CHAPTER I

INTRODUCTION

1.1 Classical Theory:

The basic principles of conservation of mass, momentum and energy hold good for all the materials irrespective of their constitution. It is expected that these principles are not sufficient to describe the behaviour of an individual material uniquely. In fact, two different material bodies having the same geometry and mass distribution, and subjected to identical external agents respond differently. These differences in response are caused by internal constitution of the matter. In order to take account of the nature of different materials, we must, therefore, find additional equations identifying the basic characteristics of the body with respect to the response sought. This is done by introducing models appropriate to the particular class of phenomena under scrutiny. Such models define ideal materials. For the ideal material to represent a physical material adequately, there exist certain physical principles, which must be satisfied by all such models. After these requirements are met, there still remain certain unknown response coefficients which are determined by experiments. Once this is done, the theory is fully constructed and ready for application. It remains acceptable until such times as once again the phenomena predicted by the theory do not agree with the experimental results.

The classical theory of isotropic viscous incompressible fluids is based on a linear constitutive equation

\[ \sigma'_{ij} = 2\mu_1 e_{ij}, \quad e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \]  

(1.1.1)
where \( \mu_l \) is the coefficient of viscosity and \( v_i \) are velocity components, \( \sigma_{ij}^\prime \) is the extra stress related to stress tensor \( \sigma_y \) as

\[
\sigma_{ij} = -p \delta_{ij} + \sigma_{ij}^\prime
\]  

(1.1.2)

where \( p \) is indeterminate hydrostatic pressure and \( \delta_{ij} \) is kronecker delta. The constitutive equation (1.1.1) for isotropic fluids was suggested by Newton (1687). A fluid obeying the equation (1.1.1) is called a Newtonian fluid. The classical linear theory of isotropic, viscous fluids based on this constitutive equation (1.1.1) provides a satisfactory approximation to that of ordinary mobile liquids like water, air, glycerine, honey and many thin oils. It is capable of explaining the phenomena of lift, skin friction, form drag, separation and secondary flows etc. However, it fails, to explain a number of phenomena observed in several other fluids. These fluids show marked deviations from Newtonian behaviour.

1.2 Development of Theory of Non-Newtonian Fluid Flows:

The subject of Fluid Dynamics has made tremendous advances since Euler gave his famous equation of fluid flow for perfect (non-viscous incompressible and compressible) fluid flow in 1755 in his paper “General principles of the motion of fluids”. The perfect or ideal fluids are characterized by the assumption that for them the stress tensor is a linear function of the rate of strain tensor and the stress vector or a plane surface in contact with a fluid is normal to the surface. Euler’s partial differential equations were non-linear but solutions were obtained flows past circular and elliptic cylinders and past spheres and ellipsoids. The ideal fluid theory gave excellent results for certain classes of problems such as wave formation and tidal motions.
The ideal fluid theory however led to the conclusion that when a solid moves through a fluid at infinity, it experiences no drag. This was in contradiction with observations (D'Alembert's paradox). To explain the drag, the concept of a viscous Newtonian fluid (in which the stress tensor is a linear function of the rate of strain tensor but stress vector on a surface is not normal to the surface with which it is in contact) was introduced. Navier in 1821 and Stokes in 1845 obtained the equations of motion for these fluids independently and from different considerations.

Using the Navier-Stokes equations and the no-slip conditions at the surfaces past which viscous fluids flow, exact solutions were obtained for flows between plates and in pipes and annuli between translating and rotating coaxial cylinders. Exact solutions were also obtained for flows past suddenly accelerated plane walls near oscillating flat plates, near rotating discs and for stagnation flows. Good approximate solutions were also obtained for very slow motions and for lubrication flows. The solutions obtained were interesting but exact solutions for problems which were important from the engineering point of view were not yet available.

A great breakthrough came in 1904, when Prandtl proposed his assumptions that when viscous fluid flows past a surface, the viscous effect is dominant in a thin layer (called the boundary layer) near the surface and outside this layer, the flow may be regarded as that of an ideal fluid. The boundary layer theory had tremendous achievements to its credit and transformed fluid dynamics into a discipline of great engineering importance. It gave correct predictions for resistance to motion of ships and for lift and drag on aeroplane wings.

About five decades ago phenomena were observed which could not be explained
on the basis of viscous fluid theory. When the assumptions of this theory were examined, it was considered feasible to relax the assumption of linearity between stress and rate of strain tensor. More general non-linear relations between stress and rate of strain were postulated. At a later stage, it was found that flow behaviour of some materials could be explained by postulating linear and non-linear relations between stress vector and strain tensor, rate of strain tensor and even rate of stress tensor. It immediately led to the development of theory of non-Newtonian fluid flows.

1.3 Inadequacy of Linear Theory and Generalization:

A number of significant experiments have been carried out revealing non-Newtonian characteristics of liquids. In these experiments, a number of new phenomena have been observed in a vast number of liquids of great technological and industrial importance such as thick oils, paints, lubricants, starch solutions, rubber toluene solutions, colloids and suspensions and poly-isobutylene solutions in mineral oils.

Merrington (1943) observed that when a solution of rubber in mineral oil is forced through a straight pipe, the fluid swells after emerging out of the tube. This effect is called Merrington effect. Garner and Nissan (1946) noticed that if a cylindrical rod is rotated in a vessel containing metallic soap, hydrocarbon gels, high polymer solutions or a variety of other such materials, the fluid rises up at the rod to a considerable extent. This phenomena was demonstrated experimentally by Weissenberg (1947, 1949) and is known as Weissenberg effect. Weissenberg (1947, 1949) performed experiments with various types of liquids. In his experiments, the liquids were sheared in a gap between an outer vessel rotated at various constant angular velocities and an inner cylinder that could be either fixed rigidly...
or allowed to move up and down. The phenomena of rising the liquid in a direction perpendicular to the plane of shearing is called normal stress effect (poynting effect), and that of drawing of the liquid towards the axis of rotation against the action of the centrifugal force, when the liquid is sheared between two rotating plates is called centripetal pump effect. Reiner (1957) found that this effect is present even in air if the distance between the bases of the cylinders is less than a certain critical distance. The classical linear theory is incapable of explaining any of the above mentioned phenomena.

George Cantor had stated that the essence of Mathematics lie in its freedom. This freedom from the bondage of linearity of relationship between stress and strain rate tensors led to a great deal of creative work in non-Newtonian fluid flow theory. There are two types of thoughts prevailing in the literature. The first type is that the new constitutive equation should emerge from the generalization of the one-dimensional constitutive equation suggested empirically by experiments. Oldroyd, Walters and their co-workers share this opinion. On the otherhand Truesdell, Noll, Coleman, Ericksen and their co-workers have obtained the constitutive equations to account for the non-Newtonian phenomena by using the general idea of fluidity. Using the available Mathematical tools, they have approximated these constitutive relations to give the behaviour of certain real fluids.

Below, we present some of the important constitutive equations which are in frequent use in the literature.

