CHAPTER 4
THE METHODOLOGY
OF MONTE CARLO EXPERIMENTS

1. Introduction: There are two approaches to studying the properties of a mathematical
system (representing a real-world system): the first, analytical and the second,
experimental. In the analytical approach, the mathematical system is described in great
details (so as to represent the real-world system as accurately as possible). All the axioms
and assumptions underlying the system are exhaustively and explicitly described with
utmost accuracy. All definitions relevant to the system are accurately and explicitly
stated. All the operations that the mathematical system would be subjected to are
accurately and exhaustively described in their details. Then, appropriate mathematical
methods are applied to draw conclusions from the axioms, assumptions and statements
that describe the said mathematical system. All the conclusions drawn in this process are
the implications of the mathematical system so described. Nevertheless, mathematical
methods applied to draw conclusions have some impact on the conclusions. A careful
attempt is made, therefore, to apply mathematical methods such that they modify the
conclusions as little as possible. These conclusions are deductive in nature and have
general applicability.

However, complicated systems cannot be described in very accurate and
exhaustive details. Even if the system is described in details, appropriate methods to draw
conclusions may be either unavailable or intractable due mainly to the complexity of the
system. To investigate into the properties of such a mathematical system by analytical
methods is very difficult, if not impossible. Experimental methods are applied to
investigate into the properties of such systems. The conclusions drawn from experimental
methods provide a great insight into the nature of the system. However, these conclusions have lesser degree of certainty as well as applicability. These conclusions are inductive inferences – generalizations based on the study of samples – and, therefore, probabilistic in nature.

II. The Monte Carlo Method: The Monte Carlo method of investigating into the properties of a mathematical system is basically experimental and numerical in nature. Numerical methods that are known as ‘Monte Carlo methods’ can be loosely described as statistical simulation methods, where statistical simulation is defined in quite general terms to be any method that utilizes sequences of random numbers to perform the simulation. Monte Carlo methods have been used for centuries, but only in the past several decades has the technique gained the status of a full-fledged numerical method capable of addressing the most complex applications. The credit for inventing the Monte Carlo method often goes to Stanislaw Ulam, a Polish born mathematician who worked for John von Neumann on the United States’ Manhattan Project during World War II. Ulam is primarily known for designing the hydrogen bomb with Edward Teller in 1951. Quoted in Eckhardt (1987), Ulam describes the incident as:

The first thoughts and attempts I made to practice [the Monte Carlo Method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than “abstract thinking” might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a
succession of random operations. Later … [in 1946, I] described the idea to John von Neumann, and we began to plan actual calculations.

Working with John von Neumann and Nicholas Metropolis, he developed algorithms for computer implementations, as well as exploring means of transforming non-random problems into random forms that would facilitate their solution via statistical sampling. This work transformed statistical sampling from a mathematical curiosity to a formal methodology applicable to a wide variety of problems. It was Metropolis who named the new methodology after the casinos of Monte Carlo. Ulam and Metropolis published their paper on Monte Carlo method in 1949.

Monte Carlo method is now used routinely in many diverse fields, from the simulation of complex physical phenomena such as radiation transport in the earth's atmosphere and the simulation of the esoteric sub-nuclear processes in high energy physics experiments, to the mundane, such as the simulation of simple games. The analogy of Monte Carlo methods to games of chance is a good one, but the game is a physical system, and the outcome of the game is not a pot of money or stack of chips (unless simulated) but rather a solution to some problem. The winner is the scientist, who judges the value of his results on their intrinsic worth, rather than the extrinsic worth of his holdings.

Statistical simulation methods may be contrasted to conventional numerical discretization methods, which typically are applied to ordinary or partial differential equations that describe some underlying physical or mathematical system. In many applications of Monte Carlo, the physical process is simulated directly, and there is no need to even write down the differential equations that describe the behavior of the system. The only requirement is that the physical (or mathematical) system be described
by probability density functions (pdf's). Assuming that the evolution of the physical system can be described by probability density functions, then the Monte Carlo simulation can proceed by sampling from these pdf's, which necessitates a fast and effective way to generate random numbers uniformly distributed on the interval [0,1]. The outcomes of these random samplings, or trials, must be accumulated or tallied in an appropriate manner to produce the desired result, but the essential characteristic of Monte Carlo is the use of random sampling techniques (and perhaps other algebra to manipulate the outcomes) to arrive at a solution of the physical problem. In contrast, a conventional numerical solution approach would start with the mathematical model of the physical system, discretizing the differential equations and then solving a set of algebraic equations for the unknown state of the system.

