CHAPTER III

TWO SERVER POISSON QUEUE WITH BULK SERVICE

3.0 INTRODUCTION

While BAILEY (1954) and others confined to single server bulk queueing system, ARORA (1964), GHARE (1968) etc. considered multi-server system. Their discussions confine to the usual rule for bulk service wherein a server takes into service, in a batch, the whole queue length or a number of units equal to the capacity $b$ of the server, whichever is less.

In this chapter we study the model $M/M/2$ under the general bulk service rule and deduce the results of ARORA and GHARE as particular cases.

Here we consider the transient and steady state behavior of a two server queueing system under the following assumptions:

(i) Customers arrive one by one at the service facility (consisting of two servers arranged in parallel) in a Poisson
stream with mean rate $\lambda$ and form a queue if the servers are busy.

(ii) The queue discipline is FCFS.

(iii) The capacity of each of the server is $b$.

(iv) Customers are served in batches in accordance with the general policy for bulk service (discussed in Chapter II). At the service epoch of each server, if the queue consists of more than $b$ units, the facility takes the first $b$ units or equal to for service; if the queue length $k$ is less than or equal to $b$ but not less than $a$, then a batch of size $k$ ($a \leq k \leq b$) is taken from the head of the queue for service (whenever a server is free; $a$ being an integer less than or equal to $b$). But if the queue is less than $a$, the server remains idle until the queue length reaches $a$. If both the servers are idle and the queue length is less than $a$, it is assumed that the next batch, whenever it reaches the size $a$, will join any one of the two channels.

(v) The service times for the two servers are assumed to be independent each being distributed exponentially with mean $1/\mu$.

(vi) The interarrival times and the service times are independent of each other.
The following results have been obtained:

(i) the L.T. of the time-dependent queue length probabilities

(ii) the steady state probabilities corresponding to (i)

(iii) the mean queue lengths when none of the server is busy, one is busy and both are busy

(iv) the distribution of the busy period

(v) the results due to GHARE (1968) (for c = 2 in his paper) and ARORA (1964) and those for fixed size bulk service as particular cases.

3.1 NOTATIONS AND DEFINITIONS

Define

\[ P_{r,n}(t) = \text{Probability that at time } t, \text{ there are } n \text{ units waiting in the queue and } r \text{ channels are busy.} \]

Clearly, it is non-zero only when \( r = 0 \) or \( 1, 0 \leq n \leq a-1) \); \( r=2, n \geq 0 \)

\[ P(z,t) = \sum_{n=0}^{\infty} P_{2,n}(t)z^n \text{ is the generating function of the probabilities } P_{2,n}(t), (n \geq 0) \]
\[ \rho = \frac{\lambda}{2bn} \text{ is the traffic intensity of the system} \]

\[ E_r(v) = \text{The average queue length in steady state when } r(0 \leq r \leq 2) \text{ servers are busy} \]

\[ E(v) = \text{The average queue length in steady state irrespective of whether the servers are busy or not} \]

\[ E(T) = \text{The first moment about origin of the busy period distribution} \]

\[ E(T^2) = \text{The second moment about origin of the busy period distribution} \]

\[ P_{r,n} \text{ is the steady state expression corresponding to } P_{r,n}(t) \]

### 3.2 FORMULATION OF THE EQUATIONS

Elementary probability reasoning leads to the following set of difference-differential equations:

\[
\frac{d}{dt} P_{2,n}(t) = (\lambda + 2\mu)P_{2,n}(t) + \lambda P_{2,n-1}(t) + 2\mu P_{2,n+b}(t)
\]

\[ (n \geq 1) \quad (3.2.1) \]

\[
\frac{d}{dt} P_{2,0}(t) = - (\lambda + 2\mu)P_{2,0}(t) + \lambda P_{1,a-1}(t) + 2\mu \sum_{k=a}^{b} P_{2,k}(t)
\]

\[ (3.2.2) \]
(d/dt) \( P_{1,0}(t) \) = \(- (\lambda + \mu) P_{1,0}(t) + \lambda P_{0,a-1}(t) \) 
\[ + 2 \mu P_{2,0}(t) \] 

(3.2.3)

(d/dt) \( P_{1,j}(t) \) = \(- (\lambda + \mu) P_{1,j}(t) + \lambda P_{1,j-1}(t) \) 
\[ + 2 \mu P_{2,j}(t); (1 \leq j \leq a-1) \] 

(3.2.4)

(d/dt) \( P_{0,m}(t) \) = \(- \lambda P_{0,m}(t) + \lambda P_{0,m-1}(t) + \mu P_{1,m}(t) \) 
\[ (1 \leq m \leq a-1) \] 

(3.2.5)

(d/dt) \( P_{0,0}(t) \) = \(- \lambda P_{0,0}(t) + \mu P_{1,0}(t) \) 

(3.2.6)

It is to be noted that when \( a=1 \), equations (3.2.4) and (3.2.5) will not occur.

3.3 SOLUTION OF THE EQUATIONS

Let the time be reckoned from the instant when both the servers are busy and none is waiting in the queue, i.e.

\[ P_{2,0}(0) = 1 \] 
\[ (3.3.7) \]

Then, also

\[ P(z,0) = 1 \] 
\[ (3.3.8) \]
Multiplying (3.2.1) and (3.2.2) by the appropriate powers of \( z \) and adding over all \( n \), we get

\[
\frac{d}{dt} P(z,t) = \left[ \lambda + 2 \mu - \lambda z - 2 \mu z^{-b} \right] P(z,t) + 2 \mu \sum_{k=a}^{b-1} P_{2,k}(t)(1 - z^{k-b})
\]  

(3.3.9)

\[
- 2 \mu \sum_{k=0}^{a-1} z^{k-b} P_{2,k}(t) + \lambda P_{1,a-1}(t).
\]

Now taking the L.T. of (3.3.9) and then using (3.3.8), we get on simplification

\[
P^*(z,s) = \frac{1}{(s + \lambda + 2 \mu - \lambda z)z^b - 2 \mu}
\]

\[
\left[ z^b + 2 \mu \sum_{k=a}^{b-1} (z^b - z^{k-b})P^*_{2,k}(s) \right] (3.3.10)
\]

\[
- 2 \mu \sum_{k=0}^{a-1} z^{k-b} P^*_{2,k}(s) + \lambda z^b P^*_{1,a-1}(s).
\]
The zeros of the denominator of (3.3.10) can be obtained from the solution of the equation

\[(s + \lambda + 2 \mu - \lambda z) z^b - 2 \mu = 0. \quad (3.3.11)\]

Let

\[f(z) = (s + \lambda + 2 \mu) z^b\]
\[g(z) = -2 \mu - \lambda z^{b+1}\]

Both \(f(z)\) and \(g(z)\) are analytic inside and on the contour \(|z| = 1\). Also, on \(|z| = 1\) for \(\text{Re}(s) > 0\), we have

\[|f(1)| = |(s + \lambda + 2 \mu)z^b| = |s + \lambda + 2 \mu| > \lambda + 2 \mu > |2 \mu + \lambda z^{b+1}| = |g(1)|\]

Thus we find that all the conditions of Rouche's theorem are satisfied. Hence the equation (3.3.11) has \(b\) solutions inside and one solution outside the unit circle \(|z| = 1\). Let the outside solution be denoted by \(z_0\) (Note that \(z_0 = z_0(s)\) is a function of \(s\)).