1.4 Oldroyd and Walters Fluid:

Einstein (1906, 1911) tried to explain the behaviour of colloids and suspensions by considering that they consist of a large number of rigid spherical particles of equal radius
suspended in a Newtonian fluid of viscosity $\eta$. Frohlich and Sack (1946) discussed the flow of such a system but they assumed the suspended spherical particles to be elastic. Maxwell (1867) proposed the following constitutive equation on the basis of the combination of mechanical models for Newtonian fluid (a dash pot) a Hookian solid (a spring) to describe the behaviours of colloids and suspensions,

$$\left(1 + \lambda_1 \frac{d}{dt}\right) \sigma' = 2 \mu_1 e \quad (1.4.1)$$

where $\lambda_1$ is a material constant called stress relaxation time. The equation (1.4.1) shows that if strain becomes constant, the stress decays exponentially with time, thus showing the stress relaxation effect. Jeffrey (1929) generalized the relation (1.4.1) to take account of strain-rate relaxation effects also and proposed the following constitutive equation

$$\left(1 + \lambda_1 \frac{d}{dt}\right) \sigma' = 2 \mu_1 \left(1 + \lambda_2 \frac{d}{dt}\right) e \quad (1.4.2)$$

where $\lambda_2$ is strain-rate relaxation time. Frohlich and Sack (1946) have shown that the equation (1.4.2) can explain the behaviour of dilute colloids and suspensions. Toms and Strawbridge (1953) have determined the material constants $\mu_1, \lambda_1, \lambda_2$ experimentally for solutions of polymethyl metacrylate in $n$-butyl acetate. Oldroyd (1950) suggested that the generalization of one-dimensional empirical rheological equations of state, so that it is universally valid, should be done by using the coordinate system which is convected with the material (i.e. embedded with the material and deformed with it). He defined the convected differentiation of a mixed tensor $b_{i...k...}$ as

$$\frac{\delta}{\delta t} b_{i...k...} = \frac{\partial}{\partial t} b_{i...k...} + v^m b_{i...k...m} + \sum v^m_{ij} b_{i...k...m} - \sum v^m_{ik} b_{i...k...m} \quad (1.4.3)$$
where \( \Sigma, \Sigma' \) denote a sum of all similar terms, one for each covariant (contravariant) suffix.

With these ideas, Oldroyd generalized the equation (1.4.2) in many ways by replacing ordinary derivative with a convected one. Two of the possible generalizations are

\[
\left(1 + \lambda_1 \frac{\delta}{\delta t}\right) \sigma'_{ik} = 2 \mu_1 \left(1 + \lambda_2 \frac{\delta}{\delta t}\right) e_{ik} \quad (1.4.4)
\]

and

\[
\left(1 + \lambda_1 \frac{\delta}{\delta t}\right) \sigma''_{ik} = 2 \mu_1 \left(1 + \lambda_2 \frac{\delta}{\delta t}\right) e''_{ik}. \quad (1.4.5)
\]

The constitutive equations (1.4.4) and (1.4.5) represent different liquids through the one-dimensional model for both of them is same. The liquids governed by the constitutive equations (1.4.4) and (1.4.5) are called Oldroyd's liquid \( A \) and Oldroyd's liquid \( B \) respectively. The liquid \( A \) does not exhibit Weissenberg effect while liquid \( B \) does so. Both the constitutive equations (1.4.4) and (1.4.5) are linear. Oldroyd (1958) proposed a non-linear generalization of these two equations as

\[
\left(1 + \lambda_1 \frac{\delta}{\delta t}\right) \sigma_{ij} - \lambda_1 \left(\sigma_{ik} e_{kj} + \sigma_{kj} e_{ik}\right) + \mu_0 \sigma_{kk} e_{ij} \\
+ \nu_1 \sigma_{kl} e_{kl} \delta_{ij} = 2 \eta_0 \left[ e_{ij} + \lambda_2 \left(\frac{\delta}{\delta t} e_{ij} - 2 e_{ik} e_{kj} + \nu_2 e_{kl} e_{kl} \delta_{ij}\right) \right]. \quad (1.4.6)
\]

The equation (1.4.6) represents Oldroyd liquid \( A \) when

\[
\eta_0 > 0, \quad \lambda_1 = -\mu_1 > \lambda_2 = -\mu_2 \geq 0, \quad \mu_0 = \nu_1 = \nu_2 = 0 \quad (1.4.7)
\]

and liquid \( B \) when
\[ \eta_0 > 0, \quad \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geq 0, \quad \mu_0 = \nu_1 = \nu_2 = 0. \]  

(1.4.8)

Hence the liquid governed by equation (1.4.6) can be called generalized Oldroyd liquid.

By suitable choice of physical constants, the liquid (1.4.6) gives many of the essential features observed in non-Newtonian liquids. The effect of the elastic as well as non-linear terms in the equation (1.4.6) have been studied in a number of flow problems by Tanner (1962, 1963), Laslie (1961), Frater (1964), Sharma (1959) and Nanda (1963) etc. The effect of elasticity is to check the flow and it appears from the results that elastic strings are formed in the fluid.

Walters (1960) considered infinite number of Maxwell elements connected in parallel and proposed the following constitutive equation

\[ \sigma'_k(x,t) = 2 \int_{-\infty}^{t} \psi(t-t') e_{ik}(x,t') dt' \]  

(1.4.9)

where

\[ \psi(t-t') = \int_{0}^{\infty} \frac{N(\lambda)}{\lambda} \exp \left\{ - (t-t')/\lambda \right\} d\lambda. \]  

(1.4.10)

\( N(\lambda) \) being the distribution function of the relaxation time. To make the relation (1.4.9) universally valid, Walters replaced the ordinary integral in (1.4.9) with a convected one defined by Oldroyd (1950) and suggested the following two constitutive equations (1964):

\[ \sigma'^{ik}(x,t) = 2 \int_{-\infty}^{t} \psi(t-t') \frac{\partial x^m}{\partial x'^i} \frac{\partial x'^r}{\partial x^k} e_{mr}(x',t') dt', \]  

(1.4.11)

\[ \sigma''^{ik}(x,t) = 2 \int_{-\infty}^{t} \psi(t-t') \frac{\partial x^m}{\partial x'^i} \frac{\partial x'^m}{\partial x^k} e^{mr}(x',t') dt'. \]  

(1.4.12)
where \( x' = x'(x, t, t') \) is the position of time \( t' \) of the element which is instantaneously at the point \( x' \) at time \( t \). He referred (1.4.11) and (1.4.12) as liquids \( A' \) and \( B' \) respectively.

Walters has worked out a number of flow problems using the equation (1.4.12). Thomas and Walters have worked out the flow of the fluid governed by equation (1.4.12) through curved pipe of circular (1963) and elliptic (1965) cross sections. Beard and Walters (1964) have derived the two dimensional boundary layer equations and using them discussed the flow near a stagnation point. It has been shown by Walters (1962) that in the case of liquids with short memories (i.e. short relaxation times) the equation of state can be simplified to

\[
\sigma^{ik} = 2\eta_0 e^{ik} - 2k_0 \frac{\delta}{\delta t} e^{ik} \quad (1.4.13)
\]

where \( \eta_0 = \int_0^\infty n(\sigma) d\sigma \) is the limiting viscosity at small rate of shear, \( k_0 = \int_0^\infty \sigma N(\sigma) d\sigma \) and \( \frac{\delta}{\delta t} \) denotes the convected differentiation of a tensor quantity, which for any contravariant tensor \( b^{ik} \) is given by

\[
\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x^m} - \frac{\partial v^k}{\partial x^m} b^{im} - \frac{\partial v^l}{\partial x^m} b^{mk} \quad (1.4.14)
\]

where \( v_l \) is the velocity vector. This idealised model is a valid approximation of Walters liquid (Model \( B' \)) taking very short memories into account so that terms involving

\[
\int_0^\infty \sigma^n N(\sigma) d\sigma , \ n \geq 2 \quad (1.4.15)
\]

have been neglected.
1.5 Reiner-Rivlin Fluid:

Reiner (1945) used the Stoke's definition (1845) of fluidity that the stress $\sigma'$ is a function of strain-rate $e$, i.e.

$$\sigma' = f(e)$$  \hspace{1cm} (1.5.1)

using isotropy and homogeneity of the material, stress can be given by matrix polynomial in $e$, i.e.

$$\sigma' = \sum_{r=1}^{\infty} \mu_r (e)^r .$$  \hspace{1cm} (1.5.2)

Using Cayley-Hamilton theorem for matrices of rank 3, he showed that the relation (1.5.2) reduces to a polynomial of second degree and is given by

$$\sigma' = \mu_0 U + \mu_1 e + \mu_3 e^2$$  \hspace{1cm} (1.5.3)

where $\mu_0, \mu_1, \mu_3$ are functions of $a_r$'s and the invariants I, II, III of the strain-rate matrix $e$. In (1.5.3) $U$ denotes the unit matrix.