Nevertheless, this general description of Monte Carlo methods may not directly apply to some applications. It is natural to think that Monte Carlo methods are used to simulate random, or stochastic, processes, since these can be described by pdf's. However, this coupling is actually too restrictive because many Monte Carlo applications have no apparent stochastic content, such as the evaluation of a definite integral or the solution of a system of linear equations. However, in these cases and others, one can pose the desired solution in terms of pdf's, and while this transformation may seem artificial, this step allows the system to be treated as a stochastic process for the purpose of simulation and hence Monte Carlo methods can be applied to simulate the system. Therefore, one takes a broad view of the definition of Monte Carlo methods and includes in the Monte Carlo rubric all methods that involve statistical simulation of some underlying system, whether or not the system represents a real physical process. The range of applications that have been addressed with statistical simulation techniques is enormous, from the simulation of
galactic formation to quantum chromodynamics to the solution of systems of linear
equations.

**Major Components of a Monte Carlo Algorithm:** Given the working definition of
Monte Carlo method, let us now describe briefly the major components of a Monte Carlo
method. These components comprise the foundation of most Monte Carlo applications.
The primary components of a Monte Carlo simulation method include the following:

*Probability distribution functions (pdf's)* - the physical (or mathematical) system must be
described by a set of pdf's.

*Random number generator* - a source of random numbers uniformly distributed on the
unit interval must be available.

*Sampling rule* - a prescription for sampling from the specified pdf's, assuming the
availability of random numbers on the unit interval, must be given.

*Scoring (or tallying)* - the outcomes must be accumulated into overall tallies or scores for
the quantities of interest.

Additionally, an estimate of the statistical error (variance) as a function of the
number of trials and other quantities have to be determined. To enhance the speed at
which the experiments are carried out, methods for reducing the variance in the
estimated solution (resulting into reduction of the computational time for Monte Carlo
simulation) are applied. For complicated and large systems, the *Parallelization and
vectorization*- based algorithms are used such as to allow the Monte Carlo method to be
implemented efficiently on advanced computer architectures.

Thus, the Monte Carlo method is an artificial sampling method, which may be
used for solving complicated problems in analytic formulation and for simulating purely
statistical problems. The method is being used more and more in recent years, especially
in those cases in recent years, where the number of factors included in the problem is so large that an analytical solution is either extremely involved or rather impossible. The main idea behind the Monte Carlo method is either to construct a stochastic model that is in agreement with the whole problem analytically or to simulate the whole problem directly. In both cases, an element of randomness has to be introduced according to some well-defined rules. Then a large number of trials are performed, the results are observed, and finally a statistical analysis is undertaken in the usual way. The advantages of the method are, above everything that even very difficult problems can often be treated very easily, and desired modifications can be applied without much trouble. However, the disadvantages of Monte Carlo method are the relatively poor (vis-à-vis analytical methods) precision and the large number of trials that are necessary. Now, when very fast computers are readily available, the cumber of calculation in large number of trials is no longer a serious matter. This facility may be utilized to increase the number of trials sufficiently large so as to attain an acceptable degree of precision.

Monte Carlo Method and Study of Properties of Estimators of a Linear Model: A number of attempts have been made to investigate into the small sample properties of various estimators of linear model. We find a detailed discussion on the application of Monte Carlo method to study the properties of the estimation methods of multi-equation models in Intriligator, 1978, pp. 416-420.

In our endeavour to gauge the performance of LAD estimator vis-a-vis LS estimator, we propose to use of Monte Carlo experiments. The essence of the Monte Carlo study is that instead of estimating unknown parameters using a specific technique, known parameters, which are chosen before hand, are estimated using different techniques. Comparisons between estimated and true parameters are subsequently used to
make inference about the different techniques. It simulates the process of estimating parameters using a controlled setting in which the true parameter values are known, like a ‘laboratory’ in which controlled experiments on econometric estimators can be studied.