Since the degree of the denominator in (3.3.10) is one greater than that of the numerator, the expression \(P^*(z, s)\) can readily be expanded in terms of the external
solution $z_0$ as

$$
P^*(z,s) = \frac{Q(s)}{(z_0 - s)} = \sum_{r=0}^{\infty} \frac{Q(s)}{z_0^{r+1}} z^r , \quad (3.3.12)
$$

where $Q(s)$ is a function of $s$ to be evaluated.

From (3.3.12), we get

$$
P^*_{2,n}(s) = \frac{Q(s)}{z_0^{n+1}} , \quad n > 0 \quad (3.3.13)
$$

The L.T. of (3.2.4) is

$$(s + \lambda + \mu)P^*_{1,j}(s) = \lambda P^*_{1,j-1}(s) + 2\mu P^*_{2,j}(s) .$$

Solving the above recursively we get with the help of (3.3.13)

$$
P^*_{1,j}(s) = \left( \frac{\lambda}{s + \lambda + \mu} \right)^j P^*_{1,0}(s) + 2\mu \sum_{k=1}^{j} \lambda^{j-k} z_0^{-(k+1)} (s + \lambda + \mu)^{k-j-1} Q(s) \quad (3.3.14)
$$

$$1 \leq j \leq a-1.$$
Similarly from (3.2.5), we get on using (3.3.14)

\[ p_{0,m}^*(s) = \left( \frac{\lambda}{s+\lambda} \right)^m \left[ p_{0,0}^*(s) + \left\{ 1 - \left( \frac{s+\lambda}{s+\lambda+\mu} \right)^m \right\} p_{1,0}^*(s) \right. \]
\[ \left. + 2 \mu^2 \sum_{r=1}^{m} \sum_{k=1}^{r} \lambda^{-k} z_0^{-(k+1)} (s+\lambda)^{r-1} \right] (s+\lambda+\mu)^{k-r-1} Q(s), \]

\[ 1 \leq m \leq a-1. \]

and, in particular,

\[ p_{0,a-1}^*(s) = \left( \frac{\lambda}{s+\lambda} \right)^{a-1} \left[ p_{0,0}^*(s) + \left\{ 1 - \left( \frac{s+\lambda}{s+\lambda+\mu} \right)^{a-1} \right\} p_{1,0}^*(s) \right. \]
\[ \left. + 2 \mu^2 \sum_{r=1}^{a-1} \sum_{k=1}^{r} \lambda^{-k} z_0^{-(k+1)} (s+\lambda)^{r-1} \right] (s+\lambda+\mu)^{k-r-1} Q(s), \]

\[ (3.3.16) \]

Now, substituting from (3.3.16) into the L.T. of (3.2.3) and using (3.3.13), we get

\[ p_{1,0}^*(s) = M(s) \lambda^a (s+\lambda)^{1-a} (s+\lambda+\mu)^{-1} p_{0,0}^*(s) \]
\[ + \phi_0(s) Q(s), \] 

\[ (3.3.17) \]
where \( M(s) \) (which is a function of \( s \)) is given by

\[
\frac{1}{M(s)} = 1 - \mu^{a}(s + \lambda + \mu)^{-a} (s + \lambda)^{1-a} \tag{3.3.18}
\]

\[
= \left[ (s + \lambda + \mu)^{a-1} - (s + \lambda)^{a-1} \right]
\]

and \( \phi_{0}(s) \) (a function of \( s \)) is obtained by putting \( j=0 \) in the expression for \( \phi_{j}(s) \) in (3.3.21) and omitting the last term.

Substituting from (3.3.17) into (3.3.14) and (3.3.15), we get respectively

\[
P_{1,j}(s) = M(s) \mu^{a-j} (s + \lambda)^{1-a} (s + \lambda + \mu)^{-(j+1)} P_{0,0}(s)
\]

\[
+ \phi_{j}(s) Q(s) \tag{3.3.19}
\]

and

\[
P_{0,m}^{*}(s) = \left( \frac{\lambda}{s + \lambda} \right)^{m} \left[ 1 + \left\{ 1 - \left( \frac{s + \lambda}{s + \lambda + \mu} \right)^{m} \right\} M(s) \mu^{a} \right.
\]

\[
\times \left[ (s + \lambda)^{1-a} (s + \lambda + \mu)^{-1} \right] P_{0,0}(s) + \left( \frac{\lambda}{s + \lambda} \right)^{m} Q(s)x
\]

\[
\left[ \left\{ 1 - \left( \frac{s + \lambda}{s + \lambda + \mu} \right)^{m} \right\} \phi_{0}(s) + 2 \mu^{2} \sum_{r=1}^{m} \sum_{k=1}^{r} \mu^{k} \lambda^{r-k} \right]
\]

\[
\times (k+1) z_{0} (s + \lambda)^{r-1} (s + \lambda + \mu)^{k-r-1} \right\} \tag{3.3.20}
\]
where

\[ \phi_j(s) = 2M(s)(s+\lambda+\mu)^{-(j+1)} \lambda^j \left\{ \mu^2 \sum_{r=1}^{a-1} \sum_{k=1}^{r} \lambda^{-k} (s+\lambda+\mu)^{k-r-1} z_0^{-(k+1)} + \mu/z_0 \right\} \]

\[ + 2\mu \sum_{k=1}^{j} \lambda^{-k} z_0^{-(k+1)} (s+\lambda+\mu)^{k-j-1} \]

\[ 1 \leq j \leq a-1. \]