Rivlin (1947) also obtained a similar constitutive equation by assuming that the stress is a function of the velocity gradient and then satisfying the invariant requirement for an isotropic incompressible fluid, the equation suggested by Reiner and Rivlin as follows:

$$\sigma_{ij} = -p \delta_{ij} + \mu_1 e_{ij} + \mu_3 e_{ia} e^a j.$$  \hspace{1cm} (1.5.4)

A fluid governed by this constitutive equation (1.5.4) is known as Reiner-Rivlin fluid. The material coefficients $\mu_1$ and $\mu_3$ are generally regarded as constants.

The Reiner's theory based on (1.5.4) is capable of explaining Merrington effect,
Weissenberg effect, Poynting effect, Centripetal pump effect etc. Much theoretical work has been done on this fluid by a number of workers such as Reiner (1951), Braun and Reiner (1952), Srivastava (1958, 1958, 1959, 1961), Jain (1962, 1962) and Jones (1960, 1964) etc.

But the new developments have shown that the constitutive equation (1.5.4) represents no real fluid. In simple shearing flows, it gives that the normal stresses perpendicular to and along the plane of shearing are equal. This is in contradiction with the experimental fact that for most of the real fluids, the difference of the two normal stresses is a function of rate of shear. We shall see, later that this equation (1.5.4) does not give complete second-order corrections to linear equation (1.1.1). Rivlin and Ericksen presented a more general theory of fluids which succeeded to an extent to cover the defects of this theory.

1.6 Rivlin-Ericksen Fluid:

Rivlin and Ericksen (1955) proposed that the stress depends upon the spatial gradients of velocity, acceleration, 2nd acceleration, . . . , (M-1)th acceleration. By using the invariant requirements they showed that the stress must be given by an isotropic function of the tensor $A_{(N)ij}$, $N = 1, 2, \ldots, M$ as

$$\sigma'_{ij} = f_{ij} \left[ A_{(1)kl}, A_{(2)kl}, \ldots, A_{(M)kl} \right]$$  \hspace{1cm} (1.6.1)

where $A_{(N)ij}$ are known as Rivlin-Ericksen tensors and can be defined by successive material differentiation of the squared arc element $ds^2$ as

$$\frac{D^n}{Dt^n} (ds^2) = A_{(N)ij} \, dx^i \, dx^j$$  \hspace{1cm} (1.6.2)
where \( \frac{D}{Dt} \) is the material derivative defined as

\[
\frac{D\chi}{Dt} = \frac{\partial \chi}{\partial t} + v^i \chi_{,i}.
\] (1.6.3)

The recurrence formula for \( A_{(N)}_{ij} \) may be written as

\[
A_{(1)}_{ij} = v_{i,j} + v_{j,i} = 2 \epsilon_{ij},
\]

\[
A_{(2)}_{ij} = a_{i,j} + a_{j,i} + 2 \nu_m \nu_{m,j}, \quad \left( a_i \equiv \frac{Dv_i}{Dt} \right)
\]

and

\[
A_{(N+1)}_{ij} = A_{(N)k} \nu_{k,j} + A_{(N)}_{kj} \nu_{k,i} + \frac{D}{Dt} A_{(N)}_{ij}.
\] (1.6.4)

A fluid governed by the constitutive equation (1.6.1) is called Rivlin-Ericksen fluid of complexity \( M \). The simplest of such fluids are Reiner-Rivlin fluids when \( A_{(N)} = 0 \) for \( N \geq 2 \). Such fluids have been mentioned in the preceding article. The next important class of Rivlin-Ericksen fluids have the constitutive equation of the form

\[
\sigma'_{ij} = f_{ij} \left[ A_{(1)kl}, A_{(2)kl} \right].
\] (1.6.5)

For isotropic fluid, the equation (1.6.5) can be shown to reduce to

\[
\sigma'_{ij} = \mu_0 U + \mu_1 \left[ A_1 \right] + \mu_2 \left[ A_2 \right] + \mu_3 \left[ A_{(1)} \right]^2 + \mu_4 \left[ A_{(2)} \right]^2 + \mu_5 \left[ \left[ A_1 \right] \left[ A_2 \right] \right] + [A_2] \left[ A_1 \right] + \mu_6 \left[ \left[ A_{(1)} \right]^2 \left[ A_{(2)} \right] + \left[ A_{(1)} \right] \left[ A_{(2)} \right] \right]^2 \right] + \mu_7 \left[ \left[ A_{(1)} \right] \left[ A_{(2)} \right] \right]^2 + \mu_8 \left[ \left[ A_{(1)} \right]^2 \left[ A_{(2)} \right]^2 + \left[ A_{(2)} \right]^2 \left[ A_{(1)} \right]^2 \right] \right)
\] (1.6.6)

where, \( \mu_m, m = 0, 1, \ldots, 8 \) are scalar functions of the ten invariants of tensors \( A_{(1)} \).
Viscometric flows have been defined by Coleman and Noll (1959) and a general analysis of such flows have been given by Coleman and Noll (1959). Examples of viscometric flows are steady shearing flow, steady Coutte flow, steady Poiseuille flow, and steady helical flow in an annulus.

For viscometric flows, this class of fluids is indistinguishable from general Rivlin-Ericksen fluid since for such fluids, all tensors \([A(\alpha_{\ell})]\) except \([A(\alpha)]\) and \([A(2)]\) vanish. Rivlin (1956) found the exact solutions for viscometric flows using the constitutive equation (1.6.6) and expressed them in eight material constants. Markovitz (1957) observed that \(\mu_m, m = 4, 5, \ldots, 8\) may be omitted without affecting the solutions. Then the reduced constitutive equation takes the form

\[
\sigma_{ij} = -p \delta_{ij} + \mu_1 A_{(1)}^{ij} + \mu_2 A_{(2)}^{ij} + \mu_3 A_{(1)i\alpha} A_{(1)}^{\alpha j} \tag{1.6.7}
\]

where \(p\) is indeterminate isotropic pressure.

Although the general Rivlin-Ericksen fluid (1.6.1) accounts for shear dependent viscosity and normal stress effects it shears with its special case, the Newtonian fluid of the serious shortcoming of not accounting for the phenomena which gives rise to Boltzmann's Theory of linear viscoelasticity, that; gradual stress relaxation. When \([A(\alpha_{\ell})] = 0\) for \(\alpha = 1, 2, \ldots, M\), the extra stress \([\sigma']\) on a Rivlin-Ericksen fluid cannot change in time, but it does in actual relaxation experiments on viscoelastic materials such as high polymeric fluids. Stated differently when a Rivlin-Ericksen fluid is brought to rest, its memory does not fade gradually, but rather precipitously \([\text{Coleman (1962)}]\).
1.7 Simple Fluid:

The definition of a simple fluid introduced by Noll(1955, 1958) expressed a concept of fluidity more general than expressed by most definitions of non-Newtonian fluids previously. Noll (1958) defined a simple fluid as a substance such that all local states of equal mass and density are intrinsically equivalent in response, with all observable differences in response being due to definite differences in past history. The simple fluid is more general than even Rivlin-Ericksen fluid. It can exhibit the phenomena of "shear dependent viscosity", "normal stress difference" and "gradual stress relaxation". It represents many real materials not covered by the constitutive equations derived earlier.