III. Random Numbers, their Distributions and Generation: As mentioned before, random numbers play a very important role in the Monte Carlo method. Random numbers are almost always generated on computers by using some mathematical method. Such random numbers (generated by using some mathematical formula) are in fact pseudo-random numbers, so called because there exists a rule (the mathematical formula) for generating them and they have a well-defined periodicity. However, if the period is large enough and the numbers so generated satisfy certain statistical tests, then these numbers are random for all practical purposes. There are three leading methods for generating rectangular or uniformly distributed (pseudo) random numbers, namely the Mid-square method, the Fibonacci method and the Power method. These methods can be combined or hybridized also. Random numbers with various statistical distributions are obtained by suitable transformation of rectangular distributed random numbers. In most cases, first the rectangular distributed random numbers are transformed into normally distributed (standard) random numbers by an application of the Central Limit Theorem (i.e., the average of sufficiently large number of rectangular distributed random numbers has normal distribution) and suitable scaling.

Alternatively, if \( r_1 \) and \( r_2 \) are the two uniformly distributed random numbers lying between zero and unity, then \( u = \sqrt{-2 \ln(r_1)} \cos(2\pi r_2) \) follows standard normal distribution. Here \( \pi = 4 \tan^{-1}(1) \approx 3.1415926535897932 \) (Texas Instruments, p. 54). Then these normally distributed random numbers are transformed into other random numbers with desired distribution.
Pseudo-random Number Generators: Some of the pseudo-random number generators are discussed below.

(i) Generator 1. (Mid-square method): A Sequence of the pseudo-random numbers \( \{r_i\} \), \( i = 1, 2, \ldots \), where each \( r_i \) is a 38 bit number is generated by the recursive algorithm defining

\[
r_{i+1} = \text{mid (part of) 38 bits of } r_i^2
\]

\( r_1 = 01000010111011101010010011011110 \)

\( i = 1, 2, \ldots \) to any large integer.

This sequence terminates in zero for \( i = 750000 \) and has been very thoroughly tested for randomness and found useful in many practical problems.

The sequence of random numbers generated by this method may exhaust in the case of large problems. In such circumstances, the same sequence can be used repeatedly without repetition of results, provided the initial \( r_1 \) is not used at the identical place it was first called.

Alternatively, the random numbers generated can be increased two or four times by shifting the available random numbers. This procedure involves using in sequence not only \( r_i \) but also the fractional part of \( 2^n r_i \), \( 2^{2n} \left( \frac{2^n r_i}{2^n} \right) \), etc as random numbers before squaring \( r_i \) to obtain \( r_{i+1} \). (It should however, be noted that through such shifts the left most digit is lost).

(ii) Generator 2 (Mid-square-bit method): Algorithm for generating a sequence of pseudo-random numbers \( \{r_i\}, i=1, 2, \ldots \) is given as follows (Krishnamurthy & Sen, p. 303).

Let \( r_1 \) be an 8 bit random number.
(b). Pick up its 8th (or 9th) bit, place it in the 8th-bit position of r2.

(c). Take \( r_{11} = \text{mid (part of)} \) 8 bits of \( r_1^2 \). Square \( r_{11} \). Pick up the 8th (or 9th) bit of \( r_{11}^2 \), place it in the 7th-bit position of \( r_2 \).

(d). Take \( r_{12} = \text{mid (part of)} \) 8 bits of \( r_{11}^2 \). Square \( r_{12} \). Pick up the 8th (or 9th) bit of \( r_{11} \), place it in the 6th-bit position of \( r_2 \).

(e) Continue the process five more times to obtain an 8-bit \( r_2 \).

Using this algorithm, \( r_3 \) can now be generated from \( r_2 \). The process can be continued to generate \( r_t \). Although, such generation is computationally more time consuming, the method has been tested and found to be very useful.

