CHAPTER 6

SOME INTRINSIC RESULTS IN TENSOR PRODUCTS OF INVOLUTIVE BANACH ALGEBRAS
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This chapter is mainly concerned with some special results on the
theory of tensor products of involutive Banach algebras. It is known that
many important Banach algebras naturally carry an involution. The C*-algebras, which is an essential tool in the theory of operator *-algebras, also play a crucial role in the study of representation of a very extensive class of involutive Banach algebras [23]. It seems that this theory is also important for the formal calculus with operator rings, the unitary representation theory of groups, a quantum mechanical formalism and abstract ring theory.

In [23], [106] and [121], there is an interesting discussion on various results regarding the representation theory of C*-algebras with applications to the group representations. In Section-I of this chapter, we want to apply this representation theory to study the ideal spaces of the tensor product of C*-algebras. We prove that if $M_1$ and $M_2$ are maximal regular left ideals in the separable C*-algebras $X_1$ and $X_2$ respectively, and $\alpha$ is a C*-norm on $X_1 \otimes X_2$ such that $X_1 \otimes_\alpha X_2$ is a type-I C*-algebra, then corresponding to $M_1$ and $M_2$,

This chapter is based on our research paper [28].
there is a unique maximal regular left ideal in $X_1 \otimes \alpha X_2$. We establish the converse of this in case of unital C*-algebras $X_1$ and $X_2$. As application, the structure of the Jacobson radical in $X_1 \otimes \alpha X_2$ is also discussed here.

In Section-II, we give a special consideration to the involutive measure algebra $M^1(G)$ for a locally compact group $G$. The following are some of the results presented in this section.

(i) Let $(A, G_1, \theta)$ and $(A, G_2, \xi)$ be two given C*-dynamical systems and let $f$ and $g$ be two states of the C*-algebra $A$ which are stationary for the group morphisms $\theta$ and $\xi$ respectively. If $x \rightarrow \theta(x)$ is a representation of $G_1$ and $y \rightarrow \xi(y)$ is a representation of $G_2$, then there exists a representation $\sigma$ (uniquely determined by $f$ and $g$) of $M^1(G_1) \otimes_\gamma M^1(G_2)$ on the Hilbert space tensor product $H_{\pi f} \otimes_\gamma H_{\pi g}$. Corresponding to this representation $\sigma$, a positive form $\phi$ on $M^1(G_1) \otimes_\gamma M^1(G_2)$ is also derived here.

(ii) Let $\rho_1$ be the representation of $M^1(G_1)$ associated with the above representation $\sigma$, and let $S$ be a positive operator on the space $H_{\pi f}$ which commutes with $\rho_1(M^1(G_1))$, and $\|S\| \leq 1$. For $p \otimes q \in H_{\pi f} \otimes_\gamma H_{\pi g}$, we construct a positive form $\psi_S$ on $M^1(G_1) \otimes_\gamma M^1(G_2)$.

(iii) The relations between the collection of all positive Hermitian operators on $H_{\pi f}$ and $H_{\pi g}$ with the set of all above mentioned type of representations $\psi_S$ are also established in this section. Moreover, some other interesting results are discussed with fruitful outcomes.
Before highlighting the main results of this section, we present a brief discussion on various tensor norms and tensor products on C*-algebras.

Let $\alpha$ be an algebra norm defined on an algebra $X$. $\alpha$ is called a C*-norm (refer to [7]) on $X$ if there is an involution on the $\alpha$-completion of $X$ making it into a C*-algebra. The greatest C*-norm, $\| \cdot \|_{\text{max}}$ and the least C*-norm $\| \cdot \|_{\text{min}}$ on the tensor product of C*-algebras $X_1$ and $X_2$ are already mentioned in Chapter-1.

On $X_1 \otimes X_2$, another important norm is the Haagerup norm, and it is defined by:

$$
\| u \|_h = \inf \left\{ \sum_{i=1}^{n} x_i^* x_i : \sum_{i=1}^{n} y_i^* y_i : u = \sum_{i=1}^{n} x_i \otimes y_i, x_i, y_i \in X_1 \otimes X_2 \right\}.
$$

The Haagerup tensor product, $X_1 \otimes_h X_2$ is the completion of $X_1 \otimes X_2$ with respect to this norm [7].

For a C*-algebra $X$, let $M_n(X)$ denote $X \otimes M_n$ with its unique C*-norm, and it is regarded as the involutive algebra of $n \times n$ matrices with elements in $X$. We consider the C*-algebras $X_1$ and $X_2$. For positive integers $m$ and $n$, let $P = (x_{ij}) \in M_m(X_1)$ and $Q = (y_{ij}) \in M_n(X_2)$. The tensor product $P \otimes Q$ is defined as the $mn \times mn$ matrix: $P \otimes Q = (x_{ij} \otimes y_{kl}) \in M_{mn}(X_1 \otimes X_2)$, using the relation $(M_m \otimes X_1) \otimes (M_n \otimes X_2) \cong M_{mn}(X_1 \otimes X_2)$. Every element $u \in X_1 \otimes X_2$ can be expressed as $u = \omega(P \otimes Q) \xi$, for some $\omega \in M_{1, mn}, P \in M_m(X_1), Q \in$
$M_{n}(X_{2})$, $\xi \in M_{mn,1}$, where $M_{pq}$ denotes the space of complex $p \times q$ matrices over $\mathbb{C}$. Now, on $X_{1} \otimes X_{2}$, the operator space projective tensor norm, $\| \cdot \|$ is defined by, $\| u \|_{\lambda} = \inf \{ \| \omega \|, \| P \|, \| Q \|, \| \xi \| : u = \omega (P \otimes Q)\xi$ is a decomposition as above}. The operator space projective tensor product, $X_{1} \otimes \lambda X_{2}$ is the completion of $X_{1} \otimes X_{2}$ with respect to this norm [79].

The various norms on $X_{1} \otimes X_{2}$ are ordered as:

\[ \| \cdot \|_{h} \leq \| \cdot \|_{\min} \leq \| \cdot \|_{\max} \leq \| \cdot \|_{h} \leq \| \cdot \|_{\lambda} \leq \| \cdot \|_{\gamma}. \]

In [7], Blecher presented the following geometrical characterizations of $C^{*}$-norms.

**Lemma 6.01** [7] Let $X$ be a $C^{*}$-algebra and $Y$ be a Banach algebra. Let $T : X \rightarrow Y$ be a contraction with dense range, and it maps some two-sided approximate identity for $X$ to a two-sided approximate identity for $Y$. Then there exists an involution on $Y$ such that $Y$ is a $C^{*}$-algebra and $T$ is involution preserving.

**Lemma 6.02** [7] Let $X$ and $Y$ be $C^{*}$-algebras. Then any algebra norm on $X \otimes Y$ which is dominated by $\| \cdot \|_{\max}$ is a $C^{*}$-norm.

For a $C^{*}$-algebra $X$, let $SL(X)$ denote the structure space for left ideals of $X$, i.e., the collection of all maximal regular left ideals of $X$. Our aim is to find out a relationship between the structure spaces for left ideals of two $C^{*}$-algebras $X_{1}$ and $X_{2}$ with that of their tensor product. We need the following Lemmas for this purpose.
Lemma 6.03 [23] Let $X$ be an involutive Banach algebra having an approximate identity. Let $f$ be a continuous positive form on $X$ and $\pi$ be the representation of $X$ defined by $f$. Then $\pi$ is non-trivial and topologically irreducible if and only if $f$ is pure.

[A continuous positive form $f$ on a normed involutive algebra $X$ is called pure, if $f \neq 0$ and if every continuous positive form on $X$ which is dominated by $f$ is of the form $\lambda f$ ($0 \leq \lambda \leq 1$).]

Lemma 6.04 [23] Every C*-algebra has an increasing approximate identity bounded by 1.

Lemma 6.05 [23] Let $X$ be a C*-algebra. If $X$ is separable and $I$ is a closed two-sided ideal of $X$, then $I$ is a primitive if and only if $I$ is prime.

Lemma 6.06 [23] Let $X$ be a C*-algebra. The primitive two-sided ideals of $X$ are just the kernels of the non-zero topologically irreducible representations of $X$ in a Hilbert space.

Lemma 6.07 [23] Let $X$ be a C*-algebra.

(i) Let $f$ be a positive form on $X$ and $N_f$ is the left ideal of those $x \in X$ such that $f(x^*x) = 0$. Then $N_f$ is maximal regular if and only if $f$ is pure.

(ii) The map $f \rightarrow N_f$ is a bijection from the set of pure states onto the set of maximal regular left ideals.

(iii) Every closed left ideal of $X$ is the intersection of the maximal regular left ideals containing it.
Lemma 6.08 Let $X_1$ and $X_2$ be two separable C*-algebras. Then for any C*-norm $\alpha$, $X_1 \otimes_{\alpha} X_2$ is separable.

