CHAPTER 7

MOTIVATIONS

AND

OPEN

RESEARCH PROBLEMS
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In this short chapter, we wish to concentrate on some motivations of the results described in this monograph. These motivations give rise to many interesting research problems in the field of our study.

In chapter-2, we have established some enlightening results regarding radicals in the tensor product $X_1 \otimes y X_2$ considering $X_1$ and $X_2$ as unital Banach algebras. Now, the following problems can be raised:

1) Can we prove similar results if the Banach algebras $X_1$ and $X_2$ are without unity or having an approximate identity?

2) The results in chapter-2 are proved only in case of the projective tensor product of two Banach algebras. Can we carry out analogous study in case of the injective tensor product of two Banach algebras?

3) We obtained the structure of the Baer's lower radical in $X_1 \otimes y X_2$ if the structure of the same for $X_1$ and $X_2$ are given. Now, what can be said about the converse of this?

In chapter-3, we have discussed the metric approximation property (MAP) in the dual space of tensor product, and have proved the following result:

Let $X_1$ and $X_2$ be two Banach spaces such that $X_1^*$ and $X_2^*$ have the metric approximation property. If $F(X_1) \otimes F(X_2)$ is dense in $F(X_1 \otimes X_2)$,
and the dual norm of $\gamma$ is uniform on $X_1^* \otimes X_2^*$, then $(X_1^* \otimes X_2)^*$ has the metric approximation property.

This result can be generalized to the case of uniform reasonable cross norms on $X_1 \otimes X_2$, whose dual norms are also uniform. But it is not known whether the stated conditions are sufficient for any other tensor norm. The following problem arises in this context:

4) What are the sufficient conditions for $(X_1^* \otimes X_2)^*$ to have the metric approximation property for any tensor norm $\alpha$, if $X_1^*$ and $X_2^*$ have the metric approximation property?

Again regarding the converse of the above result, another interesting problem can be raised as:

5) Does the existence of metric approximation property in $(X_1^* \otimes X_2)^*$, where $\alpha$ is a tensor norm, ensure the existence of the same in $X_1^*$ and $X_2^*$ also?

In chapter-4, some interesting results are established on compact approximation property and its various applications in case of the projective tensor product $X_1 \otimes \gamma X_2$. If $\alpha$ is any cross norm on $X_1 \otimes X_2$, then the following problem arises:

6) Can we prove similar results regarding the compact approximation property (CAP) and the quasi approximation property (QAP) in case of $X_1^* \otimes X_2$?

Also, concerning the dual space another interesting problem is:
7) What are the sufficient conditions for \((X_1 \otimes \alpha X_2)^*\) to have the compact approximation property and quasi approximation property for any tensor norm \(\alpha\), if \(X_1^*\) and \(X_2^*\) have the same?

By Theorems 4.20 and 4.22 of chapter-4, we have proved that if \(X_1\) and \(X_2\) are two Banach spaces such that

(i) \(X_1 \otimes_{\gamma} X_2\) has a finite dimensional Schauder decomposition with \(l_p\)-norm, or

(ii) \(X_1 \otimes_{\gamma} X_2\) is a \(c_0\)-sum of finite dimensional spaces,

and further, if \(I_1\) and \(I_2\) are two closed ideals in \(X_1\) and \(X_2\) such that \(K(I_i)\) is an \(M\)-ideal in \(L(I_i)\) \((i=1,2)\), then \(K(I_1 \otimes_{\gamma} I_2)\) is an \(M\)-ideal in \(L(I_1 \otimes_{\gamma} I_2)\).

Now, we consider Banach spaces \(X_1\) and \(X_2\) such that \(X_1 \otimes_{\gamma} X_2\) satisfies none of the conditions stated in (i) and (ii). In that case, the following problem arises:

8) Can we have some other necessary conditions to be satisfied by \(X_1 \otimes_{\gamma} X_2\) so that \(K(I_1 \otimes_{\gamma} I_2)\) is an \(M\)-ideal in \(L(I_1 \otimes_{\gamma} I_2)\)?

Another immediate problem is:

9) Can we establish analogous results in case of the tensor products \(X_1 \otimes_{\alpha} X_2\) and \(I_1 \otimes_{\alpha} I_2\), \(\alpha\) being any cross norm on \(X_1 \otimes X_2\)?