An incompressible simple fluid is governed by the constitutive equation of the form

$$\sigma'(t) = \sum_{s=0}^{\infty} \left[ H_i^{(t)}(s) \right]$$

(1.7.1)

where the functional $\mathfrak{F}$ is isotropic and obeys the following identity for every orthogonal tensor $Q$

$$\mathfrak{F} \left[ Q H_i^{(t)}(s) Q^T \right] = Q \mathfrak{F} \left[ H_i^{(t)}(s) \right] Q^T$$

(1.7.2)

$Q^T$ is the transpose of $Q$ and has its trace fixed by some convection such that

$$t \cdot \mathfrak{F} \left[ H_i^{(t)}(s) \right] = 0.$$  

(1.7.3)

The history function $H_i^{(t)}(s)$ is given by

$$H_i^{(t)}(s) = F_i(t-s) F_i^T(t-s)$$

(1.7.4)
where $F_i(t-s)$ is the deformation gradient tensor at time $(t-s)$ relative to the configuration at present time $t$.

Noll (1958) has shown that the Rivlin-Ericksen tensors $A_{(N)}$ defined in article 1.6 are given by

$$A_{(N)} = \left[ (-1)^N \frac{D^N}{Ds^N} \left( H_i^j(s) \right) \right]_{s=0}.$$  \hspace{1cm} (1.7.5)

The constitutive equation (1.7.1) is equivalent to the Noll's definition (1958) of the simple fluid. There are two main approaches of approximating the constitutive equation (1.7.1) so that it results in constitutive equation of a perfect fluid, Newtonian fluid and fluids of higher order.

The first approach due to Coleman and Noll (1960) is to compare the responses of the same fluid to more and more severe retardation of a given motion. For a given history $H_i^j(s)$ they constructed a new history $H_a(s)$ as

$$H_a(s) = H_i^j(\alpha s), \quad 0 \leq s \leq \alpha.$$ \hspace{1cm} (1.7.6)

The retardation history $H_a(s)$ corresponds to a kinematical history essentially the same as that which give rise to $H_i^j(s)$ with the exception that this new history is carried out as lower rate. The extra stress $\sigma'_{(a)}$ corresponding to the retarded history $H_{(a)}(s)$ obey the approximation formulae

$$\sigma'_{(a)} = \mu_1 A_{(1)} y_j,$$ \hspace{1cm} (1.7.7)

$$\sigma'_{(a)} = \mu_1 A_{(1)} y_j + \mu_2 A_{(2)} y_j + \mu_3 A_{(3)} A_{(1)}^{\alpha} j,$$ \hspace{1cm} (1.7.8)

which are complete within terms of order one and two respectively in $\alpha$. The equation
(1.7.7) is same as (1.1.1) and represents a Newtonian fluid and equation (1.7.8) is same as (1.6.7) and represents an incompressible second-order fluid. We can have higher order fluids in a similar manner.

The second approach due to Truesdell (1964) is to compare the responses to the same deformation of different fluids having smaller and smaller natural time. In a general analysis of viscometric flow in incompressible simple fluids, Coleman and Noll (1959, 1959, ) have shown that the solution of any viscometric flow may be expressed in terms of three material constants \( \eta(k) \), \( \tau_1(k) \) and \( \tau_2(k) \), \( k \) being the shear-rate in simple shearing flow. For this flow the stresses \( \sigma_{yz} \) and \( \sigma_{zx} \) vanish and

\[
\begin{align*}
\sigma_{xy} &= k \eta(k), \\
\sigma_{xx} - \sigma_{zz} &= \tau_1(k), \\
\sigma_{yy} - \sigma_{zz} &= \tau_2(k).
\end{align*}
\]

For consistency with hydrostatics, \( \tau_1(0) = \tau_2(0) = 0 \). We assume therefore that \( \eta(k) > 0 \) for all \( k \) and continuously differentiable at \( k = 0 \) and \( \tau_1(k) \) and \( \tau_2(k) \) are twice continuously differentiable at \( k = 0 \).

Truesdell (1964) constructed a time function \( \lambda \) for simple fluid as

\[
\lambda(k) = \frac{\sqrt{[\tau_1(k)]^2 + [\tau_2(k)]^2}}{k^2 \eta(k)}
\]

(1.7.9)

\( \eta(k) \) is non-negative, even function and vanishes identically, since \( \eta(k) > 0 \) for such fluids and only for such fluids as show no normal stress effect in viscometric flows. He also defined the natural viscosity \( \mu \) and natural time \( \lambda \) as the values of \( \eta(k) \) and \( \lambda(k) \) in the limit of slow shearing
\[ \mu = \eta(0), \quad \lambda = \lim_{k \to 0} \left[ \lambda(k) \right]. \]  

(1.7.10)

Truesdell (1964) supposed the flow given once and for all but compare the responses of a suitable sequence of fluids. Namely, using the dimensional invariant specification of (1.7.1), we keep \( \mu \), \( k \) and dimensionless functional fixed but let the natural time \( \lambda \) approaches zero. Coleman-Noll's approach is that \( \lambda \) is kept fixed but \( k \) tends to zero. In the theory of a general simple fluid the dimensionally invariant statement requires the specification of \( \lambda, \mu \) and the dimensionless functional of a dimensionless tensor argument function of a dimensionless variable. So, we allow the constitutive functional \( \mathcal{F} \) to depend upon \( \lambda \) and \( \mu \) as scalar parameters, since none of the various deductions made from it will be invalidated because they all concern transformations leaving invariants to make it dimensionally invariant. Then it can be shown that

\[ \sigma' = \mathcal{F} \left[ H_i^j(s); \mu; \lambda \right] \]  

(1.7.11)

may be replaced by

\[ \frac{\lambda}{\mu} \sigma' = \mathcal{F}(0) \left[ H_i^j(s) \right] \]  

(1.7.12)

where \( \mathcal{F}(0) \) is a dimensionless isotropic functional. Now, if \( H_i^j(s) \) is a fixed deformation in the class of \( \mathcal{F} \) functions \([H]\) and is \( n \) times continuously differentiable at \( s = 0 \), and again if we consider that \( \mathcal{F}(0) \) is a fixed dimensionless constitutive functional which is obviously in the class \( \mathcal{F} \) and continuous at zero, and a fixed \( \mu \) but let \( \lambda \) approaches to zero, then with arbitrary small error

\[ \sigma' \approx \mathcal{F}_2 \left[ \lambda^1 A_{(1)}, \lambda^2 A_{(2)}, \ldots, \lambda^r A_{(r)}, \ldots, \lambda^n A_{(n)} \right] \]  

