PART- I

QUEUEING SYSTEM WITH TWO PHASES OF HETEROGENEOUS SERVICE

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CHAPTER II *

A POISSON INPUT QUEUE WITH TWO PHASES OF HETEROGENEOUS SERVICE

2.1 Introduction

The classical $M/G/1$ queueing systems have been discussed extensively in the past due to their practical applications in many areas, e.g. see Kendall (1953) and Cox [1955(a)]. Recently there has been considerable amount of work done with $M/G/1$ type of queues in which the server may provide a second phase of service. Doshi (1991) considered such a model to assign a control of queue where the execution of two phases of service is required by a central server with application in a distributed system. Madan [2000(a)], in a recent paper considers an $M/G/1$ queue where the server provides a first essential service to all the arriving customers, whereas only some of them receive a second phase of optional service. The first essential service time follows a general distribution but the second optional service is assumed to be exponentially distributed. Some examples of queueing situations where such service mechanism can arise are also given. Medhi (2002) generalized this model by considering that second optional service is also governed by general distribution. More recently, Choudhury [2003(a)] investigated the same model in more depth. In all these studies, the well known supplementary variable technique has been applied to carry out the various performance analysis of this type of model. However, a simple derivation even for this model is also possible. Hence, in the present study, we have generalized Madan's work in the sense that both the services are governed by general distributions with different distribution functions. In most of the previous studies, only the solutions in terms of generating

* Some parts of this chapter have been published in "Journal of Indian Statistical Association", (2004), 42(1), 63-74. [ see Ref. 56 of Bibliography ]
functions are given. However, in this study, we develop a more detailed study, which includes the recursive computation of limiting probabilities. To this end, the mathematical methodology will be based on a combination of embedded Markov chain analysis and Markov regenerative process.

The following results have been obtained under the present study of this chapter-

(i) The queue size distribution at departure epoch
(ii) The queue size distribution due to busy periods of the server
(iii) The busy period distribution
(iv) The waiting time distribution
(v) The recursive solution of the departure point queue size distribution
(vi) The numerical illustration

2.2 The system

We consider an $M/G/1$ queueing system, where arrivals occur according to a Poisson process with mean arrival rate $\lambda$ and the server serves the units in two phases of heterogeneous service on first come first serve basis. The first phase of service (FPS) is essential for all the units. As soon as the FPS of an unit is completed it may leave the system with probability $(1 - p)$ or may immediately opt for a second phase of service (SPS) in an additional service channel with probability $p(0 \leq p \leq 1)$. Assuming that the service times $B_i$ ($i = 1, 2$) of two channels are mutually independent of each other having general law of distribution with distribution function (d.f.) $B_i(x)$ and Laplace Stieltjes transform (LST) $B_i^*(s)$ with finite moment $\beta_i^{(k)}, k \geq 1$ for $i = 1, 2$ (denoting first and second phase of service channels respectively). Further it is also assumed that the same server serves both the channels or if there are two different servers, only one unit can be taken for service in either channel at any time and that no other can be taken for service until the unit being served leaves the system after final service completion. Thus the time required by an unit to complete the service cycle is given by
\[ B = \begin{cases} B_1 + B_2 & \text{with probability 'p'} \\ B_1 & \text{with probability 'q' (=1 - p)} \end{cases} \]

By \( g_j \) and \( h_j \) we denote the probabilities that \( j \) units arrive during a FPS time \( B_1 \) and SPS time \( B_2 \), respectively. Hence

\[ g_j = \sum_{i=0}^{\infty} \frac{(\lambda x)^i e^{-\lambda x}}{i!} dB_1(x) = \frac{(-\lambda)^j B_1^{(j)}(\lambda)}{j!}; \quad j = 0, 1, 2, \ldots \]

and \( h_j = \sum_{i=0}^{\infty} \frac{(\lambda x)^i e^{-\lambda x}}{i!} dB_2(x) = \frac{(-\lambda)^j B_2^{(j)}(\lambda)}{j!}; \quad j = 0, 1, 2, \ldots \)

where \( B_i^{(0)}(s) = \frac{d^i B_i'(s)}{ds^i}; \quad i = 1, 2. \)

Let \( G(z) \) and \( H(z) \) be the probability generating function's (PGF's) of \( \{g_j; j \geq 0\} \) and \( \{h_j; j \geq 0\} \) respectively, then we have

\[ G(z) = \sum_{j=0}^{\infty} z^j g_j = B_1'(\lambda - \lambda z) \]

and \( H(z) = \sum_{j=0}^{\infty} z^j h_j = B_2'(\lambda - \lambda z); \) respectively.