For \( j = a-1 \), (3.3.19) yields

\[ P^*_1,a-1(s) = M(s) \lambda^{2a-1} (s+\lambda)^{1-a} (s+\lambda+\mu)^{-a} P^*_0,0(s) \]

\[ + \phi_{a-1}(s) Q(s). \]

Rewriting (3.3.10) and (3.3.12) at \( z=1 \), we get

\[ P^*(1,s) = Q(s)/(z_0 - 1) \]

\[ = \frac{1 - 2\mu \sum_{k=0}^{a-1} P^*_2,k(s) + \lambda P^*_1,a-1(s)}{s}. \]

\[ (3.3.23) \]
Substituting from (3.3.13) and (3.3.22) into (3.3.23) and simplifying, we have

\[ Q(s) = R(s) \left[ 1 + M(s) \lambda^{2a} (s+\lambda)^{1-a} (s+\lambda+\mu)^{-a} p_{0,0}^*(s) \right], \]

where

\[ \frac{1}{R(s)} = \frac{s}{s_0 - 1} + 2 \mu \sum_{k=0}^{a-1} s_0^{-k+1} - \lambda^{a} \phi_{a-1}(s). \]

(3.3.25)

Now, substituting the value of \( Q(s) \) given by (3.3.24) into (3.3.13), (3.3.17), (3.3.19) and (3.3.20), the L.T. of the state probabilities i.e., \( P_{0,n}^*(s), P_{1,0}^*(s), P_{1,j}^*(s), P_{0,m}^*(s) \) are derived explicitly as

\[ P_{2,n}^*(s) = s_0^{-(n+1)} R(s) \left[ 1 + M(s) \lambda^{2a} (s+\lambda)^{1-a} (s+\lambda+\mu)^{-a} p_{0,0}^*(s) \right]. \]
\[
\begin{align*}
\mathbb{P}^*_{1,0}(s) &= \left[ M(s) \lambda^a(s+\lambda)^{1-a} (s+\lambda+\mu)^{-1} \\
&+ \phi_0(s) R(s) M(s) \lambda^{2a}(s+\lambda)^{1-a} \\
(s+\lambda+\mu)^{-a}\right] \mathbb{P}^*_{0,0}(s) + \phi_0(s) R(s) \\
(3.3.25)
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}^*_{1,j}(s) &= \left[ M(s) \lambda^{a+j}(s+\lambda)^{1-a}(s+\lambda+\mu)^{-(j+1)} \\
&+ \phi_j(s) R(s) M(s) \lambda^{2a}(s+\lambda)^{1-a} \\
(s+\lambda+\mu)^{-a}\right] \mathbb{P}^*_{0,0}(s) + \phi_j(s) R(s) \\
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}^*_{0,m}(s) &= \left( \frac{\lambda}{s+\lambda} \right)^m \left[ 1 + \left\{ 1 - \left( \frac{s+\lambda}{s+\lambda+\mu} \right)^m \right\} \right] \\
&\left\{ M(s) \lambda^a(s+\lambda)^{1-a} (s+\lambda+\mu)^{-1} + \phi_0(s) R(s) \times \\
M(s) \lambda^{2a}(s+\lambda)^{1-a} (s+\lambda+\mu)^{-a} \right\} \\
&+ 2 \mu^2 R(s) M(s) \sum_{r=1}^{m} \sum_{k=1}^{r} \lambda^{2a-k} \\
\end{align*}
\]
\[ x z_0^{-(k+1)}(s+\lambda)^{r-a} (s+\lambda + \mu)^{k-r-a-1} \] \[ p_{0,0}^*(s) \]

\[ + \left( \frac{\lambda}{s+\lambda} \right)^m \left[ 2 \mu^2 R(s) \sum_{r=1}^{m} \sum_{k=1}^{r} \lambda^{-k} z_0^{-(k+1)} \right] \]

\[ (s+\lambda)^{r-1} (s+\lambda + \mu)^{k-r-1} + \phi_0(s) R(s) \times \]

\[ \left\{ 1 - \left( \frac{s+\lambda}{s+\lambda + \mu} \right)^m \right\} \].

Since the L.T. of the sum of all probabilities at any time \( t \) is \( 1/s \),

\[ p_{0,0}^*(s) + \sum_{m=1}^{a-1} p_{0,m}^*(s) + \sum_{j=0}^{a-1} p_{1,j}^*(s) + \sum_{n=0}^{\infty} p_{2,n}^*(s) = \frac{1}{s}. \]

(3.3.26)

Substituting from (3.3.25) into (3.3.26), an explicit expression for \( p_{0,0}^*(s) \) can be derived easily.
3.4 STEADY STATE SOLUTIONS

To obtain the results in steady state, the following expressions are considered:

\[ k = \lim_{s \to 0} z_0(s) \]

\[ M = \lim_{s \to 0} M(s) = \left[ 1 - \frac{\lambda}{\lambda + \mu} + \left( \frac{\lambda}{\lambda + \mu} \right)^a \right]^{-1} \]

\[ \phi_0 = \lim_{s \to 0} \phi_0(s) = 2 M \left( \frac{\lambda}{\lambda + \mu} \right) \left[ k^{b-1}(k^b - 2)^{-1} \right] \]

\[ + \frac{\mu}{(\lambda k)} \]

\[ + \mu/(\lambda k) \]  \hspace{1cm} (3.4.1)\]

\[ \phi_j = \lim_{s \to 0} \phi_j(s) = \left( \frac{\lambda}{\lambda + \mu} \right)^j \left[ \phi_0 + 2k^{b-1}/(k^b - 2) \right] \]

\[ - 2k^{b-1-j}/(k^b - 2) , \]

\[ 1 \leq j \leq a - 1 \]
where \( k(>1) \) satisfies the equation

\[
\tilde{b}_t = \frac{x^{-b}(x^b - 1)}{(x - 1)}
\]