**Generator 3 (Power residue method):** The power residue method is the most commonly used method to generate pseudo-random sequences. A sequence of integers \( r_1, r_2, \ldots \) is defined recursively by

\[
 r_{i+1} \equiv \alpha r_i \pmod{\mu}
\]

where \( r_1 = \) a starting value, \( \alpha \) and \( \mu \) are certain integers and the congruential notation means that \( r_{i+1} \) is the (positive) remainder when \( \alpha r_i \) is divided by \( \mu \). Now, the sequence \( r_1, r_2, \ldots \) will be a periodic sequence whose period cannot exceed \( \mu \), since division by \( \mu \) can produce at most \( \mu \) different remainders (viz, 0, 1, 2, ..., \( \mu - 1 \)). The integers \( \alpha \) and \( \mu \) should be selected in such a manner that they are capable of producing a very long period relative to the number of random numbers required in computation.

The Power residue algorithm for the generation of pseudo-random numbers is:

(1) Let the given computer have a word length of \( K \) bits. Also let the arithmetic be carried out with the binary point to the extreme right or, equivalently, let the arithmetic be the integer.

(2) Choose \( \alpha = \) an integer of the form \( 8x \pm 3 \) and close to \( 2^{K/2} \).
(3) Choose \( \mu = 2^K \)

(4) Choose \( r_1 = \) any odd integer. Now \( \alpha r_1 = 2^K \) bit number.

(5) Discard the \( K \) high-order bit. The residue \( \mu_2 \) is then the \( K \) low-order bit number.

The process is iterated. To obtain a pseudo random number \( r_i \) in \([0,1]\), the binary point is considered to be at the extreme left. The period of the sequence \( r_1, r_2, \ldots \) is \( 2^{K-2} \) for a 32-bit computer, this sequence has a period \( 2^{30} \approx 10^9 \).

On most of the Personal Computers, we may generate uniformly distributed random numbers lying between 0 and 1 by the following algorithm:

(1). Declare \( IU \) and \( IV \) as the two 2-byte integers.

(2). Feed a seed, \( IU \) (preferably odd and of five digits, but less than 32766).

(3). Define \( IV = 259 \times IU \)

(4). If \( IV \geq 0 \) then \( R = IV \); Re-initiate \( IU = IV \); Standardize \( R = R \times 0.3051851E-04 \).

\[ \text{Store/print the random number } R \]

\[ \text{Go to step (3) to generate the next random number with the re-initiated } IU. \]

\[ \textbf{Else} \]

(5). Replace \( IV \) by \( IV + 32767+1 \); \( R = IV \); Re-initiate \( IU = IV \);

\[ \text{Standardize } R = R \times 0.3051851E-04. \]

\[ \text{Store/print the random number } R \]

\[ \text{Go to step (3) to generate the next random number with the re-initiated } IU. \]

**Distributions of Random Numbers:** There are different types of random numbers, some discrete and others continuous. They follow different types of distribution. Data collected from nature following random processes throw up different types of distribution. Karl Pearson identified eight types of empirical distributions (Kapur & Saxena, pp. 296-300).
Additionally, there are umpteen number of theoretical distributions. Here we give the salient features of some distributions relevant for our purpose.

**The Normal Distribution**: A continuous random variable, e, has the normal distribution if it has the probability density function (pdf)

\[ f(e) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left( \frac{e - \mu}{\sigma} \right)^2 \right\}; \text{where} \quad -\infty < e < \infty \]

In the standard normal distribution the mean is taken as zero and the standard deviation as unity. The pdf now reduces to

\[ f(e) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} e^2 \right) \]

The important features of this function are:

1. The normal probability curve is bell shaped and symmetrical about the ordinate at \( e = \mu \).

2. The ordinate decreases rapidly as \( e \) increases numerically. Curve extends to infinity on either side of the mean. The maximum ordinate is at \( e = \mu \) and is given by

\[ Y_{\text{max}} = \frac{1}{\sigma \sqrt{2\pi}}. \]

3. The ordinate at \( e = \mu \) divides the area under the normal curve into two equal parts.

   Thus the median of the distribution coincides with the mean and the mode.

4. The two points of inflexion of the normal curve are equidistant from the mean.

5. For large values of \( \sigma \), the normal curve tends to be flatter, while for small values of \( \sigma \), it has a small peak.
6. No portion of the curve lies below the X-axes since normal probability function cannot be negative.