### 2.3 Queue size distribution at Departure epoch

In this section we derive the PGF of the queue size distribution at departure epoch, by applying embedded Markov chain technique as a simple alternative approach.
Before going to derive the PGF of it we first prove the following theorem for existence of steady state conditions.

**Theorem 2.3.1.1.** Let $\beta_i^{(0)} < \infty$; for $i=1,2$, and $B_i$ is the service time random variable for $i$-th. phase of service, then the system is ergodic if and only if $\rho = \lambda[\beta_1^{(0)} + p\beta_2^{(0)}] < 1$.

**Proof.** Let $t_n$ be the $n$-th. service completion epoch of the total service, then $L_n = N(t_n + 0); n \geq 1$ (where $N(t_n)$ represents number of units in the system at time instant $t_n$), is the number of customers left behind by the $n$-th. departing customer. Clearly, the Markov chain $\{L_n; n \geq 1\}$ is aperiodic and irreducible with its transition probability matrix

$$P = (P_{i,j}) = pP_1 + (1-p)P_2,$$

where

\[
P_1 = \begin{pmatrix}
c_0 & c_1 & c_2 & c_3 & \cdots \\
c_0 & c_1 & c_2 & c_3 & \cdots \\
0 & c_0 & c_1 & c_2 & \cdots \\
0 & 0 & c_0 & c_1 & \cdots \\
0 & 0 & 0 & c_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
g_0 & g_1 & g_2 & g_3 & \cdots \\
g_0 & g_1 & g_2 & g_3 & \cdots \\
0 & g_0 & g_1 & g_2 & \cdots \\
0 & 0 & g_0 & g_1 & \cdots \\
0 & 0 & 0 & g_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]

$c_j = \text{Prob}[\text{'}j\text{'}\text{ units arrive during the service time '}B_1 + B_2\text{'}]\]

\[
= \sum_{i=0}^{j} g_i h_{j-i}; \quad j = 0,1,2,\ldots \quad (2.3.1)
\]

and $P_{i,j} = \text{Prob}[L_{n+1} = j | L_n = i]$.

Clearly, the transition probabilities are given by
Then the drift also known as Lyapounov function [see Pakes (1969)] is defined as \( d(i) = E[L_i - L_n | L_n = i] \) for \( i \in S \), where \( S = \{0, 1, 2, \ldots\} \) is the state space (i.e., number of units in the system). Now for our model the drift is given by

\[
d(0) = \lambda \beta_1 = \lambda \beta_1^{(0)} + p \beta_2^{(0)}
\]

\[
d(i) = \lambda \beta_1 - 1; \quad \text{for} \quad i \geq 1.
\]

Then there exist an \( \eta > 0 \) such that \( d(i) < -\eta \) for all \( i \neq 0 \). Hence by Foster's criterion [e.g., see Sennott et al. (1983) or section 1.7 of Chapter 1] the condition \( \lambda \beta_1 = \lambda \beta_1^{(0)} + p \beta_2^{(0)} = \rho \) (say) < 1 is sufficient for ergodicity. This completes the proof.

Now we are in a position to obtain the PGF of the departure point queue size distribution under the ergodic (steady state) condition. Let \( \psi_j (j \geq 0) \) be the steady state probability that \( j \) customers are left behind by a departing customer, then

\[
\psi_j = \lim_{n \to \infty} \Pr[L_n = j]; \quad j \geq 0.
\]

Now, \( \{\psi_j ; (j \geq 0)\} \) can be obtained by solving the system of equations

\[
\psi^t P = \psi ;
\]

where \( \psi = [\psi_0, \psi_1, \psi_2, \ldots] \) is a column vector.