Using the property \( \lim_{t \to \infty} P_{r,n}(t) = \lim_{s \to 0} \mathcal{L} P_{r,n}(s) \),

the steady state probabilities are obtained from (3.3.25) in terms of the above expressions:

\[
P_{2,n} = k^{-(n+1)} A \ P_{0,0}
\]

\[
P_{1,0} = \left[ M \left( \frac{\lambda}{\lambda + \mu} \right) + A \phi_0 \right] P_{0,0}
\]

\[
P_{1,j} = \left[ M \left( \frac{\lambda}{\lambda + \mu} \right)^j + A \phi_j \right] P_{0,0} \tag{3.4.2}
\]

\[
P_{0,m} = \left[ 1 + \left\{ 1 - \left( \frac{\lambda}{\lambda + \mu} \right)^m \right\} \left[ M \left( \frac{\lambda}{\lambda + \mu} \right) + A \phi_0 \right.ight.
\]

\[
+ 2A \left( \frac{k^b - 1}{k^b - 2} \right) - 2 \mu A \left( \frac{k^{b-1}m - 1}{\lambda(k^b - 2)(k - 1)} \right) \left\} \right] P_{0,0}
\]
where

\[ A = R M \lambda^{a+1} (\lambda + \mu)^{-a}. \]

Since the sum of all probabilities at any time \( t \) in steady state is 1, we have

\[
\sum_{m=0}^{a-1} P_{0,m} + \sum_{j=0}^{a-1} P_{1,j} + \sum_{n=0}^{\infty} P_{2,n} = 1
\]

\[ (3.4.3) \]

Substituting from (3.4.2) into (3.4.3) and simplifying, we get

\[
1/P_{0,0} = a + a \left\{ M \left( \frac{\lambda}{\lambda + \mu} \right) + A \phi_0 + 2A \frac{k^{b-1}}{k^b - 2} \right\}
\]

\[- \left[ \left\{ 2 \frac{k^{b-1}}{\lambda(k-1)(k^b-2)} \right\} \left\{ a - \frac{k^a - 1}{k^{a-1}(k-1)} \right\} \right. \]

\[
+ \left\{ \frac{2}{k - 1} \right\} \left\{ \frac{k^{b-a} (k^a - 1)}{k^b - 2} - \frac{1}{2} \right\} \right\} A.
\]

\[ (3.4.4) \]
Average queue length

The average queue lengths $E_0(v)$, $E_1(v)$ and $E_2(v)$ are respectively

\[ E_0(v) = A(k - 1)^{-2} p_{0,0} \]

\[ E_1(v) = \left\{ \begin{array}{l}
M(a-1)x^{a+2} + \left\{ A(\phi_0 + D)(a-1) - M a \right\} x^{a+1}
\end{array} \right. \]

\[ - A(\phi_0 + D)ax^{a} + M x^2 + A(\phi_0 + D)x \left( \frac{\mu}{\lambda + \mu} \right)^{-2} \]

\[ + AD \left\{ ax^a - (a - 1) k^{a-1} - k^{2a-1} \right\} / \]

\[ \left\{ k^2(a-1) (k - 1)^2 \right\} p_{0,0} \]

(3.4.5)
\[ E_2(v) = \left[ \frac{a(a-1)}{2} + \left\{ Mx + AD \left( \frac{\mu}{\lambda(k-1)} \right) + A \Phi \right\} x \right. \\
\left. \{ \frac{a(a-1)}{2} - ((a-1)x^{a+1} - ax^a + x)\left(\frac{\mu}{\lambda + \mu}\right)^2 \} \right. \\
+ AD \frac{\mu}{\lambda(k-1)} \left\{ \frac{k^a - ak + (a-1)}{k^{a-1}(k-1)^2} \right\} \\
- \frac{xk^a - ax^a k + (a-1)x^{a+1}}{k^{a-1}(k-x)^2} \right\} P_{0,0}, \]

where

\[ x = \frac{\lambda}{\lambda + \mu} \]
\[ D = \frac{2k^{b-1}}{k^{b-2}} \]

The average queue length \( E(v) \) can be obtained by adding \( E_0(v) \), \( E_1(v) \) and \( E_2(v) \).
(I) The system $M/M_1,b^2$:

Here $a=1$ i.e. service occurs in batches of size at most $b$ (i.e. usual bulk service). We have

$$M = 1$$

$$\phi_0 = \frac{2\mu}{k(\gamma + \mu)}$$

$$R = \frac{k(\gamma + \mu)}{2\mu^2}$$

$$A = \frac{(k/2)(\gamma/\mu)^2}{(1/2) k^{-\alpha} (\gamma/\mu)^{\alpha}}$$

and therefore,

$$P_{2,a} = (1/2) k^{-\alpha} (\gamma/\mu)^{\alpha} P_{0,0}$$

$$P_{1,0} = (\gamma/\mu) P_{0,0}$$

$$P_{0,0} = \frac{1}{1 + (\gamma/\mu) + (1/2)(\gamma/\mu)^2 (1 - k^{-1})^{-1}}$$
\[ E_0(\nu) = E_1(\nu) = 0 \]

\[ E(\nu) = E_2(\nu) = 2kb^2\rho^2/[ (k - 1)\{2b^2\rho^2k + (k - 1)(1+2b\rho)\}] \]

which is in agreement with the corresponding results of GHARE (1968) (for \( \sigma = 2 \) in his paper).

Further, if service occurs one by one, then

\[ \frac{1}{k} = \frac{\lambda}{2\mu} = \rho \]

and therefore,

\[ P_{2,n} = 2(1 - \rho)\rho^{n+2}/(1 + \rho) \]

\[ P_{1,0} = 2\rho(1 - \rho)/(1 + \rho) \]

\[ P_{0,0} = (1 - \rho)/(1 + \rho) \]

\[ E(\nu) = 2\rho^3/(1 - \rho^2) \]

which agree with the classical results of \( M/M_1,1/2 \) queueing system (see MORSE (1958)).
(II) The system $M/M_b,b/2$:

Here $a=b$ i.e. service takes place in batches of fixed size $b$. We have

$$
M = \left[ 1 - \left( \frac{\lambda}{\lambda + \mu} \right) + \left( \frac{\lambda}{\lambda + \mu} \right)^b \right]^{-1}
$$

$$
\phi_0 = 2M \left( \frac{\lambda}{\lambda + \mu} \right) \left[ \left( \frac{k^{b-1}}{k^b - 2} \right) \left\{ 1 - \left( \frac{\lambda}{\lambda + \mu} \right)^{b-1} \right\} - \frac{\mu (k^{b-1} - 1)}{\lambda k^{b-1} (k-1)} \right] + \frac{\mu}{\lambda k}
$$

$$
\phi_j = \left( \frac{\lambda}{\lambda + \mu} \right)^j \left[ \phi_0 + \frac{2k^{b-1}}{k^b - 2} \right] - \frac{2k^{b-1}}{k^j (k^b - 2)}
$$

$$
1 \leq j \leq b-1
$$

$$
R = \left[ \frac{2\mu (k^{b-1})}{k^b (k-1)} - \lambda \phi_{b-1} \right]^{-1}
$$

On substituting these expressions in (3.4.2) and (3.4.4) and putting $a=b$ in (3.4.5) the results for fixed size service can be obtained.
3.5 THE DISTRIBUTION OF THE BUSY PERIOD

For a two-server queueing system, the distribution of the busy period may be defined in two ways. It may be the distribution of the interval during which (a) both the servers are busy and (b) at least one of the servers is busy. We consider below both the cases:

(a) Both the servers are busy:

In this case the length of the busy period is the length of the interval between the arrival of an unit that makes both the channels busy and the subsequent instant at which one of the channel becomes idle leaving \( r \) \( (0 \leq r \leq a-1) \) customers in the waiting line.