For the study of group representation theory and the theory of singular integral operators, the type-I C*-algebras play an important role. In [23] and [121], there are several characterizations of type-I C*-algebras.

Lemma 6.09 [121] Let $X$ be a C*-algebra. The following conditions are equivalent.

(i) $X$ is a type-I C*-algebra.

(ii) $X$ is smooth, i.e., for every non-zero irreducible representation $(\pi, H)$ of $X$, $\pi(X)$ contains a non-zero compact operator on $H$.

(iii) Any quotient C*-algebra of $X$ does not satisfy the condition of Glimm, i.e., for every non-zero quotient C*-algebra $Z$ of $X$, there exists a positive element $h$ in $Z$ such that the rank of $\pi(h) \leq 1$ for all irreducible representations $(\pi, H)$ of $Z$.

Imposing the condition that the tensor product, $X_1 \otimes_{\alpha} X_2$, of the C*-algebras $X_1$ and $X_2$ is a type-I C*-algebra, we give our first result in this section as,

Theorem 6.10 Let $X_1$ and $X_2$ be two separable C*-algebras and $\alpha$ be a C*-norm on $X_1 \otimes X_2$ such that $X_1 \otimes_{\alpha} X_2$ is a type-I C*-algebra. Let $M_1$ be a maximal regular left ideal in $X_1$ and $M_2$ be a maximal regular left ideal in $X_2$. Then corresponding to $M_1$ and $M_2$, there is a unique maximal regular left ideal in $X_1 \otimes_{\alpha} X_2$. 
Proof. By Lemma 6.07, let $M_1 = N_f$ and $M_2 = N_g$, where $f$ is a pure state on $X_1$ and $g$ is a pure state on $X_2$.

If $X_1$ does not contain an identity element, let $\tilde{X}_1$ be the C*-algebra obtained by adjoining an identity to $X_1$. For each element $(\lambda, x) = (\lambda + x) \in \tilde{X}_1$, $(\lambda \in \mathbb{C}, x \in X_1)$, we take $\tilde{f}(\lambda + x) = \lambda \|f\| + f(x)$. Then $\tilde{f}$ is a positive form on $\tilde{X}_1$ extending $f$.

Let $N = \{x \in \tilde{X}_1 : f(x^*x) = 0\}$. Then $N$ is a left ideal of $\tilde{X}_1$. We consider the pre-Hilbert space $\tilde{X}_1/N$ and let $X_{1f}$ be the Hilbert space which is the completion of $\tilde{X}_1/N$. For each $x \in \tilde{X}_1$, let $\pi'(x)$ be the operator in $\tilde{X}_1/N$ obtained from left multiplication by $x$ in $\tilde{X}_1$ by passage to the quotient. Let $\xi$ be the canonical image of unity in $\tilde{X}_1/N$. Now, from [23], we have the following:

(i) Each $\pi'(x)$ has a unique extension to a continuous linear operator $\pi(x)$ in $X_{1f}$.

(ii) The map $x \to \pi(x)$, $(x \in X_1)$ is a representation of $X_1$ in $X_{1f}$.

(iii) $\xi$ is a cyclic vector for $\pi(X_1)$, i.e., the closed subspace generated by $\{\pi(x)\xi : x \in X_1\}$ is $X_{1f}$.

(iv) $f(x) = \langle \pi(x)\xi, \xi \rangle$ for each $x \in X_1$.

We denote the above representation of $X_1$ defined by $f$ in the Hilbert space $X_{1f}$ by $\pi_f$.
Similarly, let \( \pi_g \) be the representation of \( X_2 \) defined by \( g \) in the Hilbert space \( X_{2g} \). Now, \( X_1 \) and \( X_2 \) are having approximate identity, and \( f \) and \( g \) are (continuous) pure positive forms on \( X_1 \) and \( X_2 \) respectively. So, by Lemma 6.03, \( \pi_f \) and \( \pi_g \) are non-trivial and topologically irreducible.

We take \( P_f = \ker \pi_f \) and \( P_g = \ker \pi_g \). Since \( \pi_f \) and \( \pi_g \) are non-zero topologically irreducible representations, so, using Lemma 6.06, we have, \( P_f \) and \( P_g \) are primitive ideals in \( X_1 \) and \( X_2 \) respectively. Clearly, \( P_f \) and \( P_g \) are closed. \( X_1 \) and \( X_2 \) being separable, by Lemma 6.05, \( P_f \) and \( P_g \) are prime ideals in \( X_1 \) and \( X_2 \) respectively.

Let \( P = P_f \otimes X_2 + X_1 \otimes P_g \). Then \( P \) is a prime ideal in \( X_1 \otimes \alpha X_2 \). Let
\[
\left( \sum_i x_{ni} \otimes y_{ni} \right)
\]
be a sequence in \( P \) such that \( \sum_i x_{ni} \otimes y_{ni} \xrightarrow{\ast} \sum_k a_k \otimes b_k \) in \( X_1 \otimes \alpha X_2 \). Now, \( \sum_i x_{ni} \otimes y_{ni} \in P = P_f \otimes X_2 + X_1 \otimes P_g \ \forall \ n \)
\[\Rightarrow x_{ni} \in P_f \text{ or } y_{ni} \in P_g \ \forall \ n, i\]
So, \( \pi_f(x_{ni}) = 0 \) or \( \pi_g(y_{ni}) = 0 \ \forall \ n, i \)

We have, \((\pi_f \otimes \pi_g)(\sum_k a_k \otimes b_k) = \sum_k \pi_f(a_k) \otimes \pi_g(b_k)\)
\[\Rightarrow (\pi_f \otimes \pi_g)(\lim_{n \to \infty} \sum_i x_m \otimes y_m) = \sum_k \pi_f(a_k) \otimes \pi_g(b_k)\]
\[\Rightarrow \sum_k \pi_f(a_k) \otimes \pi_g(b_k) = \lim_{n \to \infty} \sum_i \pi_f(x_m) \otimes \pi_g(y_m) = 0\]
\[\Rightarrow \pi_f(a_k) = 0 \text{ or } \pi_g(b_k) = 0 \ \forall \ k\]
\[\Rightarrow a_k \in \ker \pi_f \text{ or } b_k \in \ker \pi_g \ \forall \ k\]
\[ \Rightarrow a_k \in P_f \text{ or } b_k \in P_g \forall k \]
\[ \Rightarrow a_k b_k \in P_f \otimes X_2 + X_1 \otimes P_g = P \forall k \]
\[ \Rightarrow \sum_k a_k b_k \in P. \]

So, \( P \) is a closed prime ideal in \( X_1 \otimes \alpha X_2 \). Since \( X_1 \otimes \alpha X_2 \) is separable, so, by Lemma 6.05, \( P \) is a primitive ideal in \( X_1 \otimes \alpha X_2 \). Let

\[ \text{Prim} (X_1 \otimes \alpha X_2) : \text{the set of all primitive two-sided ideals of } X_1 \otimes \alpha X_2. \]

\[ (X_1 \otimes \alpha X_2)^\wedge : \text{the set of classes of non-trivial topologically irreducible representations of } X_1 \otimes \alpha X_2. \]

Since \( X_1 \otimes \alpha X_2 \) is a type-I C*-algebra, so, from [23], we have, the mapping \( T : (X_1 \otimes \alpha X_2)^\wedge \to \text{Prim} (X_1 \otimes \alpha X_2) \) given by \( T(\pi_H) = \ker \pi_H \) is bijective.

Now, \( P \) is an element of \( \text{Prim} (X_1 \otimes \alpha X_2) \). So, \( P = \ker \pi_{H_P} \) for some non-trivial topologically irreducible representation \( \pi_{H_P} \) (uniquely determined by \( P \)) of \( X_1 \otimes \alpha X_2 \) in a Hilbert space \( H_P \), say. For \( \xi \in H_P \), let \( f_P \) denote the positive form on \( X_1 \otimes \alpha X_2 \) defined by: \( f_P (z) = \langle \pi_{H_P} (z) \xi , \xi \rangle, \ z \in X_1 \otimes \alpha X_2. \)

Since \( \pi_{H_P} \) is non-trivial and topologically irreducible and \( X_1 \otimes \alpha X_2 \) is having approximate identity, using Lemma 6.03, we have, \( f_P \) is pure.