(1.7.13)
where \( S(2) \) is isotropic and so can be reduced to a polynomial with arbitrary small error. This polynomial when arranged as a polynomial in \( \lambda \) becomes

\[
\sigma' = \mu \sum_{r=1}^{n} \lambda^{r-1} Q_r \left[ A_{(1)}, A_{(2)}, \ldots, A_{(n)} \right] \quad (1.7.14)
\]

where dots stand for terms of degree greater than \((n-1)\) in \( \lambda \) and \( Q_r \) is most general isotropic polynomial of degrees \( a_1, a_2, \ldots, a_n \) in the components of tensors \( A_{(1)}, A_{(2)}, \ldots, A_{(n)} \) such that

\[
a_1 + 2a_2 + 3a_3 + \ldots + n a_n = q. \quad (1.7.15)
\]

Putting \( q = 1 \), we get from equation (1.7.15) that

\[
\alpha_1 = 1, \quad \alpha_2 = \alpha_3 = \ldots = 0
\]

and hence from equation (1.7.14) we get,

\[
\sigma + p U = \mu \alpha A_{(1)}. \quad (1.7.16)
\]

If \( \alpha = 1 \), then the constant of proportionality for such a fluid is \( \mu \) between \( A_{(1)} \) and deviatoric stress tensor. This equation (1.7.16) represents Newtonian fluid. Taking \( q = 2 \), the equation (1.7.15) gives \( \alpha_3 = \alpha_4 = \ldots = 0 \) and the equation (1.7.14) can be written as

\[
\sigma + p U = \mu \left[ A_{(1)} + \lambda \left\{ \beta A^2_{(1)} + \beta' A_{(2)} \right\} \right]. \quad (1.7.17)
\]

The equation (1.7.17) reduces to the equation (1.7.8) if

\[
\mu = \mu_1, \quad \mu \lambda \beta' = \mu_2, \quad \mu \lambda \beta = \mu_3
\]

and so it represents an incompressible second-order fluid. We can write the constitutive equations of higher orders in this way. All the three material constants can be determined
from the viscometric flows for any real material. Hence the constitutive equation of state for any material behaving as a second-order fluid can be determined completely. This property is not shared by fluids of higher order and viscometric flows do not suffice to give all the material constants. Markovitz and Coleman (1964) proved that $\mu_2$ is negative, from thermodynamic considerations and experimentally it has been found so.

In this thesis, we have taken for our study Walters liquid (Model B') with short memories governed by the constitutive equation (1.4.13) and an incompressible second-order fluid governed by the constitutive equation (1.7.8) or (1.7.17) in preference to other models whose constitutive equations are more complicated. The constitutive equation of an incompressible second-order fluid in tensor notation is as follows

$$\sigma_{ij} = -p\delta_{ij} + \mu_1 A_{(1)ij} + \mu_2 A_{(2)ij} + \mu_3 A_{(1)ij} A_{(1)\alpha} \delta_{\alpha j}$$

(1.7.18)

where

$$A_{(1)ij} = v_{i,j} + v_{j,i},$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v^m_{,i} v_{m,j}. $$

Here $v_i$ and $a_i$ are velocity and acceleration components, $p$ is an indeterminate pressure which differs, in general form the mean pressure $\overline{p} = -\frac{1}{3} \sigma_{ii}$. The material coefficients $\mu_1, \mu_2, \mu_3$ are taken constants with $\mu_1$ and $\mu_3$ as positive and $\mu_2$ as negative.

1.8 Dynamical Equations For Walters Fluid (Model B') and Incompressible Second-order Fluid:

In this article, we present the dynamical equations for Walters fluid (Model B') and
incompressible second-order fluid:

**(a) Equations in Cartesian Coordinates:**

Let \( x, y, z \) be the rectangular cartesian coordinates and \( u, v, w \) be the velocity components in these directions. Then the equation of continuity is

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\] (1.8.1)

The equations of motion are

\[
\rho a_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z},
\]

\[
\rho a_y = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z},
\]

\[
\rho a_z = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z},
\]

(1.8.2)

where

\[
a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z},
\]

\[
a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z},
\]

\[
a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z},
\]

are the acceleration components in \( x, y, z \) directions and the components of stress for Walters fluid (Model B') are given by
\[
\begin{align*}
\sigma_{xx} &= -p + 2\eta_0 \frac{\partial u}{\partial x} - 2k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial x} - \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial x} \right] + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial y} + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \frac{\partial u}{\partial z} \\
\sigma_{yy} &= -p + 2\eta_0 \frac{\partial v}{\partial y} - 2k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \frac{\partial v}{\partial y} - \left( \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \frac{\partial v}{\partial y} \right] + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial x} + \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial v}{\partial z} \\
\sigma_{zz} &= -p + 2\eta_0 \frac{\partial w}{\partial z} - 2k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \frac{\partial w}{\partial z} - \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \right) \frac{\partial w}{\partial z} \right] + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial w}{\partial x} + \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z} \right) \frac{\partial w}{\partial y} \\
\sigma_{xy} &= \eta_0 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \frac{\partial v}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \frac{\partial u}{\partial y} \\
\sigma_{xz} &= \eta_0 \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \right) \right] + \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x} \right) \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \right) \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \right) \frac{\partial u}{\partial z} \\
\sigma_{yz} &= \eta_0 \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) - \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \right) \right] + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial v}{\partial z} 
\end{align*}
\]
\[
- \left\{ \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial w}{\partial y} \right\} - \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial v}{\partial x} \right\}
\]

\[
+ \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right) \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} - \left\{ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right) \frac{\partial v}{\partial y} \right\}
\]

\[
\frac{(dw)^2}{dz} + \frac{1}{4} \frac{dc}{dz} + \frac{1}{2} \frac{dv}{dy} + \frac{1}{2} \frac{dw}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{du}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{du}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{dv}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{dv}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{dw}{dy} \frac{dz}{dy} + \frac{1}{2} \frac{dw}{dy} \frac{dz}{dy}
\]

(1.8.3)

The stress components for incompressible second-order fluid are given by

\[
\sigma_{xx} = -p + 2\mu_1 \frac{\partial u}{\partial x} + 2\mu_2 \left[ \frac{\partial^2 x}{\partial x^2} + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right] + \mu_3 \left[ 4 \left( \frac{\partial u}{\partial x} \right)^2 \right]
\]

\[
+ \left[ \frac{\partial v}{\partial x} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial y} \right) \right]^2
\]

\[
\sigma_{yy} = -p + 2\mu_1 \frac{\partial v}{\partial y} + 2\mu_2 \left[ \frac{\partial^2 y}{\partial y^2} + \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] + \mu_3 \left[ 4 \left( \frac{\partial v}{\partial y} \right)^2 \right]
\]

\[
+ \left[ \frac{\partial v}{\partial y} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial x} \right) \right]^2
\]

\[
\sigma_{zz} = -p + 2\mu_1 \frac{\partial w}{\partial z} + 2\mu_2 \left[ \frac{\partial^2 z}{\partial z^2} + \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \mu_3 \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right]
\]

\[
+ \left[ \frac{\partial v}{\partial z} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial w}{\partial z} \left( \frac{\partial w}{\partial y} \right) \right]^2
\]

\[
\sigma_{xy} = \mu_1 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \mu_2 \left[ \frac{\partial^2 y}{\partial x \partial y} + 2 \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right]
\]

\[
+ \mu_3 \left[ 2 \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2 \frac{\partial v}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right]
\]

\[
\sigma_{yz} = \mu_1 \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \mu_2 \left[ \frac{\partial^2 y}{\partial z \partial y} + 2 \left( \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \frac{\partial w}{\partial y} \right) \right]
\]
Here \( \rho \) is the density of the fluid.

