Chapter 4
SINGLE USE CONFIDENCE REGIONS
When $\Sigma = \sigma^2 I$

4.1 Introduction

In this chapter we derive confidence regions for the parameters $\theta$ and $\eta$ (unknown value of the explanatory variable). Here we assume that the variance covariance matrix of the explanatory variable is of the form $\sigma^2 I$ where $\sigma^2$ is unknown. For computing the confidence region, we develop some approximations that allow easy computation of the confidence region.

4.2 Canonical Forms

We shall first consider the models (1.1.3) and (1.1.4) where $\Sigma = \sigma^2 I_p$. Suppose $Y_1$, $S$, $A_1$ and $\theta_1$ be as defined in (2.2.1).

Let,

\[ Y_1 = YX^1 (XX^1)^{-\frac{1}{2}}, \quad s^2 = \text{tr}(S) = \text{tr}[Y(I - X^1 (XX^1)^{-1}X) Y^1] \quad (4.2.1) \]

\[ A_1 = A (XX^1)^{\frac{1}{2}}, \quad \theta_1 = (XX^1)^{\frac{1}{2}} \theta \]

Then

\[ Y_1 \sim N (A_1, \sigma^2 I_q \otimes I_p), \quad y \sim N (A_1 \theta_1, \sigma^2 I_p), \quad s^2 \sim \sigma^2 \chi^2_{(N-q)p} \quad (4.2.2) \]

where $N > q$ and $p \geq q$. 
We use $\chi^2_r$ to denote a central chisquared random variable with $r$ degrees of freedom. Note that $Y_1$ and $s^2$ form sufficient statistics for the model (1.1.3) when $\Sigma = \sigma^2 I_p$. For the models (1.1.3) and (1.1.5) the canonical form is,

$$ Y_1 \sim N(A_1, \sigma^2 (I_q \otimes I_p)) \quad s^2 / \sigma^2 \sim \chi^2_{p(N-q)}, $$

and

$$ y \sim N(A_1 t_1(\eta), \sigma^2 I_p) \quad (4.2.3) $$

where $\eta$ is an $r \times 1$ vector, and $t_1(\eta)$ is an $q \times 1$ vector valued function of $\eta$. $Y_1$ and $A_1$ are as defined in (2.2.1) and $t_1(\eta)$ as defined in (2.4.1) and $s^2$ is defined in (4.2.1). Also $Y_1$, $y$ and $s^2$ are independently distributed.

It is to be noted that $\theta$ and $\eta$ are invariant under the transformation,

$$ Y_1 \rightarrow cOY_1, \; y \rightarrow cOy \; \text{and} \; s^2 \rightarrow c^2 s^2 \quad (4.2.4) $$

where $c$ is a positive scalar and $O$ is a $p \times p$ orthogonal matrix.

### 4.3 Confidence Regions

We shall first consider the model (4.2.2). First we obtain a confidence region for $\theta_1$, then we use the transformation in (2.2.1) to obtain a confidence region for $\theta$. The confidence regions that we shall construct is based on the pivot $K(\theta_1)$ given by
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$$K(\theta_1) = \frac{(N-q)p}{q} \frac{(y - Y_1\theta_1)' Y_1 (Y_1'Y_1)^{-1} Y_1'(y - Y_1\theta_1)}{s^2} \quad (4.3.1)$$

Note that given $Y_1$, $y - Y_1\theta_1 \sim N((A - Y_1)\theta_1, \sigma^2 I_p)$ and furthermore, $Y_1(Y_1'Y_1)^{-1} Y_1'$ is a symmetric idempotent matrix of rank $q$. Hence given $Y_1$, the quantity $(y - Y_1\theta_1)' Y_1(Y_1'Y_1)^{-1} Y_1'(y - Y_1\theta_1) / \sigma^2$ is distributed as non-central chi-square with $q$ degrees of freedom and non-centrality parameter given by

$$\theta_1'(Y_1 - A_1)' Y_1(Y_1'Y_1)^{-1} Y_1'(Y_1 - A_1) \theta_1 / \sigma^2. \quad (4.3.2)$$

Consequently given $Y_1$,

$$K(\theta_1) \sim F^{(A)}_{q,(N-q)p}, \quad (4.3.2)$$

the non-central $F$-distribution with degrees of freedom $(q, (N - q) p)$ and non-centrality parameter $\Lambda$ given by

$$\Lambda = \theta_1'(Y_1 - A_1)' Y_1(Y_1'Y_1)^{-1} Y_1'(Y_1 - A_1) \theta_1 / \sigma^2 \quad (4.3.3)$$