Let \( M_{f_P} = \{ z \in X_1 \otimes \alpha X_2 : f_P (z^* z) = 0 \} \)

\[ = \{ z \in X_1 \otimes \alpha X_2 : \langle \pi_{H_P} (z^* z) \xi , \xi \rangle = 0 \} \]

By Lemma 6.07, \( M_{f_P} \) is a maximal regular left ideal in \( X_1 \otimes \alpha X_2. \) \( \square \)
Now, we proceed for the converse of the above Theorem. Here, we assume that the C*-algebras $X_1$ and $X_2$ are having the unit elements. The result can be stated as:

**Theorem 6.11** Let $X_1$ and $X_2$ be two separable type-I C*-algebras with unit elements $e_1$ and $e_2$ respectively. Let $\alpha$ be a C*-norm such that $X_1 \otimes_\alpha X_2$ is also a type-I C*-algebra. If $M$ is a maximal regular left ideal in $X_1 \otimes_\alpha X_2$, then corresponding to $M$, the maximal regular left ideals $M_1$ and $M_2$ can be uniquely constructed for the spaces $X_1$ and $X_2$ respectively.

**Proof.** Since $M$ is a maximal regular left ideal in $X_1 \otimes_\alpha X_2$, so, $M = N^f$, where $f$ is a pure state on $X_1 \otimes_\alpha X_2$. Let $\pi_f$ be the representation of $X_1 \otimes_\alpha X_2$ defined by $f$ (refer to [23]). Now $X_1 \otimes_\alpha X_2$ having approximate identity, by Lemma 6.03, $\pi_f$ is non-trivial and topologically irreducible, i.e., $\pi_f \in (X_1 \otimes_\alpha X_2)^\wedge$. So, $P = \ker \pi_f \in \text{Prim} (X_1 \otimes_\alpha X_2)$. Clearly, $P$ is closed. Since $X_1 \otimes_\alpha X_2$ is separable, so, by Lemma 6.05, $P$ is a prime ideal in $X_1 \otimes_\alpha X_2$.

We define $\phi_1: X_1 \rightarrow (X_1 \otimes_\alpha X_2)/P$ such that $\phi_1(x) = (x \otimes e_2)/P$ and $\phi_2: X_2 \rightarrow (X_1 \otimes_\alpha X_2)/P$ such that $\phi_2(y) = (e_1 \otimes y)/P$.

Let $P_1 = \ker \phi_1$ and $P_2 = \ker \phi_2$. Then $P_1$ and $P_2$ are closed prime ideals in $X_1$ and $X_2$ respectively. Again $X_1$ and $X_2$ being separable, $P_1 \in \text{Prim} (X_1)$ and $P_2 \in \text{Prim} (X_2)$.

Let $\hat{\mathcal{X}}_i$ $(i=1,2)$ denote the set of classes of non-trivial topologically irreducible representations of $X_i$ $(i=1,2)$ respectively. Now, we consider the
mappings $T_1 : \hat{X}_1 \to \text{Prim } (X_1)$ and $T_2 : \hat{X}_2 \to \text{Prim } (X_2)$. Since $X_1$ and $X_2$ are type-I C*-algebras so, the mappings $T_1$ and $T_2$ are bijective [23]. Thus, there exists a unique non-trivial topologically irreducible representation $\pi_1$ of $X_1$ in some Hilbert space $H_{\pi_1}$, say, such that $P_1 = \ker \pi_1$. Similarly, there exists a unique $\pi_2 \in \hat{X}_2$ such that $P_2 = \ker \pi_2$.

For $\xi_1 \in H_{\pi_1}$, let $f_1$ denote the positive form on $X_1$ defined by $f_1(x) = \langle \pi_1(x)\xi_1, \xi_1 \rangle$. Since $\pi_1$ is non-trivial and topologically irreducible and $X_1$ is having approximate identity, using Lemma 6.03, we have, $f_1$ is pure.

Let $M_1 = \{ x \in X_1 : f_1(x^*x) = 0 \}
= \{ x \in X_1 : \langle \pi_1(x^*x)\xi_1, \xi_1 \rangle = 0 \}$.

Then, by Lemma 6.07, $M_1$ is a maximal regular left ideal in $X_1$. Similarly, we can construct a maximal regular left ideal $M_2$ for $X_2$ also. \quad \square

**Corollary 6.12** Let $X_1$ and $X_2$ be two commutative C*-algebras and let $\Omega$ be the spectrum of $X_1$. For $M_1 \in SL(X_1)$ and $M_2 \in SL(X_2)$, there exists a unique $M \in SL(C_0(\Omega, X_2))$. If $X_1$ and $X_2$ are also unital, then $M \in SL(C_0(\Omega, X_2))$ uniquely determines $M_1 \in SL(X_1)$ and $M_2 \in SL(X_2)$.

[$C_0(\Omega, X_2)$ denotes the Banach space of all $X_2$-valued continuous functions $f$ vanishing at infinity on $\Omega$, with the norm: $\| f \| = \{ \sup || f(t) || : t \in \Omega \}$.

**Proof.** By [121], we have, for commutative C*-algebra $X_1$ and for any C*-algebra $X_2$, the least C*-norm $\| . \|_{\text{min}}$ on $X_1 \otimes X_2$ coincides with the injective tensor norm $\lambda$, and $X_1 \otimes_\lambda X_2 \cong C_0(\Omega, X_2)$. Again $X_1 \otimes_\lambda X_2$ is
commutative and so, it is a liminal C*-algebra [23], i.e., for every irreducible representation \(\pi\) of \(X_1 \otimes \kappa X_2\), \(\pi(x)\) is compact for each \(x \in X_1 \otimes \kappa X_2\). So, by Lemma 6.09, \(X_1 \otimes \kappa X_2\) is a type-I C*-algebra. Similarly, both \(X_1\) and \(X_2\) are also type-I C*-algebras. Now, the result follows by Theorems 6.10 and 6.11. □

**Corollary 6.13** Let \(X_1\) and \(X_2\) be two unital commutative C*-algebras and let \(\Omega_i\) be the spectrum of \(X_i\) \((i=1, 2)\). For \(M_1 \in SL(X_1)\) and \(M_2 \in SL(X_2)\), there exists a unique \(M \in SL(C_0(\Omega_1 \times \Omega_2))\). Conversely, \(M \in SL(C_0(\Omega_1 \times \Omega_2))\) uniquely determines \(M_1 \in SL(X_1)\) and \(M_2 \in SL(X_2)\).

**Proof.** If \(X_1\) and \(X_2\) are commutative C*-algebras and \(X_2\) is unital, then by [121], there exists only one C*-norm \(\alpha\) on \(X_1 \otimes X_2\), and \(\alpha = \lambda\). Moreover, \(X_1 \otimes \kappa X_2 \cong C_0(\Omega_1 \times \Omega_2)\). The rest of the proof follows now as in the above Corollary 6.12. □

As an application of the above results, we now derive the structure of the Jacobson radical in the tensor product of two C*-algebras. The idea of three different radicals (viz., nil, Levitzki and Baer's lower radical) has already been encountered in Chapter-2. Relating the property of right quasi-regularity in an algebra and the Jacobson radical, we mention here the following results (refer to [22]).

**Lemma 6.14** Let \(X\) be an algebra and \(I\) be a right quasi-regular right ideal of \(X\). If \(x\) is a right quasi-regular element in \(X\) and if \(y \in I\), then \(x + y\) is right quasi-regular.
Proof. Since $x$ is right quasi-regular, we have, for some $t$, $x + t + xt = 0$.

We consider the element $y + yt \in I$. Since $I$ is a right quasi-regular right ideal, so, $y + yt$ is right quasi-regular, and let $z$ be its right quasi-inverse.

Then, $y + yt + z + (y + yt)z = 0$. Now, it can be easily shown that $t + z + tz$ is a right quasi-inverse for $x + y$. Thus, $x + y$ is a right quasi-regular element in $X$. 

Corollary 6.15 If $I_1$ and $I_2$ are two right quasi-regular right ideals of an algebra $X$, then $I_1 + I_2$ is also a right quasi-regular right ideal of $X$.

Let $J$ be the sum of all the right quasi-regular right ideals of an algebra $X$. Then $J$ is a right quasi-regular right ideal and it contains every right quasi-regular right ideal of $X$. Moreover, we have:

Lemma 6.16 $J$ is a two-sided ideal of $X$.

Lemma 6.17 Let $X$ be a right quasi-regular algebra. Then every homomorphic image of $X$ is also a right quasi-regular algebra.

Lemma 6.18 The quotient algebra $X/J$ is semi-simple with respect to right quasi-regularity.

[The proofs are easy and so omitted.]

Thus, we have,

Theorem 6.19 Right quasi-regularity is a radical property for the algebra $X$.

The corresponding radical $J$ is the Jacobson radical for $X$. From [22], we get the following representation of $J$. 
Lemma 6.20 [22] The Jacobson radical of any algebra $X$ is equal to

(i) the intersection of all the maximal regular right ideals of $X$.

(ii) the intersection of all the maximal regular left ideals of $X$.

(iii) \{a: ax \text{ is right quasi-regular, for every } x \text{ in } X\}.