Moreover, some interesting problems regarding various approximation properties are:

10) In [16], Choi and Kim introduced the properties of \textit{weak\textsuperscript{*}-density} (\textit{w*D}) and \textit{bounded weak\textsuperscript{*}-density} for the dual space of a Banach space \(X\).
and proved that if $X^*$ has the CAP and the W*D property, then $X$ has the CAP. Now, does the existence of W*D property in $X_1^*$ and $X_2^*$ imply the existence of the same in $(X_1 \otimes X_2)^*$, and conversely?

11) We know that if $M$ is a closed subspace of a Banach space $X$, then the pair $(X, M)$ has the three-space property for the AP, whenever $M$ is complemented in $X$ (refer to [16]). Now, if $M$ is any closed subspace (not necessarily complemented) of $X_1 \otimes X_2$, does $(X_1 \otimes X_2, M)$ have the three-space property for the AP, and also for the CAP?

12) In [17], it is proved that the WAP and QAP are inherited to the complemented subspaces, although their inheritance to the general subspaces is much harder. It is a well-known fact that if $M$ is a closed subspace of a Banach space $X$ and $M$ is complemented in $X$, then $M^\perp$ is complemented in $X^*$. But the converse is false in general. Now, we can pose the following problem:

Let $X_1$ and $X_2$ be two Banach spaces having WAP, and $\alpha$ be a cross norm on $X_1 \otimes X_2$. If $M$ is a closed subspace of $X_1 \otimes_\alpha X_2$ such that $M^\perp$ is complemented in $(X_1 \otimes_\alpha X_2)^*$, then does $M$ possess the WAP?

Similar problem arises in case of QAP also.

A Banach space $X$ has the Hermitian approximation property (HAP) [38] if for every compact set $K$ in $X$ and every $\varepsilon > 0$, there is a compact Hermitian operator $H$ on $X$ with $\|H\| \leq 1$ such that $\|Hx-x\| \leq \varepsilon$ for all $x \in K$. Regarding HAP in the tensor product, the following problems arise:
13) If $X_1$ and $X_2$ are two Banach spaces, then does the existence of HAP in $X_1$ and $X_2$ imply the existence of the same in $X_1 \otimes X_2$, and conversely?

14) If $X_1$ and $X_2$ are two Banach spaces with HAP, then what can be said about the existence of HAP in the dual space of $X_1 \otimes \alpha X_2$, for any tensor norm $\alpha$?

15) Can we establish some relation between the HAP and the QAP in the tensor product of two Banach spaces?

From [11], we see that only four natural tensor norms preserve the Banach algebra structure. In chapter-5, we have discussed only the case of projective tensor norm out of these four. Let $X_1$ and $X_2$ be any two Banach algebras and $\alpha$ be any of the remaining three tensor norms which preserve the Banach algebra structure. In this regard, the following problems arise:

16) Does the amenability of $X_1$ and $X_2$ imply the amenability of their tensor product $X_1 \otimes \alpha X_2$, and conversely?

17) Let $\alpha$ be a cross norm on $X_1 \otimes X_2$ and $\alpha^*$ be the dual norm of $\alpha$. Now, if $X_1$ and $X_2$ are amenable, what can be said about the amenability of $X_1^* \otimes \alpha^* X_2^*$?

In Theorem 5.18, we have considered the projective tensor product of reduced group C*-algebra, $C_r^*(G)$ of a locally compact group $G$, and any Banach algebra $A$. In general, there does not exist a unique C*-norm on $L^1(G)$. Another possible C*-norm is the universal C*-norm, $\| \cdot \|_u$ on $L^1(G)$.
[80]. For every $f \in L^1(G)$, this norm is defined by, $\|f\|_u = \{ \|\lambda(f)\| : \lambda$ is a $^*$-representation of $L^1(G)$ on a Hilbert space $\}.$

The universal group C*-algebra, $C^*(G)$ is the completion of $L^1(G)$ with respect to this norm.

We have mentioned in chapter-6 that on the algebraic tensor product of two C*-algebras the Haagerup norm is defined by:

$$\|u\|_h = \inf \left\{ \left\| \sum_{i=1}^{n} x_i^* y_i \right\|^{1/2} : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

Now, another interesting problem can be posed as:

18) Can we prove similar results in case of the Haagerup tensor product of universal group C*-algebras?

In chapter-6, Theorem 6.10 is proved considering the C*-algebras $X_1$ and $X_2$ as separable and taking the C*-norm $\alpha$ such that $X_1 \otimes_\alpha X_2$ is a type-I C*-algebra. In addition to these conditions, in Theorem 6.11, we take $X_1$ and $X_2$ as unital type-I C*-algebras. However, these restrictions are very strong. So, an immediate question is:

19) Without taking the restrictions of Theorem 6.10 and 6.11, how can we prove the same results using some weaker conditions?