The transition probability distributions in this case, are given by \( f_r(t) = \frac{d}{dt} P_{1,r}(t) \), \( r = 0, 1, \ldots, a-1 \), where \( f_r(t) \) are the solutions of the following equations:

\[
\frac{d}{dt} P_{1,r}(t) = 2\mu P_{2,r}(t), \quad 0 \leq r \leq a-1
\]

\[
(3.5.1)
\]

\[
\frac{d}{dt} P_{2,0}(t) = - (\lambda + 2\mu) P_{2,0}(t) + 2\mu \sum_{k=a}^{b} P_{2,k}(t)
\]

\[
(3.5.2)
\]
\[
\frac{d}{dt} P_{2,n}(t) = - (\lambda + 2\mu) P_{2,n}(t) + \lambda P_{2,n-1}(t) + 2\mu P_{2,n+b}(t) \quad (3.5.3)
\]

As before, multiplying (3.5.2) and (3.5.3) by the appropriate powers of \( z \) and adding, we get

\[
\frac{d}{dt} P(z,t) = - \left[ \lambda + 2\mu - \lambda z - 2\mu z^{-b} \right] + 2\mu \sum_{k=a}^{b-1} (1 - z^{k-b}) P_{2,k}(t)
\]

\[
- 2\mu \sum_{k=0}^{a-1} z^{k-b} P_{2,k}(t). \quad (3.5.4)
\]

The L.T. of (3.5.4) under the initial condition given by (3.3.7) yields

\[
P^*(z,s) = \frac{1}{(s + \lambda + 2\mu - \lambda z) z^b - 2\mu}
\]

\[
\left[ z^b + 2\mu \sum_{k=a}^{b-1} (z^b - z^k) p^*_{2,k}(s) \right. \]

\[
- 2\mu \sum_{k=0}^{a-1} z^k p^*_{2,k}(s) \right]. \quad (3.5.5)
\]
Following the procedure adopted in section 3.3, it is easy to write that

\[ P^*(z,s) = \frac{T(s)}{z_0 - z} , \]  

(3.5.6)

where \( T(s) \) is an unknown constant to be evaluated and \( z_0 ( \equiv z_0(s) \) is a function of \( s \) ) is the root greater than unity of the equation

\[ (s + \lambda + 2 \mu - \lambda z)z^b - 2 \mu = 0 . \]  

(3.5.7)

As usual, from the expanded form of (3.5.6), we get

\[ P_{2,r}(s) = \frac{T(s)}{z_0^{r+1}} , \quad r \geq 0 . \]  

(3.5.8)

Setting \( z = 1 \) in (3.5.5) and (3.5.6), we get

\[ \frac{T(s)}{z_0 - 1} = \left[ 1 - 2 \mu \sum_{k=0}^{a-1} P_{2,k}^{*} \right] / s \]  

(3.5.9)

which when solved for \( T(s) \), after using (3.5.8), yields the value of \( T(s) \) as given below...
\[ T(s) = \frac{z_0^a (z_0 - 1)}{(s + 2 \mu)z_0^a - 2 \mu} \quad (3.5.10) \]

By definition, the L.T. of the pdf of the busy period is

\[ f_s^o(t) = s P_s^r(s), \quad 0 \leq r \leq a-1 \]

\[ = 2 \mu P_{2,r}^s(s) \]

\[ = \frac{2 \mu z_0^{a-r-1} (z_0 - 1)}{(s + 2 \mu)z_0^a - 2 \mu}. \quad (3.5.11) \]

The equation (3.5.11) can be expressed in the following form

\[ f_s^o(t) = \sum_{i=0}^{\infty} \frac{2 \mu}{s+2 \mu} \frac{i+1}{(z_0 - z_0^i)} \left( \frac{-(i+a+r)}{z_0} - \frac{-(i+a+r+1)}{z_0} \right) \quad (3.5.12) \]

The evaluate \( z_0 \), let us use the well known

Lagrange's formula given below:
The equation

\[ z = h + w \phi(z) \]

has one and only one root that approaches \( h \) as \( w \) approaches zero and if this root is \( y_0 \), then

\[
f(y_0) = f(h) + \sum_{j=1}^{\infty} \frac{w^j}{j!} \frac{d^{j-1}}{dh^{j-1}} \left( \phi^j(h) \cdot f'(h) \right),
\]

\( \phi(*) \) and \( f(*) \) being functions of \( \phi(z) \).

Let us now write (3.5.7) in the form

\[
z = \left( \frac{s + \lambda + 2 \mu}{\lambda} \right) - \left( \frac{2 \mu}{\lambda} \right) \left( \frac{1}{z^b} \right) \quad (3.5.13)
\]

By the above theorem, we observe that (3.5.13) has one and only one root which approaches \((1 + s/\lambda)\), the modulus of which is greater than unity, since \( \text{Re}(s) > 0 \). Also we know, by Rouche's theorem, that (3.5.7) has only one root \( z_0 \), whose modulus is greater than one. So \( z_0 \) is the same root as that obtained by Lagrange's formula.
Hence, writing \( f(z) = \frac{1}{z^r} \), \( h = \frac{(s + \lambda + 2 \mu)}{\lambda} \),\n\( w = -\left(2 \frac{\mu}{\lambda}\right) \) and \( \phi(z) = \frac{1}{z^b} \), where \( \sigma \) is any positive integer, we get (refer LUCHAK (1956) and AROHA (1964))

\[
\sigma^r z_0 = \left\{ a^{-\sigma} + \sigma \sum_{j=1}^{\infty} \left[ (-1)^{j+1} / j! \right] (2 \frac{\mu}{\lambda})^j \right\}
\]

\[
\frac{d^{j-1}}{dh^{j-1}} \left( h^{-(1+\sigma+bj)} \right) \right\}
\]

\[
= \sum_{j=0}^{\infty} \frac{\left( 2 \frac{\mu}{\lambda} \right)^j}{j!} \cdot \frac{\left[ j(b + 1) + \sigma - 1 \right]!}{(bj + \sigma)!}
\]

\[
\left[ \frac{s + \lambda + 2 \mu}{\lambda} \right] = \left\{ j(b+1) + \sigma \right\}
\]