(iv) \{a: xa \text{ is left quasi-regular, for every } x \text{ in } X\}.

In view of Theorems 6.10 and 6.11, we get that if C*-algebras $X_1$ and $X_2$ satisfy the conditions stated in these Theorems, then there exists a one-to-one correspondence between the spaces $SL(X_1) \times SL(X_2)$ and $SL(X_1 \otimes \alpha X_2)$.

For $(M_1, M_2) \in SL(X_1) \times SL(X_2)$, using Theorem 6.10, we can uniquely construct the ideal $M_{fp}$ in the space $SL(X_1 \otimes \alpha X_2)$. Let $M_{fp} = S(M_1, M_2)$, where $S: SL(X_1) \times SL(X_2) \to SL(X_1 \otimes \alpha X_2)$. Now, by Lemma 6.20, the Jacobson radical in $X_1 \otimes \alpha X_2$ can be given by,

$$J(X_1 \otimes \alpha X_2) = \cap \{M : M \in SL(X_1 \otimes \alpha X_2)\}$$

$$= \cap \{S(M_1, M_2) : (M_1, M_2) \in SL(X_1) \times SL(X_2)\}.$$

Again, if $M \in SL(X_1 \otimes \alpha X_2)$, then by Theorem 6.11, the pair $(M_1, M_2) \in SL(X_1) \times SL(X_2)$ can be uniquely determined. We take $M_i = S_i(M)$, where $S_i: SL(X_i \otimes \alpha X_2) \to SL(X_i)$, $(i=1,2)$. So, the Jacobson radical for $X_i$ $(i=1,2)$ can be given by, $J(X_i) = \cap \{S_i(M) : M \in SL(X_i \otimes \alpha X_2)\}$, $(i=1,2)$.

Thus, we have the following result:

Theorem 6.21 Let $X_1$ and $X_2$ be two separable type-I C*-algebras with unit elements. Let $\alpha$ be a C*-norm such that $X_1 \otimes \alpha X_2$ is also a type-I C*-

algebra. Then the Jacobson radical in $X_1 \otimes_a X_2$ can be determined from the structure spaces for left ideals of the spaces $X_1$ and $X_2$.

Conversely, if the structure space for left ideals of $X_1 \otimes_a X_2$ is known, then we can determine the Jacobson radical for each of $X_1$ and $X_2$.

**Remark 6.22** Since every element in a nil right ideal is right quasi-regular, so, the Jacobson radical $J$ of an algebra $X$ contains the nil radical of $X$. In general, the Jacobson radical is not nil [22]. But if $X$ is an algebraic algebra over its field, then $J$ coincides with the nil radical of $X$. [We recall that if $X$ is an algebra over the field $F$, then an element $x$ of $X$ is said to be *algebraic* [22] if it generates a finite dimensional algebra over $F$. $X$ is said to be *algebraic over $F$* if every element of $X$ is algebraic over $F$.] Now, if the $C^*$-algebras $X_1$ and $X_2$ are algebraic, then for any $C^*$-norm $\alpha$, their tensor product $X_1 \otimes_a X_2$ is also algebraic and in that case, the Jacobson radical of $X_1 \otimes_a X_2$ coincides with the nil radical of $X_1 \otimes_a X_2$.

**SECTION-II**

In this section, we discuss mainly the representation theory of the projective tensor product of involutive algebras $M^d(G)$, where $G$ is a locally compact group. Some concepts on $C^*$-dynamical systems are also related with this representation theory.

We recall that (refer to [18]) a $C^*$-dynamical system is a triple $(A, G, \theta)$ where $A$ is a $C^*$-algebra, $G$ is a locally compact group and $\theta: G \rightarrow \text{Aut}(A)$ is
a group morphism which is continuous in the sense that for each \( z \in \mathcal{A} \), the function \( x \mapsto \theta(x)z \) is a continuous function from \( G \) to \( \mathcal{A} \). Now, for the locally compact group \( G \), \( M^1(G) \) is the Banach algebra (under convolution) of bounded complex measures on \( G \), and this is an involutive algebra, where for every \( \mu \in M^1(G) \), \( \mu^* \) is defined by, \( d\mu^*(s) = d\mu(s^{-1}), s \in G \) [23].

If \( G_1 \) and \( G_2 \) are two locally compact groups, we consider the projective tensor product \( M^1(G_1) \otimes \gamma M^1(G_2) \), and for \( \mu = \sum_i \mu_i \otimes \eta_i \in M^1(G_1) \otimes \gamma M^1(G_2) \), we take \( \mu^* = \sum_i \mu_i^* \otimes \eta_i^* \). Then \( ^* \) is an involution on \( M^1(G_1) \otimes \gamma M^1(G_2) \) and thus it becomes an involutive Banach algebra. With the help of two given \( C^* \)-dynamical systems, we now proceed for constructing a representation of this involutive algebra.

**Theorem 6.23** Let \((A, G_1, \theta)\) and \((A, G_2, \xi)\) be two given \( C^* \)-dynamical systems and let \( f \) and \( g \) be two states of the \( C^* \)-algebra \( A \) which are stationary for the group morphisms \( \theta \) and \( \xi \) respectively. If \( x \mapsto \theta(x) \) is a representation of \( G_1 \) and \( y \mapsto \xi(y) \) is a representation of \( G_2 \), then there exists a representation \( \sigma \) (uniquely determined by \( f \) and \( g \)) of the projective tensor product \( M^1(G_1) \otimes \gamma M^1(G_2) \) on the space \( H_{\pi_f} \otimes \gamma H_{\pi_g} \).

**Proof.** For the \( C^* \)-dynamical system \((A, G_1, \theta)\), we have, the group morphism \( \theta: G_1 \to \text{Aut}(A) \) is continuous in the sense that for each \( z \in \mathcal{A} \), the function \( x \mapsto \theta(x)z \) is a continuous function from \( G_1 \) to the \( C^* \)-algebra \( A \). Again the state \( f \) on \( A \) is stationary for \( \theta \), i.e., \( f(\theta(x)z) = f(z) \), for each \( z \in \mathcal{A} \).
and \( x \in G_1 \). Let \( \pi_f \) be the representation defined by \( f \) and \( H_{\pi_f} \) be the associated Hilbert space. Also given that \( x \rightarrow \theta(x) \) is a representation of \( G_1 \).

Under these conditions, there exists a unique continuous unitary representation \( \rho_1 \) of \( G_1 \) (refer to [23]) in \( H_{\pi_f} \), satisfying

\[
\pi_f(\theta(x)z) = \rho_1(x) \pi_f(z) \rho_1(x^{-1}), \quad \text{for each } z \in A, \ x \in G_1.
\]

Similarly, from the C*-dynamical system \( (A, G_2, \xi) \), we get a unique continuous unitary representation \( \rho_2 \) of \( G_2 \) in \( H_{\pi_g} \), satisfying

\[
\pi_g(\xi(y)z) = \rho_2(y) \pi_g(z) \rho_2(y^{-1}), \quad \text{for each } z \in A, \ y \in G_2.
\]

Let \( \mu = \sum_i \mu_i \otimes \eta_i \in M^d(G_1) \otimes_y M^d(G_2) \). We define:

\[
\sigma(\mu) = \sum_i \int_{G_1} \int_{G_2} \rho_1(s) d\mu_i(s) \otimes \rho_2(t) d\eta_i(t) \in L(H_{\pi_f}) \otimes L(H_{\pi_g}) \subset L(H_{\pi_f} \otimes_y H_{\pi_g})
\]

Let \( x = \sum_i a_i \otimes b_i, \ y = \sum_j c_j \otimes f_j \in M^d(G_1) \otimes_y M^d(G_2) \)

Clearly, \( \sigma(x + y) = \sigma(x) + \sigma(y) \),

\[
\sigma(\lambda x) = \lambda \sigma(x), \quad \lambda \in \mathbb{C}.
\]

\[
\sigma(xy) = \sigma(\sum_{i,j} a_i c_j \otimes b_i f_j)
\]

\[
= \sum_{i,j} \int_{G_1} \rho_1(s) da_i(s) \int_{G_1} \rho_1(s) dc_j(s) \otimes \int_{G_2} \rho_2(t) db_i(t) \int_{G_2} \rho_2(t) df_j(t)
\]

\[
= \int_{G_1} \rho_1(s) \sum_i a_i \int_{G_1} \rho_1(s) dc_j(s) \otimes \int_{G_2} \rho_2(t) db_i(t) \int_{G_2} \rho_2(t) df_j(t)
\]

\[
= \sigma(x)\sigma(y)
\]