Utilizing (2.3.2) in (2.3.3), we get

\[
\psi_j = \psi_0 \left[ pc_j + (1 - p) g_j \right] + \sum_{m=0}^{j-1} \psi_m \left[ pc_{j-m} + (1 - p) g_{j-m} \right], \quad j \geq 0
\]
Let $\psi(z)$ be the PGF of $\{\psi_j; (j \geq 0)\}$, then multiplying equation (2.3.4) by appropriate power of $z^j$ and taking summation overall possible values of $j$ $(j \geq 0)$ we get

$$
\psi(z) = \sum_{j=0}^{\infty} z^j \psi_j
= \psi_0[pC(z) + (1 - p)G(z)] + [\psi(z) - \psi_0 pC(z) + (1 - p)G(z)]z^{-1}
$$

(2.3.5)

where $C(z)$ is the PGF of $\{c_j; (j \geq 0)\}$, which can be obtained from equation (2.3.1) as follows

$$
C(z) = \sum_{j=0}^{\infty} z^j C_j = G(z) H(z).
$$

(2.3.6)

Now utilizing (2.3.6) in (2.3.5), we get on simplification

$$
\psi(z) = \frac{\psi_0(1 - z)[(1 - p) + pB_1^*(\lambda - \lambda z)]B_1^*(\lambda - \lambda z)}{[(1 - p) + pB_2^*(\lambda - \lambda z)]B_1^*(\lambda - \lambda z) - z}.
$$

To obtain $\psi_0$, we utilize the normalizing condition $\psi(1) = 1$, which gives $\psi_0 = (1 - \rho)$, where $\rho = \lambda[\beta_1^{(0)} + p\beta_2^{(0)}]$ is the utilization factor of this system.

Thus we have

$$
\psi(z) = \frac{(1 - \rho)(1 - z)[(1 - p) + pB_1^*(\lambda - \lambda z)]B_1^*(\lambda - \lambda z)}{[(1 - p) + pB_2^*(\lambda - \lambda z)]B_1^*(\lambda - \lambda z) - z}.
$$

(2.3.7)

Let $L$ be the mean number of customers in the system, then

$$
L = \frac{d\psi(z)}{dz}
$$
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\[ \frac{\lambda^2 \beta_t^{(2)} + p(2 \beta_t^{(2)} + 2 \beta_{t-1}^{(1)})}{2(1-\rho)} + p \]

where \( \beta_t^{(0)} \) and \( \beta_t^{(2)} \) are defined in section 2.2 of this Chapter.

Note that for \( p = 0 \) (i.e., there is no SPS in the system), we get necessary results for a standard \( M/G/1 \) queueing model [e.g., see Kashyap and Chaudhry (1988), p. 45].

2.4 Queue size distribution due to Busy periods of the server

In this section, the stationary queue size distribution due to busy periods of first phase of service and second phase of service is investigated. The argument of Markov regenerative process (e.g., see section 1.4.3 of Chapter 1) is applied in this section for finding the result. Here we note that the state of the system at time 't' can be discussed by means of Markov process.

\[ Y(t) = (C(t), N(t), \xi(t)) \]

where \( C(t) = 1 \) or 2 according to whether the server is busy with FPS or busy with SPS at time \( t \), \( N(t) \) represents number of customer in the queue (including one in service, if any) at time \( t \). If \( C(t) \in \{1,2\} \), then \( \xi(t) \) represents the corresponding elapsed service time in progress.

By neglecting the elapsed times \( \xi(t) \) the theory of Markov regenerative process guarantees the limiting probabilities.

\[ Q_{i,j} = \lim_{t \to \infty} \text{Prob} \{ Y(t) = (i,j) \}, (i,j) \in \Omega \] exist

and are positive under some conditions of limiting probabilities.

By following the Burkes theorem [see Takagi (1991), page 7 or section 1.7 of Chapter 1] that the stationary probability \( \{Q_{i,j}; (i,j) \in \Omega\} \) are positive under the same
conditions of limiting probabilities $\{\psi_j; j \geq 0\}$ of the Markov chain $\{L_n, n \geq 1\}$ i.e. if and only if $\rho < 1$.