(3.5.14)

Taking the inverse Laplace transform of (3.5.14), (refer ERDÉLYI (1954)), we get
\[
L^{-1}\left(\frac{s^2}{s^2 + \sigma^2}\right) = \sum_{j=0}^{\infty} \left(2 \mu / \lambda\right)^j e^{-\left(\lambda + 2 \mu\right)t} \frac{1}{j! (bj + \sigma)!} (\lambda t)^{j(b+1)+\sigma} \]

(3.5.15)

where

\[
b + 1 = 2 \mu / \lambda
\]

and

\[
I_\sigma(x) = \sum_{n=0}^{\infty} \frac{1}{2^x} \left(\frac{x+n(b+1)}{n! (bn + \sigma)!} \right).
\]

Applying the formula given by (3.5.15) in (3.5.12) and using the convolution property of Laplace transform, viz.,

\[
L^{-1}\left[f_1^*(s) \cdot f_2^*(s)\right] = \int_0^t f_1(t-u) \cdot f_2(u) du,
\]

we get...
\[ f_r(t) = \sum_{i=0}^{\infty} \left( \frac{2^i}{i!} \right) e^{-2\mu t + \lambda u} \frac{(t-u)^i}{i!} \]

\[ \left[ \frac{ia + r}{ia + r} \frac{b}{p} I_{ia+r} (2\lambda pu) - \frac{(ia + r + 1)}{ia + r + 1} x \right] \]

\[ b I_{ia+r+1} (2\lambda pu) \Bigg] \] du. \quad (3.5.16)

**Moments**

The moments of the distribution of the busy period can be obtained by differentiating \( f_r(s) \) and setting \( s=0 \). The first two moments are

\[ E_r(T) = \left[ - \frac{df_r(s)}{ds} \right]_{s=0} = \frac{k^{a-r-1}}{2\mu L(k^a - 1)} x^r \]

\[ \left[ b \varphi (b + 1)k^{a+2} + (r - 2b^2\varphi - b\varphi - b)k^{a+1} + (b^2\varphi + b - r - 1)k^a + (a-r)k + r - a + 1 \right] \]

\[ 0 < r < a-1. \]
and

\[ E_r(T^2) = \left[ \frac{d^2 f_r(s)}{ds^2} \right]_{s=0} = \frac{k^{a-r-1}}{4 \mu^2 L^3 (k^a - 1)^3} \times \]

\[ \left[ 2k^{2a} (k - 1)L^3 + \{ 4a(k-1)k^{2a} 
- k^a(k^a - 1)(2M_1 + 2a(k - 1)) \}L^2 
+ \{ 2a^2 (k - 1)k^{2a} - k^a (k^a - 1) \times 
(aM_1 + a(k - 1)(a + b - 1)) + (k^a - 1)^2 \times 
(bM_1 + (a - r - 1)M_2 - a(k^a - 1)k^a M_1) \}L 
+ baks(k - 1)(k^a - 1)(b \xi + 1) - 
- b(k^a - 1)^2 (b \xi + 1) M_1 \right], \quad (3.5.17) \]

where

\[ M_i = (a - r - i)(k - 1) + ik, \quad i = 1, 2 \]
\[ L = b \theta (b + 1)k - b (b \theta + 1) \]

and \( k(> 1) \) is the root of the equation

\[ b \theta = x^{-b} (x^b - 1)/(x - 1) \]

**Particular cases**

For the usual bulk service rule i.e. for \( a = 1 \), the busy period distribution and the moments can be obtained from (3.5.11), (3.5.16) and (3.5.17) in the following form:

\[
\begin{align*}
\hat{f}^*(s) &= \hat{f}^*_0(s) = \frac{2 \mu (z_0 - 1)}{(s + 2 \mu)z_0 - 2 \mu} \\

f(t) &= f_0(t) = \sum_{i=0}^{\infty} (2 \mu)^{i+1} \int_0^t e^{-(2 \mu t + \lambda u)} e^{-(2 \mu t + \lambda u)} dx \\

&= \frac{(t - u)^i}{i!} \left[ \frac{i}{p} \right] I_i (2 \lambda pu) \\
&\quad - \frac{i + 1}{p(i+1)} I_{i+1}(2 \lambda pu) \\
&\quad du
\end{align*}
\]
\[ E(T) = E_0(T) = \frac{k}{2\mu (k - 1)} \]

\[ E(T^2) = E_0(T^2) = \frac{2k^2L + 2k}{4\mu^2L (k - 1)^2} \]

and therefore,

\[ \text{Var}(T) = E(T^2) - [E(T)]^2 = \frac{k^2L + 2k}{4\mu^2L (k - 1)^2} \]

Further, for \( b = 1 \), \( f^*(s) \) becomes

\[ f^*(s) = (2\mu/\lambda)z_1^{-1} \]

where \( z_1 (>1) \) is the root of the equation

\[ \lambda z^2 - (s + \lambda + 2\mu)z + 2\mu = 0 \]

and

\[ z_1 \rightarrow \frac{2\mu}{\lambda} \quad \text{as} \quad s \rightarrow 0. \]

Inverting it, we have
\[ f(t) = \frac{1}{\sqrt{\lambda/2 \mu}} \frac{1}{t} e^{-\left(\lambda + 2 \mu\right)t} I_1(2\sqrt{2\lambda/\mu} \, t). \]

Also,

\[ E(T) = \frac{1}{2 \mu (1 - \rho_0)}. \]

\[ E(T^2) = \frac{1}{2 \mu^2 (1 - \rho_0)^3}. \]

\[ \text{Var}(T) = \frac{(1 + \rho_0)}{4 \mu^2 (1 - \rho_0)^3}. \]

where \( \rho_0 = \frac{\lambda}{2 \mu}. \)

The above results agree with those of ARORA (1964) except for notation.
(b) One server is busy:

The busy period starts as soon as one of the channels becomes busy leaving none in the queue. The busy period continues up to the instant at which both the channels becomes idle leaving $r$ ($r=0, 1, \ldots, a-1$) units in the waiting line.