We have, \( x^* = \sum_i a_i^* \otimes b_i^* \). Let \( \varphi \otimes \psi \in H_{\pi_f} \otimes_y H_{\pi_g} \).
Now, $\langle p \otimes q, \sigma(x^*)(u \otimes v) \rangle$

$$= \left\langle p \otimes q, \sigma\left(\sum_i a_i * \otimes b_i * (u \otimes v)\right) \right\rangle$$

$$= \left\langle p \otimes q, \left(\sum_i \int G_1 \rho_1(s) da_i * (s) \otimes \int G_2 \rho_2(t) db_i * (t)\right)(u \otimes v) \right\rangle$$

$$= \left\langle p \otimes q, \sum_i \int G_1 \rho_1(s) da_i * (s) \otimes \int G_2 \rho_2(t)(v) db_i * (t) \right\rangle$$

$$= \left\langle p \otimes q, \sum_i \int G_1 \rho_1(s)(u) \otimes \rho_2(t)(v)) da_i * (s) db_i * (t) \right\rangle$$

$$= \left\langle \sum_i \int G_1 \rho_1(s)(u) \otimes \rho_2(t)(v), p \otimes q \right\rangle$$

$$= \sum_i \int G_1 \rho_1(s)(u) \otimes \rho_2(t)(v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}(u) \otimes \rho_2(t^{-1})(v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}) \otimes \rho_2(t^{-1})(u \otimes v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}) \otimes \rho_2(t^{-1})(u \otimes v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}) \otimes \rho_2(t^{-1})(u \otimes v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}) \otimes \rho_2(t^{-1})(u \otimes v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$$= \sum_i \int G_1 \rho_1(s^{-1}) \otimes \rho_2(t^{-1})(u \otimes v), p \otimes q \right\rangle da_i * (s) db_i * (t)$$

$\rho_1$ and $\rho_2$ are the continuous unitary representations of $G_1$ and $G_2$ respectively. So, $\rho_1(s^{-1}) = \rho_1((s^{-1})^*) = \rho_1(s)$.

Similarly, $\rho_2(t^{-1})^* = \rho_2((t^{-1})^*) = \rho_2(t)$. So, the expression (*) is equal to:
Thus, \( \sigma(x^*) = \sigma(x)^* \).

So, \( x \to \sigma(x) \) is a representation of \( \mathcal{M}(G_1) \otimes \mathcal{M}(G_2) \) in \( H_{\pi_f} \otimes H_{\pi_g} \). □

For commutative groups \( G_1 \) and \( G_2 \), we can give a characterization of the above representation \( \sigma \) in terms of Fourier transformation. For a locally compact group \( G \), let \( \hat{G} \) be the set of all continuous unitary representations of \( G \). For \( \mu \in \mathcal{M}(G) \), the function \( \hat{\mu} \) on \( \hat{G} \) defined by \( \hat{\mu}(\pi) = \pi(\mu), \pi \in \hat{G} \), is called the Fourier transformation of \( \mu \) (refer to [23]). Extending this concept to the tensor product, we can prove:

**Deduction 6.24** Let the locally compact groups \( G_1 \) and \( G_2 \), considered in Theorem 6.23 be abelian. Then the representation \( \sigma \) of \( \mathcal{M}(G_1) \otimes \mathcal{M}(G_2) \) can be given by: \( \sigma(\mu) = \hat{\mu}(\rho_1, \rho_2) \), where \( \hat{\mu} \) is defined as the Fourier transformation of \( \mu \in \mathcal{M}(G_1) \otimes \mathcal{M}(G_2) \), and \( \sigma, \rho_1, \rho_2 \) are as defined in Theorem 6.23.

**Proof.** Let \( \mu = \sum \mu_i \otimes \eta_i \in \mathcal{M}(G_1) \otimes \mathcal{M}(G_2) \). For \( (\pi_1, \pi_2) \in \hat{G}_1 \times \hat{G}_2 \), we define \( \hat{\mu}(\pi_1, \pi_2) = \sum \pi_1(\mu_i) \otimes \pi_2(\eta_i) \). Then \( \hat{\mu}(\pi_1, \pi_2) \) is a continuous linear operator on \( H_{\pi_f} \otimes H_{\pi_g} \).
Let \( \mu_1 = \sum_i a_i \otimes b_i, \mu_2 = \sum_j c_j \otimes d_j \in M^1(G_1) \otimes \gamma M^1(G_2). \)

Then \((\mu_1 + \mu_2)^{\gamma} = \hat{\mu}_1 + \hat{\mu}_2.\)

\[
(\mu_1 \mu_2)^{\gamma}(\pi_1, \pi_2) = (\sum_{i,j} a_i c_j \otimes b_i d_j)(\pi_1, \pi_2)
= \sum_{i,j} \pi_i(a_i) \pi_j(c_j) \otimes \pi_2(b_i) \pi_2(d_j)
= \sum_{i,j} \pi_i(a_i) \pi_2(b_i) \sum_j \pi_j(c_j) \otimes \pi_2(d_j) = (\hat{\mu}_1 \hat{\mu}_2)(\pi_1, \pi_2)
\]

Thus, \((\mu_1 \mu_2)^{\gamma} = \hat{\mu}_1 \hat{\mu}_2.\)

Also, it can be easily verified that

\((\lambda \mu)^{\gamma} = \lambda \hat{\mu}, \lambda \in \mathbb{C}, \) and \((\mu^*)^{\gamma} = (\hat{\mu})^*.\)

\[
\|\hat{\mu}(\pi_1, \pi_2)\| = \left\| \sum_i \pi_i(\mu_i) \otimes \pi_2(\eta_i) \right\| \leq \sum_i \|\pi_i(\mu_i)\| \|\pi_2(\eta_i)\| \leq \|\pi_1\| \|\pi_2\| \sum_i \|\mu_i\| \|\eta_i\|
\]

Thus, \(\sup \|\hat{\mu}\| \leq \|\mu\|\).

Let \( p = \sum_i x_i \otimes y_i \) be a non-zero element of \( M^1(G_1) \otimes \gamma M^1(G_2). \) Then there exists \( x_k, y_k \) such that \( x_k \neq 0, y_k \neq 0. \) Let \( \mathcal{K}(G_i) \) be the set of all continuous complex valued functions on \( G_i (i=1, 2) \) with compact support.

Now, there exists some \( f_i \in \mathcal{K}(G_i) (i = 1, 2) \) such that \( \delta_1 = f_1 \ast x_k \) is a non-zero element of \( L^1(G_1) \) and \( \delta_2 = f_2 \ast y_k \) is a non-zero element of \( L^1(G_2). \) Then we have some \( \pi_1 \in \hat{G}_1 \) and \( \pi_2 \in \hat{G}_2 \) (refer to [23]) such that \( \pi_1(f_1) \pi_1(x_k) = \pi_1(f_1 \ast x_k) = \pi_1(\delta_1) \neq 0, \) and \( \pi_2(f_2) \pi_2(y_k) = \pi_2(f_2 \ast y_k) = \pi_2(\delta_2) \neq 0. \) So, \( \pi_1(x_k) \neq 0 \) and \( \pi_2(y_k) \neq 0. \) Thus, \( \hat{\rho}(\pi_1, \pi_2) = \sum_i \pi_1(x_i) \otimes \pi_2(y_i) \neq 0. \)
Hence the transformation \( \mu \rightarrow \hat{\mu} \) is injective. We call \( \hat{\mu} \) as the Fourier transformation of \( \mu \).

Since \( G_1 \) and \( G_2 \) are abelian, so, \( \hat{\mu}_i(\rho_1) \) and \( \hat{\eta}_i(\rho_2) \) are the scalar operators \( \int_{\mathcal{G}_i} \rho_1(s)d\mu_i(s) \) and \( \int_{\mathcal{G}_2} \rho_2(t)d\eta_i(t) \) respectively [23]. From Theorem 6.23, we have, 
\[
\sigma(\mu) = \sum_i \int_{\mathcal{G}_i} \rho_1(s)d\mu_i(s) \otimes \int_{\mathcal{G}_2} \rho_2(t)d\eta_i(t) \\
= \sum_i \hat{\mu}_i(\rho_1) \otimes \hat{\eta}_i(\rho_2) = \sum_i \rho_1(\mu_i) \otimes \rho_2(\eta_i) \\
= \hat{\mu}(\rho_1, \rho_2).
\]

In [23], Dixmier presented a correspondence between representations and the positive forms on a given involutive algebra \( A \). Considering tensor product \( M^1(G_1) \otimes \gamma M^1(G_2) \), we derive two different types of positive forms.