Then $\{Y(t); t > 0\}$ is a Markov regenerative process with embedded Markov renewal process $\{L_n; n \geq 1\}$ hence by using the classical limiting theorems established in Ciarlet [1975(b)] to obtain

$$Q_{ij} = \frac{\sum_{n=0}^{\infty} \psi_n \tau_n(i,j)}{\sum_{n=0}^{\infty} \psi_n \sigma_n}, \quad (i,j) \in \Omega$$

(2.4.1)

where $\tau_n(i,j)$ = The expected amount of time spent by the process $\{Y(t); t > 0\}$ in the state $(i,j)$ during the service cycle given that the initial queue size is ‘n’ and $\sigma_n$ = The expected length of the service cycle given that the initial queue size is ‘n’.

Now for this model we have $\sigma_n = \beta_1^{(0)} + p \beta_2^{(0)}; n \geq 0$ (2.4.2)

and therefore $\sum_{n=0}^{\infty} \psi_n \sigma_n = \lambda^{-1}$; is the mean service cycle.

Now let us suppose that a total service ends leaving ‘n’ customers in the queue. We may distinguish two cases according to the origin of the customer who receive the next service. Let us assume this customer as primary one then its FPS starts at time (we say) $t = 0$. We observe that the time interval $(t, t + \Delta t)$ contributes to $\tau_n(1,j)$ if

(i) the FPS has not been completed before time ‘$t$’ with probability $(1 - B_1(t))$

(ii) $(j - n + 1)$ primary customers arrive during $(0, t]$.

Thus we have

$$\tau_n(1,j) = m_{j-n+1}, \text{ for } j \geq \max(0, n - 1)$$

(2.4.3)

where $g_j' = \int_0^1 P_j(t)(1 - B_1(t))dt; j \geq 0$. 

Note that \( \lambda g_j = \left(1 - \sum_{i=0}^{j} g_i\right) ; j \geq 0 \).

Then utilizing the argument of regenerative process and using (2.4.2) and (2.4.3) in (2.4.1), we get

\[
Q_{1,j} = \lambda \left[ \psi_0 g_j + \sum_{i=1}^{j+1} \psi_i g_{j+1-i} \right] ; j \geq 0
\]  
(2.4.4)

Similarly proceeding in similar manner for the case of SPS, it can be shown that

\[
Q_{2,j} = \lambda \left[ \psi_0 \sum_{n=0}^{j} g_n h_{j-n} + \sum_{i=1}^{j+1} \psi_i \sum_{n=1}^{j-i+1} g_n h_{j-i+n} \right] ; j \geq 0
\]  
(2.4.5)

where \( \lambda h_j = \left(1 - \sum_{i=0}^{j} h_i\right) ; j \geq 0 \).

A stable recursive scheme for the computation of probabilities \( \{Q_{1,j}; i = 1,2, j \geq 0\} \) in terms of \( \{\psi_j; j \geq 0\} \) follows directly by combining (2.3.4) with (2.4.4) and (2.4.5).

Now let us consider the PGF \( Q_i(z) \) of \( \{Q_{1,j}; j \geq 0\} \) for \( i \in \{1,2\} \) which follows from equation (2.4.4) and (2.4.5) on utilizing equation (2.3.7). Then after routine algebraic manipulation we get

\[
Q_1(z) = \frac{(1-p)\left[1 - B_1^*(\lambda - \lambda z)\right]}{\left[(1-p) + pB_1^*(\lambda - \lambda z)\right]B_1^*(\lambda - \lambda z) - z}
\]  
(2.4.6)

and

\[
Q_2(z) = \frac{p(1-p)\left[1 - B_1^*(\lambda - \lambda z)\right]B_1^*(\lambda - \lambda z)}{\left[(1-p) + pB_1^*(\lambda - \lambda z)\right]B_1^*(\lambda - \lambda z) - z}. \tag{2.4.7}
\]

