_{j-r} Q_r(t) , \quad a \leq j \leq b
\end{align*}
where \( \gamma_j(x)dx \) occurring in the above equations is the first order conditional probability that the service of a batch of size \( j \) (\( a \leq j \leq b \)) will be completed in the interval \((x, x+dx)\) given that it has not been completed till time \( x \): further the pdf of the overall service time distribution \( D_j(x) \) of a batch of size \( j \) is given by

\[
D_j(x) = \gamma_j(x) \exp \left\{ - \int_0^x \gamma_j(y)dy \right\}
\]

Let the initial condition be

\[
Q_r(0) = \delta_{p,r}, \quad 0 \leq r \leq a - 1
\]

\[
P_j, n(x,0) = \delta(x) \delta_{p, n+a}, \quad n \geq 0
\]

(4.7.1)
which implies that

(i) when \( p = k \) (\( k \leq a - 1 \)), time is reckoned from the instant when the server is idle and there are \( k \) units waiting in the queue,

(ii) when \( p > a \), time is reckoned from the instant when the server has taken in a batch of size \( j(a \leq j \leq b) \) leaving \( p-a \) units in the queue.

Proceeding as in model A, we have

\[
K_j(z) = 0 \quad \text{for} \quad p \leq a - 1
\]

\[
= z^{p-a} \quad \text{for} \quad p \geq a
\]

and it holds for only one value of \( j \).

\[
P_j,0(0,s) = P_j^*(0,s;z)
\]

\[
= \sum_{r=a}^{b} \left( \sum_{r=a}^{\infty} P_{r,j}(x,s) \eta_r(x)dx \right)
\]

\[
+ \lambda \sum_{r=0}^{a-1} c_{j-r} Q_r^*(s) , \quad a \leq j \leq b
\]

(4.7.2)
\[ A(s,z) = \sum_{j=a}^{b-1} \left[ D_j^* \{h(s,z)\} - z^j \right] P_j^* (0,0) + \sum_{j=a}^{b-1} K_j(z) D_j^* \{h(s,z)\} + K_b(z) z^b \]

\[ \sum_{r=0}^{a-1} \left[ \delta_{p,r} - h(s,z) z^r Q_r^*(s) \right] \]

\[ P_b(0,s;z) + K_b(z) = \frac{A(s,z)}{z^b - D_b^* \{h(s,z)\}} \]  

(4.7.3)

(4.7.4)

and

\[ P^*(s,z) = \sum_{j=a}^{b} P_j^*(s,z) \]

\[ = \frac{1 - D_b^* \{h(s,z)\}}{h(s,z)} \cdot \left[ P_b(0,s;z) + K_b(z) \right] \]

\[ + \sum_{j=a}^{b-1} \left[ P_j^*(0,s) + K_j(z) \right] \frac{1 - D_j^* \{h(s,z)\}}{h(s,z)} \]
The denominator of (4.7.4) has \( b \) zeros inside \(|z| = 1\) and therefore these zeros must vanish the numerator giving rise to \( b \) equations which are sufficient to determine the \( b \) unknowns, viz., \( P^*_j(0,s) = j = a, a+1, \ldots, b-1, Q^*_r(s), \) \( r=0, 1, \ldots, a-1. \) Thus \( P^*_b(0,s;z) + K^*_b(z) \) is completely known. Hence, \( P^*_b(s,z) \) is determined uniquely.

### 4.8 BUSY PERIOD DISTRIBUTION

To determine the distribution of the busy period of the system we consider that a busy period is the length of time between the beginning of service for a batch of size \( j (a, a+1, \ldots, b) \) and the moment when the number of units in the queue drops below \( a \) for the first time thereafter. The pdfs of the busy period, i.e., \( f^*_r(t) = (d/dt)Q^*_r(t), \) \( r=0, 1, \ldots, a-1 \) are the solutions of the following equations:

\[
(d/dt)Q^*_r(t) = \sum_{j=a}^{b} \int_0^\infty P^*_j r(x,t) \eta(x) dx ,
\]

\(0 \leq r \leq a - 1\)
\[(\partial / \partial x)P_{j,0}(x,t) + (\partial / \partial t)P_{j,0}(x,t) = -[\gamma + \eta_j(x)]P_{j,0}(x,t), \quad a \leq j \leq b\]

\[(\partial / \partial x)P_{j,n}(x,t) + (\partial / \partial t)P_{j,n}(x,t) = -[\gamma + \eta_j(x)]P_{j,n}(x,t)\]

\[+ \sum_{i=1}^{n} \lambda_i P_{j,n-i}(x,t), \quad a \leq j \leq b, \quad n > 0\]

\[P_{j,0}(0,t) = \sum_{r=a}^{b} \int_{0}^{\infty} P_{r,j}(x,t) \eta_j(x)dx, \quad a \leq j \leq b\]

\[P_{j,n}(0,t) = 0, \quad a \leq j < b, \quad n > 0\]

\[P_{b,n}(0,t) = \sum_{r=a}^{b} \int_{0}^{\infty} P_{r,n+b}(x,t) \eta_j(x)dx, \quad n > 0\]

with the initial condition (4.7.1).