**Theorem 6.25** Let \( \rho_1 \) and \( \rho_2 \) be the representations of \( G_1 \) and \( G_2 \), as defined in Theorem 6.23. Then

(a) For \( p \otimes q \in \mathcal{H}_{\pi,\gamma} \otimes \mathcal{H}_{\pi,\gamma} \), let \( \phi \) be the mapping defined on \( M^1(G_1) \otimes \gamma M^1(G_2) \) by: 
\[
\phi(\mu) = \sum_i \langle \rho_1(\mu_i)p, p \rangle \langle \rho_2(\eta_i)q, q \rangle, \quad \text{where} \quad \mu = \sum_i \mu_i \otimes \eta_i \in M^1(G_1) \otimes \gamma M^1(G_2).
\]
Then \( \phi \) is a positive form on \( M^1(G_1) \otimes \gamma M^1(G_2) \).

(b) We have, \( \rho_1(a) = \int_{\mathcal{G}_1} \rho_1(s)da(s) \), \( a \in M^1(G_1) \) defines a representation of \( M^1(G_1) \). Let \( S \) be a positive operator on the Hilbert space \( \mathcal{H}_{\pi,\gamma} \) which commutes with \( \rho_1(M^1(G_1)) \), and \( \|S\| \leq 1 \). For \( p \otimes q \in \mathcal{H}_{\pi,\gamma} \otimes \mathcal{H}_{\pi,\gamma} \), let \( \psi_S \) be the mapping on \( M^1(G_1) \otimes \gamma M^1(G_2) \) defined by:
\[ \psi_S(\mu) = \langle \sigma(\mu)(Sp \otimes q), Sp \otimes q \rangle, \ \mu \in M^d(G_1) \otimes, M^d(G_2). \]

i) \( \psi_S \) is a positive form on \( M^d(G_1) \otimes, M^d(G_2) \).

ii) \( \psi_S(\mu^*\mu) \leq \phi(\sum_i \mu_i \otimes \eta_i \otimes \eta_j), \) where \( \mu = \sum_i \mu_i \otimes \eta_i \).

**Proof.** (a) Clearly \( \phi \) is linear.

For \( \mu = \sum_i \mu_i \otimes \eta_i \in M^d(G_1) \otimes, M^d(G_2) \), we have \( \mu^*\mu = \sum_i \mu_i \otimes \eta_i \otimes \eta_j \).

Now, \( \phi(\mu^*\mu) = \sum_{i,j} \langle \rho_i(\mu_i, \mu_j), \rho_j(\eta_i, \eta_j)q, q \rangle \)

\[ \begin{align*}
&= \sum_{i,j} \langle \rho_i(\mu_i), \rho_j(\eta_i) q, q \rangle \\
&= \sum_{i,j} \langle \rho_i(\mu_i), \rho_j(\eta_i) q, \rho_j(\eta_i) q \rangle \\
&= \sum_{i,j} \langle \rho_i(\mu_i), p \otimes \rho_j(\eta_i) q, \rho_j(\eta_i) q \rangle \\
&= \langle \sum_j \rho_j(\eta_j) p \otimes \rho_j(\eta_j) q, \sum_i \rho_i(\mu_i) p \otimes \rho_j(\eta_i) q \rangle \\
&= \langle \sigma(\sum_j \eta_j(p \otimes q)), \sigma(\sum_i \mu_i \otimes \eta_i)(p \otimes q) \rangle \\
&= \langle \sigma(\mu)(p \otimes q), \sigma(\mu)(p \otimes q) \rangle = \|\sigma(\mu)(p \otimes q)\|^2 \geq 0.
\end{align*} \]

Thus, \( \phi \) is a positive form on \( M^d(G_1) \otimes, M^d(G_2) \).

(b) The mapping \( \psi_S \) on \( M^d(G_1) \otimes, M^d(G_2) \) is defined by

\[ \psi_S(\mu) = \langle \sigma(\mu)(Sp \otimes q), Sp \otimes q \rangle, \ \mu = \sum_i \mu_i \otimes \eta_i \in M^d(G_1) \otimes, M^d(G_2). \]

Clearly, \( \psi_S \) is linear. Now, \( \psi_S(\mu^*\mu) = \langle \sigma(\mu^*\mu)(Sp \otimes q), Sp \otimes q \rangle \)
\[
\begin{align*}
= (\sigma(\mu)\sigma(\mu)(Sp \otimes q), Sp \otimes q) = (\sigma(\mu)\sigma(\mu)(Sp \otimes q), Sp \otimes q) \\
= (\sigma(\mu)(Sp \otimes q), \sigma(\mu)(Sp \otimes q)) \\
= \|\sigma(\mu)(Sp \otimes q)\|^2 \geq 0.
\end{align*}
\]

Hence \( \psi_S \) is a positive form on \( M^\delta(G_1) \otimes M^\delta(G_2) \).

We have, the inner product on \( H_{\pi_f} \otimes H_{\pi_g} \) is defined by:

\[
\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle.
\]

So, \( \|a \otimes b\|^2 = \langle a \otimes b, a \otimes b \rangle = \langle a, a \rangle \langle b, b \rangle = \|a\|^2 \|b\|^2 \).

Thus, \( \| \cdot \| \) is a cross norm on \( H_{\pi_f} \otimes H_{\pi_g} \). So, for \( u \in H_{\pi_f} \otimes H_{\pi_g} \), \( \|u\| \leq \|u\|_\gamma \),

as \( \gamma \) is the greatest cross norm.

Then \( \psi_S(\mu^* \mu) = \|\sigma(\mu)(Sp \otimes q)\|_\gamma^2 \leq \|\sigma(\mu)(Sp \otimes q)\|_\gamma^2 \) \quad \text{..........................(6.1)}

We define \( T : H_{\pi_f} \otimes \gamma H_{\pi_g} \rightarrow H_{\pi_f} \otimes \gamma H_{\pi_g} \) by \( T(\sum_{i} a_i \otimes b_i) = \sum_{i} Sa_i \otimes Sb_i \).

Clearly, \( T \) is linear.

\[
\|T(\sum_{i} a_i \otimes b_i)\|_\gamma = \|\sum_{i} Sa_i \otimes Sb_i\|_\gamma \leq \|S\| \sum_{i} \|a_i\| \|b_i\| \Rightarrow \|T\| \leq \|S\| \leq 1.
\]

From (6.1), \( \psi_S(\mu^* \mu)^{1/2} \leq \|\sigma(\mu)(Sp \otimes q)\|_\gamma = \|\sigma(\sum_{i} \otimes \eta_i)(Sp \otimes q)\|_\gamma \)

\[
= \left\| \sum_{\alpha_i} \left( \int_{G_1} \rho_1(s)du_1(s) \otimes \int_{G_1} \rho_2(t)d\eta_1(t) \right)(Sp \otimes q) \right\|_\gamma
\]

\[
= \sum_{\alpha_i} \left( \int_{G_1} \rho_1(s)(Sp)du_1(s) \otimes \int_{G_1} \rho_2(t)(q)d\eta_1(t) \right) \right\|_\gamma \quad \text{..........................(6.2)}
\]

\[
= \left\| \sum_{\alpha_i} \left( \int_{G_1} \rho_1(s)du_1(s) \otimes \int_{G_1} \rho_2(t)d\eta_1(t) \right) \right\|_\gamma
\]
But $S$ commutes with $\rho_1(M^1(G_1))$. So, $S\rho_1(\mu_i) = \rho_1(\mu_i)S$, for all $i=1,2,\ldots$.

$$\Rightarrow (S\rho_1(\mu_i))(p) = (\rho_1(\mu_i)S)(p)$$

$$\Rightarrow S(\rho_1(\mu_i)(p)) = (\rho_1(\mu_i))(Sp)$$

$$\Rightarrow S(\int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s)) = \int_{\mathcal{G}_i}\rho_1(s)(Sp)d\mu_i(s)$$

So, from (6.2),

$$\psi_S(\mu*\mu)^{1/2} \leq \left\| \sum_{i} S(\int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s)) \otimes \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|$$

$$= \left\| T(\sum_{i} \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s)) \otimes \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|$$

$$\leq \|T\| \left\| \sum_{i} \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s)) \otimes \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|$$

$$\leq \sum_{i} \left\| \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s) \right\| \left\| \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|$$

$$\leq \left( \sum_{i} \left\| \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s) \right\|^2 \right)^{1/2} \cdot \left( \sum_{i} \left\| \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|^2 \right)^{1/2}$$

(By Holder's inequality)

$$\Rightarrow \psi_S(\mu*\mu) \leq \left( \sum_{i} \left\| \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s) \right\|^2 \right) \cdot \left( \sum_{i} \left\| \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right\|^2 \right)$$

$$= \left( \sum_{\mathcal{G}_i} \left( \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s) , \int_{\mathcal{G}_i}\rho_1(s) p\,d\mu_i(s) \right) \right) \cdot \left( \sum_{\mathcal{G}_2} \left( \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) , \int_{\mathcal{G}_2}\rho_2(t)q\,d\eta_i(t) \right) \right)$$
In the part (b) of Theorem 6.25, using the positive (Hermitian) operator $S$, we have obtained the representation $\psi_S$. Let $\mathcal{M}$ be the collection of all such type of representations. Now, the following question is of interest:

**Can we establish some relation between the collection of all positive Hermitian operators on $H_{\pi f}$ with $\mathcal{M}$?**

By the next result, we try to give an answer to this question.