Let \( Q(z) \) be the PGF of the queue size distribution due to total busy periods, then

\[
Q(z) = Q_1(z) + Q_2(z)
\]

\[
= \frac{(1-p)\left[1 - \left((1-p) + pB_1^*(\lambda - \lambda z)\right)B_1^*(\lambda - \lambda z)\right]}{\left[(1-p) + pB_1^*(\lambda - \lambda z)\right]B_1^*(\lambda - \lambda z) - z}. \tag{2.4.8}
\]
Hence, the relationship between \( Q(z) \) and \( \psi(z) \) is given by

\[
Q(z) = \frac{[1 - (1-p) + pB'(\lambda - \lambda z)]B_1' \psi(z)}{(1-z)[(1-p) + pB_2' \psi(z)]}
\]  

(2.4.9)

Note that for \( p = 0 \), this result is consistent with existing literature e.g. see page -58 of Kashyap and Chaudhry (1988).

2.4.1 A particular case

Suppose that the service time of FPS is general and that of the SPS is exponential with mean \( \frac{1}{\mu_2} \), the distribution being independent of each other. In this case \( B'_1(s) = \left( \frac{\mu_2}{\mu_2 + s} \right) \) and therefore from equation (2.4.6), (2.4.7), (2.4.8) and (2.4.9) we have

\[
Q_1(z) = \frac{(1-p)(\mu_2 + \lambda - \lambda z)B_1' \psi(z)}{[(1-p)(\lambda - \lambda z) + \mu_2]B_1' \psi(z) - z(\mu_2 + \lambda - \lambda z)}
\]

\[
Q_2(z) = \frac{p(1-p)(\mu_2 + \lambda - \lambda z)B_1' \psi(z)}{[(1-p)(\lambda - \lambda z) + \mu_2]B_1' \psi(z) - z(\mu_2 + \lambda - \lambda z)}
\]

\[
Q(z) = \frac{(1-p)[(1-p)B_1' \psi(z)/\mu_2 + \lambda - \lambda z] + p(\lambda - \lambda z)B_1' \psi(z)}{[(1-p)(\lambda - \lambda z) + \mu_2]B_1' \psi(z) - z(\mu_2 + \lambda - \lambda z)}
\]

and \( \psi(z) = \frac{(1-p)(1-z)[(1-p)(\lambda - \lambda z) + \mu_2]B_1' \psi(z)}{[(1-p)(\lambda - \lambda z) + \mu_2]B_1' \psi(z) - z(\mu_2 + \lambda - \lambda z)} \)
Note that the above results are consistent with the result obtained by Madan [2000(a)].

2.5 Busy period distribution

The LST of the busy period distribution of this $M/G/1$ queue with two phases of service channel can be derived as follows. We define busy period as a length of time interval that makes the server busy and it continues to the instant when the server becomes free again and denote

$$T_b = \text{length of the busy period}$$

and $I = \text{length of the idle period}$.

Let $T^*_b(s)$ be the LST of the busy period distribution, then utilizing the argument of Takács (1963) we have

$$T^*_b(s) = \beta^*[s + \lambda T^*_s(s)]$$

where $\beta^*(s) = [1 - p + pB^*_2(s)]B^*_1(s)$ is the LST of $B$ i.e. our modified service time distribution.

The expected length of the busy period is given by

$$E(T_b) = \left. \frac{d}{ds} T^*_b(s) \right|_{s=0}$$

$$= \frac{\beta^{(0)}}{(1 - \rho)} + \frac{p\beta^{(0)}_2}{(1 - \rho)^2}$$
For \( p = 0 \) (i.e. there is no additional SPS in the system) the above equation reduces to the standard \( M/G/1 \) queue. Now since \( E(I) = \frac{1}{\lambda} \), therefore the fraction of time the server remains busy is

\[
\frac{E(T_s)}{E(I) + E(T_s)} = \lambda \left[ \frac{\beta_1^{(0)}}{1 - \rho} + \frac{p \beta_2^{(1)}}{1 - \rho} \right]
\]

\[
= \text{Prob} \ [\text{the server is busy with FPS}]
\]

\[
+ \text{Prob} \ [\text{the server is busy with additional SPS}];
\]

as expected.