(b) Equations in Cylindrical Polar Coordinates:

Let \( r, \theta, z \) be the cylindrical polar coordinates and \( u, v, w \) be the velocity components in these directions. The equation of continuity in \((r, \theta, z)\) system is

\[
\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \quad (1.8.6)
\]

The equations of motion are

\[
\rho a_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r},
\]

\[
\rho a_\theta = \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r} \sigma_{r \theta},
\]

\[
\rho a_z = \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \sigma_{rz}, \quad (1.8.7)
\]

where acceleration components are

\[
a_r = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} + \frac{v^2}{r} - \frac{u^2}{r},
\]

\[
a_\theta = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{uv}{r},
\]

\[
a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} + \frac{uv}{r},
\]
Stress components in Walters fluid (Model B') are

\[
\sigma_{rr} = -p + 2\eta_0 \frac{\partial u}{\partial r} - 2k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \frac{\partial u}{\partial r} - 2 \left( \frac{\partial u}{\partial r} \right)^2 \right]
\]

\[
+ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \left( \frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r} \right) + \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} \frac{\partial u}{\partial \theta} \right) \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta}
\]

\[
\sigma_{r\theta} = -p + 2\eta_0 \frac{1}{r} \frac{\partial u}{\partial \theta} - 2k_0 \left[ \frac{1}{r} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]

\[
- \frac{2}{r} \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial \theta} + \frac{1}{2} \left( \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} \frac{\partial v}{\partial \theta} \right) \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \right\}
\]

\[
\sigma_{r\theta} = \eta_0 \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) - 2k_0 \left[ \frac{1}{r} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]

\[
- \frac{2}{r} \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial \theta} + \frac{1}{2} \left( \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} \frac{\partial v}{\partial \theta} \right) \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \right\}
\]

\[
\sigma_{rz} = \eta_0 \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - 2k_0 \left[ \frac{1}{r} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right]
\]

\[
+ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \right\}
\]

\[
\sigma_{\theta\theta} = \frac{1}{r} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]

\[
- \frac{2}{r} \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial \theta} + \frac{1}{2} \left( \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} \frac{\partial v}{\partial \theta} \right) \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \right\}
\]

\[
\sigma_{\theta\theta} = \eta_0 \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) - 2k_0 \left[ \frac{1}{r} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right) \frac{1}{r} \frac{\partial u}{\partial \theta} \right]
\]

\[
- \frac{2}{r} \left\{ \frac{1}{2} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial \theta} + \frac{1}{2} \left( \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial r} \frac{\partial v}{\partial \theta} \right) \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \theta} \right\}
\]
\[
\sigma_{\theta z} = \eta_0 \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) - 2k_0 \left[ 2 \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} \right) \frac{\partial v}{\partial z} r \frac{\partial w}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial \theta} \right] \\
- \left[ 2 \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} \right) \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right] \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial z} \right] \\
- \left[ \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \frac{\partial v}{\partial z} + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial r} \right) \frac{\partial v}{\partial z} + \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial r} \right) \frac{\partial v}{\partial z} \right], \quad (1.8.8)
\]

and the stress components in second-order fluid are given by

\[
\sigma_{rr} = -p + 2\mu_1 \frac{\partial u}{\partial r} + 2\mu_2 \left[ \frac{\partial^2 u}{\partial r^2} + \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} \right)^2 \right] + \mu_3 \left[ 4 \left( \frac{\partial u}{\partial r} \right)^2 \right]
\]

\[
+ \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} - \frac{v}{r} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 \right]
\]

\[
\sigma_{\theta \theta} = -p + 2\mu_1 \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + u \right) + 2\mu_2 \left[ \frac{1}{r} \frac{\partial a_r}{\partial \theta} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right] + \mu_3 \left[ 4 \left( \frac{\partial v}{\partial r} + \frac{u}{r} \right)^2 \right] + \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2 \right]
\]

\[
\sigma_{zz} = -p + 2\mu_1 \frac{\partial w}{\partial z} + 2\mu_2 \left[ \frac{\partial^2 w}{\partial z^2} + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \mu_3 \left[ \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)^2 \right]
\]

\[
+ \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2 + 4 \left( \frac{\partial w}{\partial z} \right)^2 \right]
\]

\[
\sigma_{r \theta} = \mu_1 \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} - \frac{v}{r} \right) + \mu_2 \left[ \frac{\partial a_{\theta}}{\partial r} + \frac{1}{r} \frac{\partial a_r}{\partial \theta} + \frac{1}{r} \frac{\partial a_{\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial a_r}{\partial \theta} \right] + 2 \left( \frac{\partial u}{\partial r} \right)^2 \]

\[
+ \left( \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} - \frac{v}{r} \right) \right]
\]
\[
\frac{\partial v}{\partial r} \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + u \right) + \frac{1}{r} \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial \phi} + \mu_3 \left[ 2 \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + u \right) \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \phi} - \frac{v}{r} \right) \right] \\
+ \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right)
\]

\[
\sigma_{\theta z} = \mu_1 \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) + \mu_2 \left[ \frac{\partial a_\theta}{\partial z} + \frac{1}{r} \frac{\partial a_z}{\partial \theta} + 2 \frac{\partial w}{\partial \phi} \right] + \mu_3 \left[ \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{r} \right] \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right)
\]

\[
\sigma_{rz} = \mu_1 \left( \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) + \mu_2 \left[ \frac{\partial a_r}{\partial z} + \frac{1}{r} \frac{\partial a_z}{\partial \theta} + 2 \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \phi} \right) \right]
\]

\[
+ \mu_3 \left[ 2 \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \left( \frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) + \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \left( \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right].
\]

(c) **Equations in Spherical Polar Coordinates**:

Let \( r, \theta, \phi \) be the spherical polar coordinates and \( u, v, w \) be the velocity components in these directions. Then the equation of continuity is

\[
\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{2}{r} \frac{u}{r} + \frac{v}{r} + \frac{r}{r} = 0. \tag{1.8.10}
\]

The equations of motion are given by

\[
\rho a_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} \left( 2 \sigma_{rr} - \sigma_{\theta \theta} - \sigma_{\phi \phi} + \sigma_{r \theta} \cot \theta \right),
\]

\[
\rho a_\theta = \frac{\partial \sigma_{\theta \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \phi}}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi \phi}}{\partial \phi} + \frac{1}{r} \left\{ 3 \sigma_{\theta \theta} + \cot \theta \left( \sigma_{\theta \phi} - \sigma_{\phi \phi} \right) \right\},
\]
\[ \rho a_r = \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} \left( 3 \sigma_{r\phi} + 2 \sigma_{\theta\phi} \cot \theta \right), \quad (1.8.11) \]

where

\[ a_r = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial u}{\partial \phi} - \frac{v^2 + w^2}{r}, \]

\[ a_{\theta} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial v}{\partial \phi} - \frac{w^2 \cot \theta + uv}{r}, \]

\[ a_{\phi} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{uv + vw}{r} \cot \theta, \]

are the acceleration components and the stress components in Walters fluid (Model B’)

are given by

\[ \sigma_{rr} = -p + 2 \eta_0 \frac{\partial u}{\partial r} - 2 k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \frac{\partial u}{\partial r} - \left\{ 2 \left( \frac{\partial u}{\partial r} \right)^2 \right. \right. \]

\[ + \left. \left. \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \left( \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) + \left( \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r} \right) \left( \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \frac{w}{r} \right) \right\} \right], \]

\[ \sigma_{\theta\theta} = -p + 2 \eta_0 \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) - 2 k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) \right. \]