**Theorem 6.26** Let $\mathcal{H}(H_{\pi f})$ be the collection of all positive Hermitian operators $S$ on $H_{\pi f}$ which commutes with $\rho_1(M^d(G_1))$, and $\|S\| \leq 1$. Let $\mathcal{M} = \{\psi_S : S \in \mathcal{H}(H_{\pi f})\}$. If $p \otimes q$ is a cyclic vector for the representation $\mu \rightarrow \sigma(\mu)$ of $M^d(G_1) \otimes \gamma M^d(G_2)$, then there is a one to one correspondence between $\mathcal{H}(H_{\pi f})$ and $\mathcal{M}$.

**Proof.** Since $S$ is Hermitian, so, $\hat{S}$ on $H_{\pi f} \otimes H_{\pi g}$ defined by:

$\hat{S}(\sum a_i \otimes b_i) = \sum S a_i \otimes b_i$, is Hermitian (refer to [38]).

Now, $\sigma(\mu)(Sp \otimes q) = \sigma(\sum \mu_i \otimes \eta_i)(Sp \otimes q)$.
\[ \sum_{t} \int_{G_1} \rho_1(s) \pi_1(s) \check{\eta}_1(t) + \sum_{t} \int_{G_2} \rho_2(t) \check{\eta}_2(t) \]

\[ = \sum_{t} \int_{G_1} \rho_1(s) \pi_1(s) \check{\eta}_1(t) \]

\[ = \sum_{t} \int_{G_2} \rho_2(t) \check{\eta}_2(t) \]

\[ = \hat{S}(\sigma(\mu)(p \otimes q)) \]

\[ = \hat{S}(\sigma(\mu)(p \otimes q)) \] (6.3)

Let \( S_1, S_2 \in \mathcal{K}(H_{\pi_f}). \) Then

\[ \Psi_{S_1} = \Psi_{S_2} \]

\[ \Rightarrow \Psi_{S_1}(\mu) = \Psi_{S_2}(\mu), \text{ for all } \mu \in M^1(G_1) \otimes M^1(G_2). \]

\[ \Rightarrow \langle \sigma(\mu)(S_1p \otimes q), S_1p \otimes q \rangle = \langle \sigma(\mu)(S_2p \otimes q), S_2p \otimes q \rangle \]

\[ \Rightarrow \langle \hat{S}_1(\sigma(\mu)(p \otimes q)), \hat{S}_1(p \otimes q) \rangle = \langle \hat{S}_2(\sigma(\mu)(p \otimes q)), \hat{S}_2(p \otimes q) \rangle \] [using (6.3)]

\[ \Rightarrow \langle \sigma(\mu)(p \otimes q), \hat{S}_1^2(p \otimes q) \rangle = \langle \sigma(\mu)(p \otimes q), \hat{S}_2^2(p \otimes q) \rangle \]

\[ \Rightarrow \hat{S}_1^2(p \otimes q) = \hat{S}_2^2(p \otimes q) \] (6.4)

Let \( u = \sum_{j} x_j \otimes y_j \in H_{\pi_f} \otimes H_{\pi_g} \)

Now, \( \hat{S}_1(\sigma(\mu))(u) = \hat{S}_1(\sigma(\sum_{j} x_j \otimes y_j)) \)

\[ = \hat{S}_1((\sum_{t} \int_{G_1} \rho_1(s) \pi_1(s) \check{\eta}_1(t))(\sum_{j} x_j \otimes y_j)) \]

\[ = \hat{S}_1(\sum_{t} \int_{G_1} \rho_1(s) x_j \pi_1(s) \check{\eta}_1(t))(\sum_{j} y_j \check{\eta}_1(t)) \]

\[ = \sum_{j} \sum_{t} \int_{G_1} \rho_1(s) x_j \pi_1(s) \check{\eta}_1(t)(\sum_{j} y_j \check{\eta}_1(t)) \]
Thus, \( \hat{S}_1 \sigma(\mu) = \sigma(\mu) \hat{S}_1 \), for all \( \mu \in M_\gamma(G_1) \otimes \gamma M_\gamma(G_2) \).

So, \( \hat{S}_1 \in \text{Comm} (\sigma(M_\gamma(G_1) \otimes \gamma M_\gamma(G_2))) \) and then

\[ \hat{S}_1^2 \in \text{Comm} (\sigma(M_\gamma(G_1) \otimes \gamma M_\gamma(G_2))). \]

[Comm \( A \) : Commutant of \( A \)]

Similarly, \( \hat{S}_2^2 \in \text{Comm} (\sigma(M_\gamma(G_1) \otimes \gamma M_\gamma(G_2))) \). Since \( p \otimes q \) is a cyclic vector for the representation \( \mu \rightarrow \sigma(\mu) \) of \( M_\gamma(G_1) \otimes \gamma M_\gamma(G_2) \), so \( p \otimes q \) is a separating vector for the commutant of \( \sigma(M_\gamma(G_1) \otimes \gamma M_\gamma(G_2)) \) (refer to [23]).

So, from (6.4), \( \hat{S}_1^2 (p \otimes q) = \hat{S}_2^2 (p \otimes q) \Rightarrow \hat{S}_1^2 = \hat{S}_2^2 \).

Since \( S_1 \) and \( S_2 \) are Hermitian operators, so, it can be proved that both \( \hat{S}_1 \) and \( \hat{S}_2 \) are also Hermitian operators on \( H_\pi \otimes H_\pi \).

Let \( \omega_1 \in \text{Sp}(S_1) \) [Sp(\( S \)) : Spectrum of \( S \)]

Then \( S_1 - \omega_1 I_1 \) is singular. So, there exists a non-zero element \( x \in H_\pi \) such that \( (S_1 - \omega_1 I_1)(x) = 0 \).

Let \( y \) be a non-zero element of \( H_\pi \). Then \( x \otimes y \in H_\pi \otimes H_\pi \), and \( x \otimes y \) is non-zero. Now, \( (\hat{S}_1 - \omega_1 I_1)(x \otimes y) = \hat{S}_1(x \otimes y) - \omega_1 x \otimes y \)

\[ = S_1 x \otimes y - \omega_1 x \otimes y = (S_1 - \omega_1 I_1) x \otimes y = 0 \]

So, \( \omega_1 \in \text{Sp}(\hat{S}_1) \). Thus \( \text{Sp}(S_1) \subseteq \text{Sp}(\hat{S}_1) \).
Again, let $\omega \in \text{Sp}(\hat{S}_1)$. Then there exists a non-zero element $z \in H_n \otimes H_n$ such that $(\hat{S}_1 - \omega I)z = 0$. Without loss of generality, we can take
\[ z = \sum_{i} x_i \otimes y_i, \text{ where the set } \{ y_1, y_2, \ldots \} \text{ is linearly independent.} \]

Now, $(\hat{S}_1 - \omega I)z = 0 \Rightarrow (\hat{S}_1 - \omega I)(\sum_i x_i \otimes y_i) = 0$
\[ \Rightarrow \sum_i \hat{S}_i x_i \otimes y_i - \sum_i \omega x_i \otimes y_i = 0 \]
\[ \Rightarrow \sum_i (\hat{S}_i - \omega I)x_i \otimes y_i = 0 \]

..........................(6.5)

Let $h$ and $k$ be two bounded linear functionals on $H_n$ and $H_n$ respectively.

Using (6.5), \( (\sum_i (\hat{S}_i - \omega I)x_i \otimes y_i)(h, k) = 0 \)
\[ \Rightarrow \sum_i h((\hat{S}_i - \omega I)x_i)k(y_i) = 0 \]
\[ \Rightarrow k(\sum_i h((\hat{S}_i - \omega I)x_i)y_i) = 0 \]

Since $k$ is arbitrary, this gives,
\[ \sum_i h((\hat{S}_i - \omega I)x_i)y_i = 0 \]
\[ \Rightarrow h(\hat{S}_1 - \omega I)x_i = 0, \text{ for all } i, \text{ since } \{ y_1, y_2, \ldots \} \text{ is linearly independent.} \]
\[ \Rightarrow (\hat{S}_1 - \omega I)x_i = 0, \text{ for all } i, \text{ since } h \text{ is arbitrary.} \]

Now, because $z$ is non-zero, there exists at least one $i$, such that $x_i \neq 0$.