2.6 Waiting time distribution

In this section, we derive the \( LST \) of the waiting time distribution of a test customer for this model. Let \( W_q(s) \) be the distribution function of the waiting time of a test customer for this model, then utilizing the distributional form of Little's law [\( \text{e.g. see Keilson and Servi (1988) or section 1.7 of Chapter I} \)] we may write

\[
W_q(\lambda - \lambda z)B^*(\lambda - \lambda z) = \psi(z)
\]  

(2.6.1)

Now, setting \( s = \lambda(1 - z) \) in the above equation (2.6.1) and using (2.3.7) we get on simplification

\[
W_q^*(s) = \frac{s(1 - \rho)}{s - \lambda[1 - B^*(s)]}
\]

(2.6.2)

where \( B^*(s) = (1 - p + pB_2^*(s))B_1^*(s) \). This is the well known Pollaczek Khinchine transform formula for waiting time distribution.
Note that for $p = 0$, we get the LST of the waiting time distribution for the classical $M/G/1$ queueing model.

Rewriting the equation (2.6.2) as

$$W_0^*(s) = \frac{(1-\rho)}{1 - \rho G_0^*(s)}$$

(2.6.3)

where $G_0^*(s) = \frac{[1 - \beta^*(s)\beta(s)]}{s\beta_1}$ and $\beta_i = \beta_i^0 + p\beta_i^0$ is the first raw moment of our modified service time distribution.

Thus the waiting time $W_0$ involves residual service time $X$ (of the unit in service at the instant the test unit arrives) as is to be expected. Expanding (2.6.3) we get

$$W_0^*(s) = (1 - \rho) \left[ 1 + \sum_{k=1}^{\infty} \rho^k \{G_0^*(s)\}^k \right]$$

$$= (1 - \rho) + (1 - \rho)\sum_{k=1}^{\infty} \rho^k \{G_0^*(s)\}^k$$

(2.6.4)

Inverting (2.6.4) we get the probability density function $w_q(x)$ of $W_0$, we get

$$w_q(x) = (1 - \rho)\delta(x) + (1 - \rho)\sum_{k=1}^{\infty} \rho^k m^k(x), x \geq 0$$

(2.6.5)

where $\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$

and $m^k(x) = \left\{ (1 - \rho) + p\beta_1^k(x) \right\} \beta_2^k(x)$, is the probability density function of $X$ and $m^k$ denotes the k-fold convolution of $m^0(x)$ with itself.

Now having the LST of the waiting time distribution, the expected waiting time can easily be obtained from equation (2.6.2) and is given by

$$E(W_0) = \left. \frac{W_0^*(s)}{ds} \right|_{s=0}$$
\[
\frac{\lambda \beta_2}{2(1-p)};
\]

where \( \beta_2 = \beta_1^{(2)} + 2p\beta_1^{(1)} \beta_2^{(1)} + p^2 \beta_2^{(2)} \) is the second raw moment of our modified service time distribution.

### 2.7 Recursive Solution of the Departure Point Queue Size Distribution

In this section an attempt has been made to obtain the recursive solution of the queue size distribution at departure epoch. After some algebraic manipulation one may use equation (2.3.4) to compute \( \psi_j \)'s recursively.

\[
\psi_{j+1} = \frac{\psi_j}{pc_j + (1-p)g_j} \cdot \frac{\psi_0[pc_j + (1-p)g_j]}{\psi_0} \sum_{i=0}^{j} \frac{\psi_i[pc_{j-i} + (1-p)g_{j-i}]}{[pc_0 + (1-p)g_0]}; \quad j \geq 0.
\]