The theory of compact quantum groups was developed by Woronowicz in 1987 and progress has been made on this topic in recent years. By a compact quantum group, we mean the dual object of a C*-algebra of continuous functions on some quantum space that is endowed with a group-
like structure. In [131], S. Wang discussed about the tensor products of compact quantum groups relating it with Woronowicz Hopf C*-algebras.

A Woronowicz Hopf C*-algebra is a unital C*-algebra $A$ together with a dense *-subalgebra $B$ generated by $u_{ij}^\alpha$ (where $\alpha \in K$, $K$ being an index set and $i, j \in \{1,2,\ldots,d_\alpha\}$), and a C*-homomorphism $\phi : A \to A \otimes A$, and a linear algebra antihomomorphism $\psi : B \to B$ such that

(i) The matrix $u^\alpha = (u_{ij}^\alpha)$ is a unitary element of $M_{d_\alpha} \otimes A$ for all $\alpha \in K$.

(ii) For $\alpha \in K$, and $i, j \in \{1,2,\ldots,d_\alpha\}$, $\phi(u_{ij}^\alpha) = \sum_{m=1}^{d_\alpha} u_{im}^\alpha \otimes u_{mj}^\alpha$.

(iii) For $b \in B$, $\psi(\psi(b^*)^*) = b$, and for $\alpha \in K$, $(I_d \otimes \psi)(u^\alpha) = (u^\alpha)^{-1}$.

[Here $M_{d_\alpha}$ denotes the algebra of $d_\alpha \times d_\alpha$ complex matrices.]

The above Woronowicz Hopf C*-algebra is denoted by $(A, B, \phi, \psi)$ or $(A, \phi)$. From [131], we have, if $(X_1, \phi_1)$ and $(X_2, \phi_2)$ are two Woronowicz Hopf C*-algebras, then a morphism from $X_1$ to $X_2$ is a unital C*-morphism $\pi : X_1 \to X_2$ such that $(\pi \otimes \pi)\phi_1 = \phi_2 \pi$.

The Woronowicz Hopf C*-algebras form a category under these morphisms. The category of compact quantum groups is defined to be the dual category of the category of Woronowicz Hopf C*-algebras. It is known that if $X_1$ and $X_2$ are two Woronowicz Hopf C*-algebras, then their maximal (and the minimal) tensor product C*-algebra has a unique Woronowicz Hopf C*-algebra structure. Now, an interesting question is:
20) Can we characterize the C*-tensor norms $\alpha$, which preserve the Woronowicz Hopf C*-algebra structures, and what will be the nature of the compact quantum groups corresponding to such tensor products?

A solution to this problem will also help us to study the automorphisms of an interesting variety of Woronowicz Hopf C*-dynamical systems.

On a Woronowicz Hopf C*-algebra $(X, \phi)$ (with unit $e$), the Haar state $h$ is defined to be a state on $X$ with the following properties:

(i) for every continuous functional $\varphi$ on $X$, we have,

$$\varphi * h = h * \varphi = \varphi(e) h,$$

where $\varphi = (\varphi \otimes h) \phi$;

(ii) for all $x \in X$, we have $x * h = h * x = h(x) e$,

where $x * h = (h \otimes I_d) \phi(x)$, $h * x = (I_d \otimes h) \phi(x)$.

Either of the properties (i) and (ii) determines $h$ uniquely.

If $\{X_i\}$ is an arbitrary family of Woronowicz Hopf C*-algebras, and $h_i$ is the Haar state on $X_i$, then the Haar state on the maximal (and the minimal) C*-tensor product of this family is determined in [131]. Now, another problem arises:

21) Let $\alpha$ be a C*-tensor norm and $\{X_i\}$ an arbitrary family of Woronowicz Hopf C*-algebras. Can we determine the Haar state on the C*-tensor product with respect to $\alpha$, of this family $\{X_i\}$?

If this problem were solved, then it would lead us to develop further study on the Haar measures and irreducible representations of the corresponding compact quantum groups.
There are many more problems and useful aspects necessary to consider under the same roof. The limitation of time and space prevents us from dwelling upon all these matters. We are quite hopeful that the works on the above mentioned areas and problems will contribute significantly to our research field, and this will open up a promising dimension for future research works.