**Non-normal Distributions:** The random nature of the error term has prompted econometricians to assume that the disturbance term, which is an influence of innumerable many factors not accounted for in the model, approaches normality according to the Central Limit Theorem. But Bartels (1977) is of the opinion that there are limit theorems, which are just likely to be relevant when considering the sum of number of components in a regression disturbance that leads to non-normal stable distribution characterised by infinite variance. Thus, the possibility of the error term following a non-normal distribution exists. In such cases, the least squares estimator will give less efficient estimates. Non-normal distributions are those, which acquire skewed, platykurtic or leptokurtic shapes when plotted i.e., they are asymmetrical about the mean or deviate significantly from the bell-shaped normal distribution. Three important and well-known non-normal distributions that have been considered in the study are discussed below.

**Cauchy Distribution:** This distribution is named after its discoverer KL Cauchy (1789-1857). The distribution is useful in statistical theory as it is an example of a distribution for which a number of theorems fail and hence emphasizes the need for scrutiny of the conditions under which statistical theorem holds.

The pdf of the Cauchy-distributed variate, \( e \), is given by:

\[
 f(e) = \frac{1}{\pi} \cdot \frac{1}{1 + (e - \mu)^2}, \quad -\infty < e < \infty
\]
The pdf curve is symmetrical about the point $e = \mu$ and thus $e = \mu$ gives us the median of the distribution. Also the density function is obviously maximum at $e = \mu$ so that this point is also the mode. The mean is given by:

$$
\mu' = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{edde}{1+(e-\mu)^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(e-\mu)de}{1+(e-\mu)^2} + \frac{\mu}{\pi} \int_{-\infty}^{\infty} \frac{de}{1+(e-\mu)^2}
$$

Now since \( \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{de}{1+(e-\mu)^2} = 1 \), we have

$$
\mu' = \mu + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ydy}{1+y^2} = \mu + \frac{1}{\pi} \lim_{\text{int. e} \to \infty} \int_{-\infty}^{\infty} \frac{ydy}{1+y^2}
$$

If $e, e'$ approach infinity independently, the second integral does not converge to a limit and thus, in the most general sense, the mean of the Cauchy distribution does not exist. However, if $e' = e$, the integral vanishes and thus if we take the principal value of the integral, the mean of the Cauchy distribution is at $\mu$. Thus, in this sense the median and mode of the distribution coincide and the distribution is symmetrical.

The variance of the distribution is \( \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(e-\mu)^2}{1+(e-\mu)^2}de \) and the integral does not converge. Thus for Cauchy distribution, the second and higher moments about the mean do not exist. The cumulative distribution function (cdf) of the distribution of $e$ is:

$$
F(e) = \frac{1}{\pi} \int_{-\infty}^{e} \frac{de}{1+(e-\mu)^2} = \frac{1}{\pi} \left[ \tan^{-1}(e-\mu) + \frac{\pi}{2} \right]
$$

The characteristic function of the distribution about the mean is

$$
\phi(t) = E[\exp\{it(e-\mu)\}] = \frac{1}{\pi} \int_{-\infty}^{\infty} \exp\{it(e-\mu)\} \left\{ \frac{1}{1+(e-\mu)^2} \right\} de
$$
The Cauchy-distributed variate, \( e \), relates to normally distributed variate in an interesting manner which is relevant to generate \( e \) from the latter. If \( U_1 \) and \( U_2 \) are independent normal variates with means \( m_1 \) and \( m_2 \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), then the variate \( e = \frac{u_1 - m_1}{u_2 - m_2} \) is Cauchy distributed (Kapur & Saxena, 1982; p. 427). In particular, the quotient of two independent standard normal variates is Cauchy distributed.

**Gamma Distribution**: The continuous random variate, \( e \), which is distributed with pdf

\[
f(e) = \frac{m^l}{\Gamma(l)} e^{l-1} \exp(-me), \quad e \geq 0, l \geq 0, m > 0
\]

is called a Gamma Variate with parameters \( l \) and \( m \), and the distribution is called the Gamma distribution. The function \( f(e) \) represents a probability density, since its integral over the range \((0, \infty)\) is unity.