\[ - \left. \left\{ \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \right. \frac{\partial v}{\partial r} + 2 \left( \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w \cot \theta}{r \sin \theta} \right) \right. \]

\[ \left( \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} \frac{-w \cot \theta}{r} \right) \left. \right\} \right], \]

\[ \sigma_{\phi\phi} = -p + 2 \eta_0 \left( \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{u}{r} \frac{\cot \theta}{r} \right) - 2 k_0 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + \frac{w}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{u}{r} \frac{\cot \theta}{r} \right) \right. \]

\[ - \left. \left\{ \left( \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) \right. \frac{\partial w}{\partial r} + 2 \left( \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{u}{r} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{w \cot \theta}{r \sin \theta} \right) \right. \]

\[ \left( \frac{1}{r \sin \theta} \frac{\partial w}{\partial \phi} \frac{-w \cot \theta}{r} \right) \left. \right\} \right]. \]
\[
\sigma_{r\theta} = \eta_0 \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} - \frac{w}{r} \cot \theta \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \phi} + \frac{w}{r} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} \right) \right]
\]

\[
\sigma_{r\phi} = \eta_0 \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} - \frac{w}{r} \cot \theta \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \phi} + \frac{w}{r} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} \right) \right]
\]

\[
\sigma_{\theta\phi} = \eta_0 \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} - \frac{w}{r} \cot \theta \right) - 2k_0 \left[ \frac{1}{2} \left( \frac{\partial}{\partial \hat{t}} + \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \phi} + \frac{w}{r} \frac{\partial}{\partial \phi} \right) \left( \frac{1}{r \sin \theta \frac{\partial}{\partial \phi}} + \frac{\partial}{\partial r} \right) \right]
\]
\[ + \frac{1}{2} \left( \frac{1}{r \sin \theta} \right) \frac{\partial v}{\partial r} - \frac{v}{r} \cot \theta \right) \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{u + v}{r} \cot \theta \right) \right] } \]

\[- \frac{1}{2} \left( \frac{1}{r \sin \theta} \right) \frac{\partial w}{\partial r} - \frac{w}{r} \cot \theta \right) + \frac{1}{2} \left( \frac{1}{r \sin \theta} \right) \frac{\partial v}{\partial r} - \frac{w}{r} \cot \theta \right) \left( \frac{1}{r} \frac{\partial v}{\partial r} + \frac{u}{r} \right) \right] } + \frac{1}{2} \left( \frac{1}{r \sin \theta} \right) \frac{\partial v}{\partial r} - \frac{v}{r} \cot \theta \right) \right) \left( \frac{1}{r} \frac{\partial w}{\partial r} - \frac{w}{r} \cot \theta \right) \right] } , \quad (1.8.12) \]

and the stress components in second-order fluid are

\[
\sigma_{rr} = -p + 2 \mu_1 \frac{\partial u}{\partial r} + 2 \mu_2 \left[ \frac{\partial a_r}{\partial r} + \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} \right)^2 \right] + \mu_3 \left[ 4 \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{w}{r} \right)^2 \right] \]

\[
\sigma_{\theta \theta} = -p + 2 \mu_1 \left( \frac{1}{r \sin \theta} \right) \frac{\partial v}{\partial r} + \frac{u}{r} + 2 \mu_2 \left[ \frac{1}{r \sin \theta} \frac{\partial a_{\theta}}{\partial r} + \frac{a_r}{r} \left( \frac{1}{r \sin \theta} \right) \frac{\partial u}{\partial r} - \frac{v}{r} \right] + \frac{1}{r \sin \theta} \frac{\partial v}{\partial r} - \frac{w}{r} \cot \theta \right)^2 \right] \]

\[
\sigma_{\phi \phi} = -p + 2 \mu_1 \left( \frac{1}{r \sin \theta} \right) \frac{\partial w}{\partial r} + \frac{u}{r} + \frac{v}{r} \cot \theta \right) + 2 \mu_2 \left[ \frac{1}{r \sin \theta} \frac{\partial a_{\phi}}{\partial r} + \frac{a_r}{r} + \frac{a_{\theta}}{r} \cot \theta \right] + \frac{1}{r \sin \theta} \frac{\partial w}{\partial r} - \frac{u + v}{r} \cot \theta \right)^2 \right] \]

\[
+ \mu_3 \left[ \left( \frac{\partial w}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial u}{\partial r} - \frac{w}{r} \right)^2 + \left( \frac{\partial v}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial w}{\partial r} - \frac{w}{r} \right)^2 + \left( \frac{\partial w}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial w}{\partial r} - \frac{w}{r} \right)^2 \right] \]
\[ \left( 1 \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \theta} - \frac{v}{r} \right) \]. \hspace{1cm} (1.8.13)

1.9 A Brief Review of The Allied Works:

Noll, Coleman and Truesdell have derived the constitutive equation of the second-order fluid from the definition of simple fluid and have discussed the viscometric flows for it in a large number of papers. In a general analysis of viscometric flows, Coleman and Noll (1959_1, 1959_2) have shown that the solution of any viscometric flow may be expressed in terms of three material functions known as shear viscosity and normal functions. They (1962) have also shown that the solution of a steady extension of a box and that of a cylinder which are non-viscometric flows depend on the material functions other than the three functions as mentioned above. Markovitz and Coleman (1964_2) have discussed non-steady helical flows and they (1964_2) have given an account of the viscometric flows, the steady extensions and non-steady helical flows have proposed some practicable methods for experiments in determining the material constants \( \mu_2 \) and \( \mu_3 \). Pipkin (1964_1) studied the alternating flows in tubes of arbitrary cross-section and also considered the annular effect when the fluid oscillates in a pipe (1964_2). Caswell and Schwarz (1961) calculated the drag force on a sphere in a creeping motion past it. Ting (1963) considered a few non-steady problems but he assumed \( \mu_2 \) to be positive which is not justified. Niler and Pipkin (1964) have studied the propagation of finite amplitude shear-wave in all the fluids mentioned in this chapter. They have discussed the effect of stress-relaxation and have compared the results in all these fluids. Srivastava has discussed the torsional oscillations of an infinite disk, when (a) there is no other plate (1963), (b) there is a rotating plate parallel
to it (1966). He has discussed the flow and the heat transfer when the fluid is sheared between a rotating and stationary disk (1964), fluctuating flow near an axi-symmetric stagnation point (1966) and unsteady flow near a stagnation point (1966). He (1966) has also derived the thermal boundary layer equation for two-dimensional plane flow and also obtained the temperature distribution for two-dimensional flow near a stagnation point which has been discussed by Rajeswari and Rathna (1962).

Srivastava and Maiti (1966) have discussed the flow past a symmetric cylinder. Srivastava and Sharma (1970) have studied the forced flow against a flat wall. Rathna (1962) has discussed the rotation of a plane lamina. Bhatnagar and Rathna (1963) have discussed the flow between two rotating co-axial cones having the same vertex. Bhatnagar (1965) has considered the steady rotation of a sphere. In the problems (1962), (1963) and (1965) either $\mu_2$ or $\mu_3$ has been neglected. Rathna considered the fluid to be visco-elastic if $\mu_3 = 0$ and non-Newtonian if $\mu_2 = 0$. However, their discussion does not give any information about a real fluid. But, for any real fluid neither $\mu_2$ nor $\mu_3$ is zero. Ramacharyulu (1964) has discussed the steady slow rotation of a sphere in a visco-elastic liquid. He (1964) has also discussed the exact solutions of two-dimensional flows of second-order incompressible fluids. The idea of super possibility developed by Ballabh (1940) for Newtonian fluid has been extended by Sharma (1964) for the same fluids.