In this case the transition probability distributions are given by $g_j(t) = (d/dt)P_{0,j}(t)$, $j = 0, 1, \ldots, a-1$ where $g_j(t)$ are determined by the following equations:

\begin{align*}
(d/dt)P_{0,j}(t) &= \mu P_{1,j}(t) , \quad 0 \leq j \leq a-1 \quad (3.5.18) \\
(d/dt)P_{1,0}(t) &= - (\gamma + \mu)P_{1,0}(t) + 2\mu P_{2,0}(t) \quad (3.5.19) \\
(d/dt)P_{1,j}(t) &= - (\gamma + \mu)P_{1,j}(t) + \lambda P_{1,j-1}(t) \\
&\quad + 2\mu P_{2,j}(t) , \quad 1 \leq j \leq a-1 \quad (3.5.20) \\
(d/dt)P_{2,0}(t) &= - (\gamma + 2\mu)P_{2,0}(t) + \lambda P_{1,a-1}(t) \\
&\quad + 2\mu \sum_{k=a}^{b} P_{2,k}(t) \quad (3.5.21)
\end{align*}
\[(d/dt)P_{2,n}(t) = -(\gamma + 2\mu)P_{2,n}(t) + \gamma P_{2,n-1}(t) + 2\mu P_{2,n+b}(t)\]  

(3.5.22)

Since (3.5.22), (3.5.21) and (3.5.20) are the same as (3.2.1), (3.2.2) and (3.2.4) respectively, proceeding as in Section 3.3, we get

\[
\begin{align*}
P^*(z,s) &= \frac{1}{z^{b}(s + \gamma + 2\mu - \gamma z) - 2\mu} \\
&= \left[ 2\mu \sum_{k=a}^{b-1} \left( z^b - z^k \right) P_{2,k}(s) \right] \\
&\quad - 2\mu \sum_{k=0}^{a-1} k P_{2,k}(s) + \gamma z^b P_{1,a-1}(s) \\

P_{1,j}^*(s) &= \left[ \frac{\gamma}{(s + \gamma + \mu)} \right]^j P_{1,0}^*(s) \\
&\quad + 2\mu \sum_{k=1}^{j} \lambda^{j-k} (s + \gamma + \mu)^{k-j-1} P_{2,k}(s) \\
&\quad \text{for } 1 \leq j \leq a-1
\end{align*}
\]
The denominator of (3.5.23) being the same as that of (3.3.10), we can express \( P^*(z,s) \) in terms of the external solution \( z_0 \) as

\[
P^*(z,s) = \frac{B(s)}{z_0 - z} = \sum_{k=0}^{\infty} \frac{B(s)}{z_0^{k+1}} z^k , \tag{3.5.25}
\]

where \( B(s) \) is a function of \( s \) to be evaluated, and therefore,

\[
P^*_2, k(s) = \frac{B(s)}{z_0^{k+1}} , \quad k > 0 \tag{3.5.26}
\]

Rewriting (3.5.23) and (3.5.25) at \( z=1 \), we get

\[
\frac{B(s)}{z_0 - 1} = \sum_{k=0}^{a-1} \frac{B(s)}{z_0^{k+1}} P^*_2, k(s) + \lambda P^*_1, a-1(s) \tag{3.5.27}
\]

The initial condition of the system is (by assumption)

\[
P^*_1, 0(0) = 1 \tag{3.5.28}
\]

Taking the L.T. of (3.5.19), and using (3.5.26) and (3.5.28), we get
\[ p_{1,0}^*(s) = \frac{1}{s + \lambda + \mu} + \frac{2\mu}{s + \lambda + \mu} \cdot \frac{B(s)}{z_0} \quad (3.5.29) \]

Substituting from (3.5.29) and (3.5.26) into (3.5.24), we find on simplification

\[ p_{1,j}^*(s) = \frac{1}{\lambda} \left( \frac{\lambda}{s + \lambda + \mu} \right)^{j+1} + 2\mu B(s) \times \quad (3.5.30) \]

\[
\left[ \sum_{j=0}^{\infty} j \left( \frac{\lambda}{s + \lambda + \mu} \right)^j \frac{j-k}{s_0^{k+1}(s + \lambda + \mu)^{j+1-k}} \right].
\]

In particular,

\[ \lambda p_{1,a-1}^*(s) = \left( \frac{\lambda}{s + \lambda + \mu} \right)^a + 2\mu B(s) \times \quad (3.5.31) \]

\[
\left[ \sum_{k=0}^{a-1} (\frac{\lambda}{s + \lambda + \mu})^{a-k} \frac{1}{s_0^{k+1}} \right].
\]

Now, substituting from (3.5.26) and (3.5.31) into (3.5.27) and solving for \( B(s) \), we get
\[ B(s) = R(s) \left( \frac{\lambda}{s + \lambda + \mu} \right)^{a} \tag{3.5.32} \]

where

\[
\frac{1}{R(s)} = \frac{s}{z_0 - 1} + 2\mu \sum_{k=0}^{a-1} \left[ 1 - \left( \frac{\lambda}{s + \lambda + \mu} \right)^{a-k} \right] \frac{1}{z_0^{k+1}}
\]

\[
= \frac{1}{(z_0 - 1)z_0^{a}(\lambda s - s - \lambda - \mu)}
\]

\[
\left[ sz_0^{a}(\lambda s - s - \lambda - \mu)
\right]
\]

\[
+ 2\mu (z_0^{a} - 1) (\lambda s - s - \lambda - \mu)
\]

\[
- 2\lambda \mu (z_0 - 1) \left\{ (\lambda z_0)^{a} - (\lambda s)^{a} \right\}.
\]

Putting the value of \( B(s) \) given by (3.5.32) in (3.5.30), we have

\[
P_{1,j}^{*}(s) = \frac{1}{\lambda} \left( \frac{\lambda}{s + \lambda + \mu} \right)^{j+1} + 2\mu R(s) \sum_{k=0}^{j} \frac{1}{z_0} \lambda^{-(j-k)} \left( \frac{\lambda}{s + \lambda + \mu} \right)^{j+1-k+a}.
\tag{3.5.33} \]
Therefore the L.T. of the pdfs of the busy period distribution are given by

\[ g_j(s) = s P_0^*, j(s) = \mu P_1^*, j(s) \]

\[ = \frac{\mu}{\lambda} \left( \frac{\lambda}{s + \lambda + \mu} \right)^{j+1} \]

\[ + 2 \mu^2 R(s) \sum_{k=0}^{j} \lambda^{-1} \left( \frac{\lambda}{s + \lambda + \mu} \right)^{a(j+1)-k-(k+1)} x_0 \]

\[ 0 \leq j \leq a-1 \]