The moment generating function (mgf) with respect to origin for gamma distribution with \( m=1 \) is

\[
M_{e=0}(t) = \frac{e^{t-1}}{1} \exp(-e) \quad e = \frac{1}{
\int_0 e^{t-1} \exp\{-e(t-1)\} \, de = (1-t)^{-1},
\]

provided that \(|t| < 1\)

Differentiating \( M_{e=0}(t) \), \( r \) times with respect to \( t \) and putting \( t=0 \), we find

\[
\mu_r' = l(l+1)\ldots(l+r-1)
\]

In particular, \( \mu_1' = l \), \( \mu_2' = l(l+1) \) and \( \mu_2 = l, \mu_3 = 2l \) and \( \mu_4 = 6l + 3l^2 \). Hence,

\[
\beta_1 = \frac{\mu_2^2}{\mu_2} = \frac{4}{l} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3 + \frac{6}{l}}{l}
\]
Hence as $l \to \infty, \beta_1 \to 0$ and $\beta_2 \to 3$ so that Gamma distribution tends to normal distribution as the parameter tends to infinity.

**Two Important Properties of Gamma Variates:** Gamma variates have two important properties relevant to our investigation; (i) the sum of two independent Gamma Variates with parameters $l$ and $m$ is a Gamma Variate with parameter $l+m$ and (ii) if $u$ is a standard normal variate, then $e = \frac{1}{2} v^2$ is a Gamma variate with parameter $\frac{1}{2}$ (Kapur & Saxena, pp. 288, 289). Using these properties, it is possible to generate Gamma distributed random numbers from standard normal random numbers.

**Beta Distributions:** There are two types of Beta-distributed random variables, the first is called $\beta_1$-distributed, and the second is called $\beta_2$-distributed. The continuous random variate, $e$, which is distributed with pdf

$$f(e) = \frac{1}{B(l, m)} e^{l-1}(1-e)^{m-1}, 0 \leq e \leq 1; \ l, m > 0$$

$$= 0, \text{ elsewhere}$$

is a Beta variate of the first kind with parameters $l$ and $m$ and is referred to as a $\beta_1(l, m)$ variate. The mgf with respect to the origin for this distribution does not have a simple form but the moments are easily found directly.

$$\mu_r', \text{ about origin} = \frac{1}{B(l, m)} \int_0^1 e^{r(l-1)}(1-e)^{m-1} \, de$$

$$= \frac{B(r+l, m)}{B(l, m)}$$

$$= \frac{l(l+1)...(l+r-1)}{(l+m)(l+m+1)...(l+m+r-1)}, \ r = 1, 2, \ldots.$$
In particular, \( \mu'_1 = \frac{l}{l+m} \), \( \mu'_2 = \frac{l(l+1)}{(l+m)(l+m+1)} \), \( \mu_3 = \frac{lm}{(l+m)^2(l+m+1)} \)

Putting \( r = 3 \) and \( r = 4 \) we successively obtain \( \mu_3 \) and \( \mu_4 \) and hence we can show that

\[
\mu_3 = \frac{2lm(m-l)}{(l+m)^3(l+m+1)(l+m+2)}
\]

\[
\mu_4 = \frac{3lm\{lm(l+m-6)+2(l+m)^2\}}{(l+m)^4(l+m+1)(l+m+2)(l+m+3)}
\]

In case \( l = m \), \( f(e) = \frac{1}{B(l,1)} \), \( 1 = 1, 0 < e < 1 \)

which is the probability function of uniform distribution in the range \((0,1)\).

**Beta Distribution of the Second Kind:** The continuous random variate, \( e \), which is distributed with pdf

\[
f(e) = \frac{1}{B(l,m)} \frac{e^{l-1}}{(1+e)^{m+1}}, e \geq 0; \ l, m > 0
\]

\[
e = 0 \text{ for } e < 0.
\]

is a Beta variate of second kind with parameters \( l \) and \( m \) and may be referred to as \( \beta_2(l,m) \) variate. The function given above represents a probability density since

\[
\int_0^\infty f(e)\,de = \int_0^\infty \frac{1}{B(l,m)} \frac{e^{l-1}}{(1+e)^{m+1}}\,de
\]

\[
= \frac{1}{B(l,m)} \int_0^1 y^{m-1}(1-y)^{l-1}\,dy
\]

on putting \( 1+e = y^{-1}, \ de = \frac{dy}{y^2} \)

\[
\therefore \int_0^\infty f(e)\,de = 1 \quad \text{since } B(l,m) = B(m,l)
\]
Also $\mu'_1$ about origin = $\int_0^1 \frac{1}{B(l, m)} \cdot \frac{e^{ly^{-1}}}{(1 + e)^{lm}} \, dy$