Dutta (1964) and Rao (1964) have discussed the stability of the fluid moving between two rotating co-axial cylinders. Both have studied the dependence of Taylor number on $\mu_2$ and $\mu_3$. Gupta (1967) has discussed the stability of the fluid film flowing down on inclined plane. Srivastava and Saroa (1970) have studied the flow past a circular cylinder by supposing a variable boundary layer along the cylinder. Again they (1978) have
studied the heat transfer in the flow past a circular cylinder. Saroa (1971) has discussed the torsional oscillations of a disk in a rotating fluid and (1967) about a steady non-zero mean. He (1972) has also studied the oscillation of the fluid near a sphere. Choudhury (1981) has studied the flow of an second-order fluid through a contracting and expanding pipe. Saroa and Choudhury (1983) have discussed the same flow between eccentric cylinders. Again, they have discussed this flow past a sphere (1984), the induced flow of an fluid due to an oscillating disk (1985), flow and heat transfer around a circular cylinder (1992), flow between rotating eccentric discs (1993). Crewther et. al. (1991) have studied the axisymmetric and non-axisymmetric flow of a non-Newtonian fluid between coaxial rotating discs. Srivastava and Sharma (1991) have discussed the flow of a second-order fluid through a circular pipe and its surrounding porous medium. Srivastava and Hazarika (1992) have studied shooting method for the flow between non-parallel plates with magnetic field. Nguyen and Chandana (1992) have discussed the non-Newtonian MHD orthogonal steady plane fluid flows. Hazarika and Baruah (1993) have studied the non-Newtonian fluid flow along a continuous stretching surface with suction and injection. Goswami and Hazarika (1993) have considered the non-Newtonian MHD flow past a stretching sheet in a porous medium with suction and blowing. Sharma and Pradip Kumar (1993) have studied Rayleigh-Taylor instability of visco-elastic fluids through porous medium. Tsamopoulos and Borkar (1994) have studied the transient rotational flow of an Oldroyd fluid $B$ over a disk. Mafdy (1994) has discussed the magnetohydrodynamic unsteady flow of a non-Newtonian fluid past an infinite porous plate. Tak and Sankhla (1994) have discussed the numerical solution of flow of power-law fluid along the wall of convergent channel. Sharma and Harish Kumar (1994) have considered the unstudy flow of a non-Newtonian fluid down an open inclined channel. Rajagopal et. al.(1994) have studied the flow of visco-elastic flow over a
stretching sheet. Bhattacharjee and Gupta (1995) have considered the flow due to a stretching surface rotating in a non-Newtonian fluid. Choudhury and Das (1995) have studied the flow of visco-elastic fluid between rotating eccentric cylinders. Rieley (1996) have discussed the boundary layer on a rotating ellipse. Kumari et. al. (1997) have studied the flow and heat transfer of a visco-elastic fluid over a flat plate with a magnetic field and a pressure gradient. Choudhury and Das (1997) have studied the free convective heat transfer in a visco-elastic fluid confined between a long vertical wavy wall and the parallel flat wall. Kim (1997) has considered the natural convection along a wavy vertical plate to non-Newtonian fluids. Howell et. al. (1997) have studied the momentum and heat transfer on a continuous moving surface in a power-law fluid. Payvar (1997) has discussed the heat transfer enhancement in laminar flow of visco-elastic fluids through rectangular ducts. Ke Mei Zhoi and Xiu Hu (1998) have studied a mathematical model for thin plate flow of non-Newtonian viscous fluids. Choudhury and Das (1998) have discussed the flow and heat transfer of Walters liquid B' between two porous rotating discs. Adluri (1999) has considered some exact solutions of steady plane MHD non-Newtonian power-law fluid flows. Choudhury and Sharma (1999) have studied the torsional oscillations of disk in an elastico-viscous fluid. Rathod and Hosurker (1998, 1999) have studied MHD flow of Rivlin-Ericksen fluid between two parallel plates and two infinite parallel inclined plates. Andrienko et. al. (2000) have studied the resonance behaviour of visco-elastic fluids in Poiseuille flow and application to flow enhancement.

1.10 Motivation, Extent and Scope of This Work:

The solution of problems of engineering interest in the flow of visco-elastic fluids require that methods be developed for studying the properties of such fluids in configurations other
than the classical viscometric flow fields. Out of the different rheological models discussed above, we have chosen Walters fluid (Model B') and second-order fluid governed by the constitutive equations (1.4.13) and (1.7.18) respectively.

A number of steady as well as unsteady flow problem have been discussed for different geometries. The results reveal various aspects of the additional terms in the constitutive equations as compared to the Newtonian fluid. Some of the results need experimental verifications while others explain the results of the earlier experiments.

The rheometrical flow system has been discussed in chapter II. Here, we have considered the flow of Walters liquid (Model B') between two infinite disks which are rotating with different angular velocities about different axes of rotation. The solution is obtained by expanding the velocity components in terms of a suitable small parameter when the inertia effects are also assumed to be small. The force on one of the disks has been calculated and it is observed that the two components of the force can be used to determine the visco-elastic parameter of the fluid.

In chapter III, flow of Walters liquid (Model B') through an annulus has been studied. The inner surface of the annulus is a smooth rigid cylinder while the outer surface is a flexible cylinder whose radius is varying with time as well as with axial distance. Perturbation technique has been employed to obtain the solution of the problem, taking variation in the outer surface as perturbation parameter. The boundary conditions of the outer surface are suitably amended with the use of Taylor’s series expansion. The dimensionless shearing stress and volume rate of flow have been obtained at various sections of the annulus. The obtained results have been numerically worked out for different values of the elastico-viscous parameter with the combination of other flow parameters and the
results are expressed in tabular forms.

Chapter IV deals with the analysis of the laminar boundary layer along a flat plate in a non-Newtonian second-order fluid in presence of a magnetic field acting perpendicular to the plate. The problem is solved by the application of steepest descent method used by Meksyn. The non-Newtonian effect on the component of velocity which is parallel to the length of the plate and also on the displacement thickness are studied in details. The velocity component $u$ as a function of $\eta$ has been presented graphically for various values of non-Newtonian parameter.

In chapter V, the steady two-dimensional free convection flow of a Walters fluid (Model $B'$) in a vertical channel one of whose walls is wavy, has been investigated analytically. The governing equations of the fluid and the heat transfer have been solved subject to the relevant boundary conditions by assuming that the solution consists of two parts: a mean part and disturbance or perturbed part. To obtain the perturbed part of the solution, the long wave approximation has been used and to solve the mean part, a well-known approximation used by Ostrach has been utilised. The relevant flow and the heat transfer characteristics, namely the skin friction and the rate of heat transfer at both the walls have been discussed in details.

In chapter VI, the flow and heat transfer in an elastico-viscous fluid between two co-axial infinite porous rotating discs is considered for small cross flow Reynolds number. The discs are rotating with different angular velocities and the rate of injection of the fluid at one disc is different from the rate of suction at the other disc. The governing equations have been solved by perturbation method, taking cross flow Reynolds number as perturbation parameter. The analytical expressions for radial, transverse, axial velocity components
and temperature have been obtained and these results have been numerically worked out for different values of parameters involved in the solution. The Nusselt number and the Skin friction coefficient for various cases have also been calculated at both the discs and the results are expressed in a tabular form. The first-order velocity components have been presented graphically for various visco-elastic parameters.

We have concluded our present thesis by compiling a wide ranging bibliography associated with our present work.