This method runs into the problem of numerical intractability because of the subtraction involved. However equation (2.3.4) can be rearranged as follows

\[
\psi_1 = \frac{1 - [pc_0 + (1-p)g_0]}{[pc_0 + (1-p)g_0]} \psi_0
\]

\[
\psi_2 = \frac{1 - [pc_0 + (1-p)g_0] - [pc_1 + (1-p)g_1]}{[pc_0 + (1-p)g_0]} (\psi_0 + \psi_1)
\]

\[
\psi_3 = \frac{1 - [pc_0 + (1-p)g_0] - [pc_1 + (1-p)g_1] - [pc_2 + (1-p)g_2]}{[pc_0 + (1-p)g_0]} (\psi_0 + \psi_1 + \psi_2)
\]

\[
+ \frac{[pc_2 + (1-p)g_2]}{[pc_0 + (1-p)g_0]} \psi_2.
\]

Now proceeding in similar manner, we get
\[ \psi_{j+1} = \frac{1 - \left[ \sum_{i=0}^{\infty} \{ pc_i + (1 - p)g_i \} \left( \sum_{i=0}^{j} \psi_i \right) + \sum_{i=2}^{\infty} \sum_{k=j+i}^{\infty} \frac{pc_k + (1 - p)g_k}{pc_0 + (1 - p)g_0} \right]}{pc_0 + (1 - p)g_0}; \quad j \geq 2 \]

These equations, involving only sums of positive numbers are very stable and yield a good numerical method for computing \( \{ \psi_j; (j \geq 0) \} \).

Note that it even yields a simple method of truncation, for a given \( \eta > 0 \), stop the computation at \( j' \) if \( \sum_{i=0}^{j'} \psi_i \geq 1 - \eta \) and set \( \psi_k = 0 \) for all \( k \geq j \).

2.8 A Simple Numerical Illustration

In this section we present some numerical results on state probabilities for various combination of FPS, SPS and different arrival rates. For the computation of \( \psi_j \)'s, we assume that FPS as well as the SPS time distributions follow exponential distribution with probability distribution function \( B_i(x) = 1 - e^{-\mu_i x} \), for \( x > 0 \) and \( i = 1, 2 \), so that

\[
\begin{align*}
g_j &= \left( \frac{\mu_1}{\mu_1 + \lambda} \right)^j; \quad j \geq 0, \\
h_j &= \left( \frac{\mu_2}{\mu_2 + \lambda} \right)^j; \quad j \geq 0, \\
c_j &= \sum_{i=0}^{j} g_i h_{j-i}; \quad j \geq 0
\end{align*}
\]

and \( \rho = \lambda \left[ 1 + \frac{p}{\mu_1 + \mu_2} \right] \).

Now for different values of \( \mu_1 \) and \( \mu_2 \) (provided \( \mu_1 < \mu_2 \)), \( \psi_j \)'s are computed for various values of \( \lambda \) and \( p \) which are shown in the following tables.
Table 2.8.1

Different values of $\psi_j$'s for $\mu_1=1.1$, $\mu_2=2.5$, two distinct values of $X$ and for various values of $p$.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda=1.0$</th>
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Table 2.8.2

Different values of $\psi_j$'s for $\mu_1=1.8$, $\mu_2=3.2$, two distinct values of $\lambda$ and for various values of $p$.

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Table 2.8.3

Different values of $\psi_j$'s for $\mu_1 = 1.5$, $\mu_2 = 2.5$, two distinct values of $\lambda$ and for various values of $p$.

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2.9 Concluding Remark

Model considered in this chapter can be easily generalized to batch arrival cases. In case of batch arrival with Compound Poisson input instants of arrivals where the random variable $X$ denotes the batch size with PGF $a(z)$, the corresponding equation (2.3.7) will hold with $B'(\lambda - \lambda a(z))$ in place of

$$B'(\lambda - \lambda z) = (1-p) + pB'_y(\lambda - \lambda z)\beta'_y(\lambda - \lambda z)$$

and $\rho = \lambda a^{(0)}B^{(0)} + p\beta^{(0)} < 1$, where $a^{(0)}$ is the first moment of $X$. 

(1-p) + pB'_y(\lambda - \lambda z)\beta'_y(\lambda - \lambda z)$
Moreover, the recursive solution for this model with batch arrival is almost similar to the result obtained in section 2.7 of this chapter. Therefore we omit the detailed analysis of the recursive solution for the batch arrival of this type of queueing model.