= $\frac{1}{B(l, m)} \int_0^1 y^{m-1} (1-y)^{l+r-1} \, dy \cdot 1 + e = y^{-1}$

= $\frac{1}{B(l, m)} \int_0^1 Z^{l+r-1} (1-Z)^{m-r-1} \, dz$ \quad where $Z=1-y$

$\therefore \mu'_1 = \frac{B(l+r, m-r)}{B(l, m)} = \frac{l(l+1)...(l+r-1)}{(m-1)(m-2)...(m-r)} ; r < m$.

Beta-distributed random numbers are related to Gamma-distributed random numbers in a very interesting manner that may be exploited to generate Beta-distributed variates from the Gamma-distributed variates. From two independent Gamma variates, $g_1$ and $g_2$ with parameters $l$ and $m$ respectively, we may obtain $e_1 = g_1/(g_1 + g_2)$, which is a $\beta(l,m)$ distributed variate, and $e_2 = g_2/g_2$, which is a $\beta_2(l,m)$ distributed variate (Kapur & Saxena, 1982; p. 292).

**Generation of Random numbers following different Distributions:** Simulation requires generation of a large number of random numbers that follow the desired distribution. The enterprise starts with the generation of uniformly distributed random numbers lying between 0 and 1. Uniformly distributed random numbers may be transformed into normally distributed random numbers, $x \sim N(0,1)$, by the transformation $x = \sqrt{-2 \ln(u)} \cos(2\pi u)$ where $u_1$ and $u_2$ are uniformly distributed independent random numbers lying between (0,1) and $x$ is the standard normal variate (Knuth (1969), Texas Instruments Inc (1979), p 54). Alternatively, one may generate $x \sim N(0,1)$ from uniformly distributed $u(0,1)$ numbers, by using the Central Limit Theorem (Gillett 1979, p. 519; Kapur & Saxena, 1982, p. 386). However, this method is less accurate and time consuming than Knuth's method. Normally distributed variate, $x$, may be used to generate Gamma distributed variate, $g$, since, if $x$ is a standard normal variate, then $g$
\( \frac{x^2}{2} \) is a Gamma variate with parameter \( \frac{1}{2} \) (Kapur & Saxena, 1982; p 288). Due to the additive property of Gamma variates, if \( x_i (i = 1,2,\ldots,n) \) are \( n \) independent normal variates with means \( m_i \) and standard deviations \( \sigma_i \), then \( g = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - m_i}{\sigma_i^2} \right)^2 \) is a Gamma variate with parameter \( \frac{1}{2n} \) (Kapur & Saxena, 1982; p. 289). From two independent normally distributed variates \( x_1 \) and \( x_2 \) we may obtain a Cauchy distributed variate, since, if \( x_1 \) and \( x_2 \) are independent normal variates with means \( m_1 \) and \( m_2 \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), then the variate \( c = \frac{x_1 - m_1}{x_2 - m_2} \) is Cauchy distributed (Kapur & Saxena, 1982; p. 427).

In particular, the quotient of two independent standard normal variates is Cauchy distributed. From two independent Gamma variates, \( g_1 \) and \( g_2 \) with parameters \( l \) and \( m \) respectively, we may obtain \( v_1 = \frac{g_1}{g_1 + g_2} \), which is a \( \beta_1(l,m) \) distributed variate, and \( v_2 = \frac{g_2}{g_1 + g_2} \), which is a \( \beta_2(l,m) \) distributed variate (Kapur & Saxena, 1982; p. 292). In general, starting from uniformly distributed variates, we may obtain a variate with almost any kind of distribution by a sequence of suitable transformations.

Graphic presentations of different distributions generated on computer (using the procedure mentioned above) give a fairly representative view of their nature. Among these, Gamma is a very skewed distribution for small shape and scale parameters. Gamma variates are non-negative. Beta variates are non-negative and largely platy-kurtic.