Note that

$$\Lambda \leq \Lambda_0 = \theta_1'(Y_1 - A_1)' (Y_1 - A_1) \theta_1 / \sigma^2 \quad (4.3.4)$$

and since

$$(Y_1 - A_1) \theta_1 \sim N(0, \sigma^2 \theta_1' \theta_1 I_p), \Lambda_0 \sim \theta_1' \theta_1 \chi^2_p \quad (4.3.5)$$
For the construction of our confidence region we use the following theorem, and based on the theorem we develop a suitable approximation.

**Theorem 4.1**

Consider the model (4.2.2) and let \( K(\theta_1) \) be as defined in (4.2.3). Let \( f_{n_1}^{(\cdot)} \) denote the probability density function of a central chi-square random variable with \( n_1 \) degrees of freedom and further more \( F_{n_2,n_3} \) and \( F_{n_2,n_3}^{(\gamma)} \) respectively denote a central F random variable and a non-central F random variable, with non-centrality parameter \( \gamma \), both having degrees of freedom \( (n_2,n_3) \).

Then for any \( \alpha > 0 \)

\[
P[F_{q,(N-q)p} \leq \alpha] \geq P[K(\theta_1) \leq \alpha]
\]

\[
\geq \int_{0}^{\infty} P[F_{q,(N-q)p}(\theta',\theta_1 \leq \alpha)] f_{n_1}^{(n_1)}du
\]

The proof of this theorem is similar to that given in Mathew and Zha (1996).

Let \( V_{\alpha}(\theta',\theta_1) \) satisfy
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\[
\int_0^\infty p \left[ F_{q,(N-q)p}(\theta'_1 \theta_1 u) \right] \leq V_\alpha (\theta'_1 \theta_1) f_p^{(u)} du = 1 - \alpha \quad (4.3.7)
\]

Then the set \{\theta_1 : K(\theta_1) \leq V_\alpha (\theta'_1 \theta_1)\} (4.3.8)

is clearly a conservative confidence region for $\theta_1$ with coverage probability at least $1 - \alpha$.

We can generalise theorem 4.1 to the canonical form (4.2.3). The pivot used here is,

\[
K(\eta) = \frac{(N-q)p}{r} \left( Y - Y_{t_1}(\eta) \right)' T(\eta) \left( T(\eta)' T(\eta) \right)^{-1} T(\eta)' Y - Y_{t_1}(\eta) \quad (4.3.9)
\]

Then the set \{\eta : K(\eta) \leq V_\alpha (t_1(\eta)' t_1(\eta))\} (4.3.10)

is a conservative confidence region for $\eta$ with coverage probability at least $1 - \alpha$.

4.4. Computation of the Confidence Region

In the above theorem we consider the inequality

\[
P[F_{q,(N-q)p} \leq \alpha] \geq P[K(\theta_1) \leq \alpha] \\
\geq \int_0^\infty p \left[ F_{q,(N-q)p}(\theta'_1 \theta_1 u) \leq \alpha \right] f_p^{(u)} du
\]

Now we consider the approximation,
\[ F_{q,(N-q)p}^{(hu)} = e F_{q,(N-q)p} \]  \hspace{1cm} (4.4.1)

where \( h = \theta_1' \theta_1 \) and \( e = 1 + \frac{hp}{q} \). Hence (4.3.7) is of the form,

\[ P[e F_{q,(N-q)p}] \leq V_\alpha(\theta_1' \theta_1) = 1 - \alpha. \]  \hspace{1cm} (4.4.2)

We shall consider the computation of \( V_\alpha(\theta_1' \theta_1) \) satisfying (4.4.2).

Let \( V_\alpha(\theta_1' \theta_1) = V(h) \) where \( h = \theta_1' \theta_1 \).

The LHS of (4.4.2) is obtained by equating only the first moment of the F-distribution, i.e., we convert non-central F to central F by applying the above technique. Hence we have

\[ F_{q,(N-q)p}^{(hu)} = e F_{q,(N-q)p} \] (approximately), \( e = 1 + \frac{hp}{m} \)

Simulation results and application to real data sets show that the above approximation is quite satisfactory and easier for computation compared to existing methods.