Proceeding as in section 4.3, we get
\[ P_{j,0}(0,s) = P^*(0,s;z) \]  
\[ = \sum_{r=a}^{b} \int_{0}^{\infty} P_{r,j}(x,s) \eta(x)dx \]  
\[ B(s,z) = \sum_{r=a}^{b-1} \left[ D_{r}^* \{ h(s,z) \} - z^r \right] P_{r,0}^*(0,s) \]  
\[ + K_b(z)z^b + \sum_{j=a}^{b-1} K_j(z) D_j^* \{ h(s,z) \} \]  
\[ - \sum_{r=0}^{a-1} z^r P_{r,0}^*(0,s) \]  
\[ P_{b}^*(0,s;z)+K_b(z) = \frac{B(s,z)}{z^b - D_b^* \{ h(s,z) \}} \]  

and
\[
\begin{aligned}
P^*(s,z) &= \frac{1 - D_b^*\{h(s,z)}{h(s,z)} \left[ P^*_b(0,s;z) + K_b(z) \right] \\
&\quad + \sum_{j=a}^{b-1} \frac{1 - D_j^*\{h(s,z)}{h(s,z)} \left[ P^*_j,0(0,s) + K_j(z) \right]
\end{aligned}
\]

(4.8.4)

Evaluating the unknown constants occurring in (4.8.3) as before, \( P^*_b(0,s;z) + K_b(z) \) and hence \( P^*(s,z) \) can be completely determined.

If we denote by \( d_i(s) \), a function of \( s \), \( i = 1, 2, \ldots, b \) the zeros of the denominator of (4.8.3) lying inside the unit circle \( |z| = 1 \), then \( P^*_r,0(0,s) \), \( r = 1, 2, \ldots, b \) are the solutions to the system of \( b \) linear equations given by

\[
\sum_{j=a}^{b} K_j(d_i) D_j^*\{h(s,d_i)\} = \sum_{r=a}^{b-1} \left[ d_i^r - D_r^*\{h(s,d_i)\}\right] P^*_r,0(0,s) + \sum_{r=0}^{a-1} d_i^r P^*_r,0(0,s)
\]

(4.8.5)

\[
i = 1, 2, \ldots, b.
\]
For $r = 0, 1, \ldots, a-1$, $f_r^*(s) = P_r^*(0, s)$ are the transform of the pdfs of the busy period. The equation (4.8.4) gives the L.T. of the pgf of the busy period process.

**Particular cases:**

I. When the service time distribution is the same for each batch with pdf $D(x)$, then setting

$$D(x) = D_j(x), \quad j = a, a+1, \ldots, b$$

$$K(s) = K_j(z), \quad j = a, a+1, \ldots, b$$

and

$$P_j^*(x, s) \eta(x) dx = \sum_{r=a}^{b} P_r^*, j(x, s) \eta_j(x) dx,$$

we get from (4.7.2) - (4.7.5) and (4.8.1) - (4.8.4) respectively,

$$P^*(s, z) = \frac{1 - D^*[h(s, z)]}{h(s, z)} \cdot \frac{1}{z^b - D^*[h(s, z)]} x^{

\left[ \sum_{j=a}^{b-1} (z^b - z^j) \int_0^\infty P_j^*(x, s) \eta(x) dx \right]}.
\[ + z^b K(z) + \sum_{r=0}^{a-1} \left\{ \left( \sum_{j=a-r}^{b-r-1} (z^b - z^r)^j c_j \right) \right\} \]

\[ - z^r h(s,z) Q^*_r(s) + \delta_{p,r} \right] \]

and

\[ P^*(s,z) = \frac{1 - D^*\{h(s,z)\}}{h(s,z)} \cdot \frac{1}{z^b - D^*\{h(s,z)\}} \]

\[ = \sum_{j=a}^{b-1} (z^b - z^j) \left[ \int_0^\infty P^*_j(x,s) \eta(x) dx + z^b K(z) \right. \]

\[ - \sum_{j=0}^{a-1} z^j \left[ \int_0^\infty P^*_j(x,s) \eta(x) dx \right]. \]

The above results agree with the corresponding expressions in section 4.3 (for \( r = 0 \) in \( \delta_{p,r} \)) and section 4.6 (for \( K(z) = z^q \)).

II. When \( c_i = \delta_{1,i} \), i.e., arrivals occur singly, then
setting \( h(s, z) = s + \lambda - \lambda z \) in (4.7.2) - (4.7.5) and (4.8.1) - (4.8.4), we get the corresponding results. For such a case, (4.8.5) yields

\[
\sum_{j=a}^{b} K_j (d_j) D_j^*(s+\lambda - \lambda d_j) = \sum_{r=a}^{b-1} \left[ d_{r}^{r} - D_{r}^*(s+\lambda - \lambda d_{r}) \right] P_{r,0}^*(s)
\]

\[+ \sum_{r=0}^{a-1} d_{r}^{r} P_{r,0}^*(s),\]

where \( d_{r}^{r} (> 1) \) is the root of the equation \( z^b - D^*(s+\lambda - \lambda z) = 0 \).

If the busy period starts with the service of a batch of size 'a' leaving zero units in the waiting line, then the above equation reduces to

\[
D_a^*(s+\lambda - \lambda d_{a}) = \sum_{r=a}^{b-1} \left[ d_{r}^{r} - D_{r}^*(s+\lambda - \lambda d_{r}) \right] P_{r,0}^*(s)
\]

\[+ \sum_{r=0}^{a-1} d_{r}^{r} P_{r,0}^*(s).\]
which agrees with the corresponding expression of NEUT'S (1967) (equation (19) page 765).

4.9 CONCLUDING REMARKS

In this chapter we have considered the queueing system in which the arrivals and services occur in groups of varying size and the service time distribution of each batch is general. We have deduced results for some special cases of service time distributions. We can likewise consider special cases of distributions of the size of arrival batches, such as geometric, zero-truncated Poisson etc. and can obtain the solutions for such special queueing models.