It appears that the above expression cannot be put in a convenient closed form from which inversion can be carried out. However, for \( a=1 \), we can deduce the results earlier obtained by ARORA (1964) as a particular case from our results.
Particular case:

For $a = 1,$

$$R(s) = \frac{z_0(z_0 - 1)(s^2 + \lambda + \mu)}{\{s(s^2 + \lambda + \mu) + 2 \mu(s + \mu)\}z_0 - 2 \mu(s + \mu)}$$

and therefore,

$$g^*(s) = g_0^*(s) = \frac{\mu(s^2 + \lambda + \mu) z_0 - 2 \mu^2}{\{s(s^2 + \lambda + \mu) + 2 \mu(s + \mu)\}z_0 - 2 \mu(s + \mu)}.$$

The above expression can be put in the form

$$g^*(s) = \sum_{k=0}^{\infty} \frac{\mu (2 \mu)^k(s + \lambda + \mu)^k z_0^k}{[s(s^2 + \lambda + \mu) + 2 \mu(s + \mu)]^{k+1}} - \sum_{k=0}^{\infty} \frac{\mu (2 \mu)^{k+1}(s + \lambda + \mu)^k z_0^{-(k+1)}}{[s(s^2 + \lambda + \mu) + 2 \mu(s + \mu)]^{k+1}}.$$
Proceeding as in (a), we get

\[ g(t) = \sum_{k=0}^{\infty} \mu(2\mu) \binom{t}{k} \left[ \frac{-r(2\mu)}{u^p} \right] I_k(2\lambda pu) \]

\[
\sum_{m=1}^{k+1} \sum_{r=0}^{m-1} \left\{ \frac{1}{(k+1-m)! (m-1)! (k+1-r)!} \right\}

\cdot (-1)^{m-1-r} \frac{(k+m-r-1)!}{r!} (\alpha_1 - \alpha_2)^{-(k+m-r)}

\cdot (t-u)^{k+1-m} \left\{ \alpha_1 + \mu \frac{k-r}{(m+1)(\alpha_1 + \mu)} \right\}

+ \frac{\alpha_1(t-u)}{(k+1-r)\mu} e^{\alpha_1(t-u)} + (-1)^{\alpha_2 + \mu} \frac{k-r}{(k+1-r)\mu} e^{\alpha_2(t-u)} \left( (k+1)(\alpha_2 + \mu) + (k+1-r)\mu \right) \left( (k+1)(\alpha_2 + \mu) + (k+1-r)\mu \right) \]
$$\sum_{k=0}^{\infty} \mu \left(2 \mu\right)^{1+k} \int_{0}^{t} \left[ \frac{-(\lambda+2 \mu)u}{k+1} \right] \frac{b}{u^p} I_{k+1}(2\lambda pu)$$

$$\left[ \sum_{m=1}^{k+1} \sum_{r=0}^{m-1} \frac{1}{(k+1-m)! (m-1)! (k-r)!} \right]$$

$$(-1)^m \frac{(k+m-r-1)!}{r!} (\alpha_1 - \alpha_2)^{(k+m-r)}$$

$$(t-u)^{k+1-m} \left\{ \begin{array}{l} (\alpha_1 + \mu)^{k-r} e^{(t-u)} + (-1)^{k+m-r} \\ (\alpha_2 + \mu)^{k-r} e^{2(t-u)} \end{array} \right\} du$$

where $\alpha_1$ and $\alpha_2$ are the roots of the equation

$$s^2 + s(\lambda + 3 \mu) + 2 \mu^2 = 0$$
The mean is given by

\[ E(T) = \frac{(1 + \varphi b)^{k-1}}{\mu (k-1)} \]

and the variance by

\[ \text{Var}(T) = \frac{1}{\mu^2 (k-1)^2 L} \left[ \varphi b(b+1)(1+3b \varphi + b^2 \varphi^2)k^3 \right. \]

\[ - (1 + \varphi b) \left\{ b(1 + 5b \varphi + b^2 \varphi^2) + 2b \varphi \right\} k^2 \]

\[ + \left\{ b \varphi (b+2) + 2b(1+b \varphi)^2 \right\} k - (1+b \varphi) b \]

3.6 CONCLUDING REMARKS

1. ARORA considered the two server queueing system with usual bulk-service rule and obtained the pf of the number in the queue, the pdf of the busy period distribution and their first two moments. GHARE simply extended the results of the multi-server case covering only the pdf of the number in the queue. So far as our knowledge goes, further analysis
of the multi-server queueing system has not yet been done in the case of bulk service. In this context, it appears that our present analysis for the two-server case is more general as it covers the general bulk service rule. Further extension to the multi-server system seems to be rather complicated. However, we give below the equations of this system whose solutions have not been attempted here.

The equations are

\[
\begin{align*}
\frac{d}{dt} P_{a,n}(t) &= - (\lambda + a \mu) P_{a,n}(t) + \lambda P_{a,n-1}(t) \\
&\quad + c \mu P_{a,n+b}(t), \quad n \geq 1 \\
\frac{d}{dt} P_{a,0}(t) &= - (\lambda + a \mu) P_{a,0}(t) + \lambda P_{a-1,a-1}(t) \\
&\quad + c \mu \sum_{k=a}^{b} P_{a,k}(t) \\
\frac{d}{dt} P_{r,0}(t) &= - (\lambda + r \mu) P_{r,0}(t) + \lambda P_{r-1,a-1}(t) \\
&\quad + (r + 1) \mu P_{r+1,0}(t), \\
&\quad 1 \leq r \leq a-1
\end{align*}
\]
\[
\frac{d}{dt} P_{r,j}(t) = - (\gamma + r \mu) P_{r,j}(t) + \lambda P_{r,j-1}(t) \\
+ (r + 1) \mu P_{r+1,j}(t),
\]
\[
1 \leq r \leq a-1
\]
\[
1 \leq j \leq a-1
\]
\[
\frac{d}{dt} P_{0,m}(t) = - \gamma P_{0,m}(t) + \lambda P_{0,m-1}(t) + \mu P_{1,m}(t),
\]
\[
0 \leq m \leq a-1
\]
\[
P_{0,-1}(t) = 0
\]

***