$\Rightarrow \hat{S}_1 - \omega I$ is singular $\Rightarrow \omega \in \text{Sp}(S_1)$. Hence $\text{Sp}(\hat{S}_1) \subset \text{Sp}(S_1)$

Thus, $\text{Sp}(\hat{S}_1) = \text{Sp}(S_1) \subset [0, \infty)$, $[\text{as } S_1 \geq 0]$

Similarly, $\text{Sp}(\hat{S}_2) = \text{Sp}(S_2) \subset [0, \infty)$. 

So, $\hat{S}_1 \geq 0, \hat{S}_2 \geq 0$ and hence $\hat{S}_1^2 = \hat{S}_2^2 \Rightarrow \hat{S}_1 = \hat{S}_2$.

Let $y$ be a non-zero element of $H_{\pi_e}$. Then for any $x \in H_{\pi_f}$, we have,

$$\hat{S}_1(x \otimes y) = \hat{S}_2(x \otimes y) \Rightarrow S_1 x \otimes y = S_2 x \otimes y \Rightarrow (S_1 x - S_2 x) \otimes y = 0$$

$$\Rightarrow S_1 x = S_2 x$$

$$\Rightarrow S_1 = S_2$$

So, the mapping $S \rightarrow \psi_S$, $S \in \mathcal{H}(H_{\pi_f})$ of $\mathcal{H}(H_{\pi_f})$ to $\mathcal{M}$ is injective.

Thus, there is a one to one correspondence between $\mathcal{H}(H_{\pi_f})$ and $\mathcal{M}$. \(\square\)

Now, we come back to the C*-dynamical system $(A, G, \theta)$ and consider its covariant representation. A covariant representation (refer to [18], [110]) of the system $(A, G, \theta)$ is a triple $(\pi, u, H)$, where $(\pi, H)$ is a representation of $A$; $(u, H)$ is a unitary representation of $G$ and $\pi((\theta(s))(z)) = u(s) \pi(z) u(s)^*$, for all $z$ in $A$ and $s$ in $G$. Every C*-dynamical system has a covariant representation [110].

Let the C*-algebra $A$ be separable and let $\mathcal{K}(G, A)$ be the collection of all functions $f : G \rightarrow A$ with compact support. Taking $\|f\| = \int_G \|f(s)\| ds < \infty$, where "$ds$" denotes the Haar measure on $G$, we have, $L^1(G, A)$ is the completion of $\mathcal{K}(G, A)$ in this norm. The multiplication on $L^1(G, A)$ is defined by $(f, g)(t) = \int_G f(s) \theta(s)(g(s^{-1}t)) ds$, as a sort of twisted convolution, and also an involution is defined by $f^*(t) = \Delta(t)^{-1} \theta(t)(f(t^{-1})^*)$, where $f, g \in \mathcal{K}(G, A)$ and $s, t \in G$ (refer to [18]). Under these operations, $L^1(G, A)$
becomes an involutive Banach algebra and it is denoted by $L^1(G, A; \theta)$. Relating covariant representation of $(A, G, \theta)$ with the non-degenerate representations of the algebra $L^1(G, A; \theta)$, Pedersen obtained the following result.

**Lemma 6.27** [110] If $(\pi, u, H)$ is a covariant representation of $(A, G, \theta)$, then there is a non-degenerate representation $(\pi \times u, H)$ of $L^1(G, A; \theta)$ such that $(\pi \times u)(f) = \int_{G} \pi(f(s))u(s)ds$ for every $f \in K(G, A)$.

Also, the correspondence $(\pi, u, H) \mapsto (\pi \times u, H)$ is a bijection onto the set of non-degenerate representations of $L^1(G, A; \theta)$.

For the given $C^*$-dynamical systems $(A, G_1, \theta)$ and $(A, G_2, \xi)$, we now derive a positive form for the projective tensor product $L^1(G_1, A; \theta) \otimes_{\gamma} L^1(G_2, A; \xi)$.

**Theorem 6.28** Let $(A, G_1, \theta)$ and $(A, G_2, \xi)$ be the given $C^*$-dynamical systems with the conditions as stated in Theorem 6.23. For every element $u \in H_{\pi f} \otimes \chi H_{\pi g}$, there is a positive form $\tau$ on $L^1(G_1, A; \theta) \otimes_{\gamma} L^1(G_2, A; \xi)$ with $\|\tau\| = \sum_j \|p_j\|^2 \|q_j\|^2$, where $u = \sum_j p_j \otimes q_j$.

**Proof.** For the $C^*$-dynamical system $(A, G_1, \theta)$, there exists a unique continuous unitary representation $\rho_1$ of $G_1$ in $H_{\pi f}$, satisfying

$$\pi_f(\theta(x)z) = \rho_1(x) \pi_f(z) \rho_1(x^{-1}) = \rho_1(x) \pi_f(z) \rho_1(x)^*, \text{ (since } \rho_1 \text{ is unitary) for each } z \in A, \ x \in G_1. \text{ Thus } (\pi_f, \rho_1, \theta) \text{ is a covariant representation for the }.$$


system \((A, G_1, \theta)\). By Lemma 6.27, there is a non-degenerate representation 
\((\pi_f \times \rho_1, H_{\pi_f})\) of \(L^1(G_1, A; \theta)\) such that
\[
(\pi_f \times \rho_1)(x) = \int_0^1 \pi_f(x(s))\rho_1(s)ds \quad \text{for every } x \in \mathcal{K}(G_1, A).
\]

Similarly, for the continuous unitary representation \(\rho_2\) of \(G_2\) in \(H_{\pi_{g_2}}\), we 
get a non-degenerate representation \((\pi_g \times \rho_2, H_{\pi_{g_2}})\) of \(L^1(G_2, A; \xi)\).

We denote the representations \(\pi_f \times \rho_1\) and \(\pi_g \times \rho_2\) by \(\pi_1\) and \(\pi_2\) respectively.

Now, for \(u = \sum_j p_j \otimes q_j \in H_{\pi_f} \otimes H_{\pi_{g_2}}\), we define \(\tau\) on \(L^1(G_1, A; \theta) \otimes_L L^1(G_2, A; \xi)\)
by 
\[
\tau(\sum_i x_i \otimes y_i) = \sum_{i,j} \langle \pi_1(x_i) p_j, p_j \rangle \langle \pi_2(y_i) q_j, q_j \rangle.
\]

Then \(\tau\) is a positive form on \(L^1(G_1, A; \theta) \otimes_L L^1(G_2, A; \xi)\).

We know that for each \(p_j\), the representation \(\pi_1\) defines a positive form 
\(f_{1j}: x \rightarrow \langle \pi_1(x) p_j, p_j \rangle\) on \(L^1(G_1, A; \theta)\).

Similarly, for each \(q_j\), we have, \(f_{2j}: y \rightarrow \langle \pi_2(y) q_j, q_j \rangle\) is a positive 
form on \(L^1(G_2, A; \xi)\). Thus, \(\tau(\sum_i x_i \otimes y_i) = \sum_{i,j} f_{1j}(x_i) f_{2j}(y_i)\).

Now, 
\[
\|\tau(\sum_i x_i \otimes y_i)\| \leq \sum_{i,j} \|f_{1j}(x_i) f_{2j}(y_i)\| \leq \sum_j \|f_{1j}\| \|f_{2j}\| \sum_i \|x_i\| \|y_i\|.
\]

Again, \(\|f_{1j}(x)\| \leq \|\pi_1(x)\| \|p_j\| \leq \|x\| \|p_j\|\) and so, \(\|f_{1j}\| \leq \|p_j\|^2\).

Similarly, \(\|f_{2j}\| \leq \|q_j\|^2\).

Then \(\|\tau\| \leq \sum_j \|f_{1j}\| \|f_{2j}\| \leq \sum_j \|p_j\|^2 \|q_j\|^2\).
By Lemma 6.04, we have that the C*-algebra $A$ has an increasing approximate identity bounded by 1. So, using Theorem 5.06 [Chapter-5], $L^1(G_1, A; \theta)$ has an approximate identity, say, $\{e_i\}_i$ bounded by 1. Similarly, let $\{u_k\}_k$ be the approximate identity in $L^1(G_2, A; \xi)$. So, $\{e_i \otimes u_k\}_i \otimes u_k$ is an approximate identity for $L^1(G_1, A; \theta) \otimes_y L^1(G_2, A; \xi)$.

By [23], $\|\tau\| = \lim_{i, k} \tau(e_i \otimes u_k)$

$$= \lim_{i, k} \sum_j \langle \pi_1(e_i) p_j, p_j \rangle \langle \pi_2(u_k) q_j, q_j \rangle$$

$$= \sum_j \lim_{i, k} \langle \pi_1(e_i) p_j, p_j \rangle \lim_{k} \langle \pi_2(u_k) q_j, q_j \rangle$$

$$= \sum_j \langle p_j, p_j \rangle \langle q_j, q_j \rangle, \quad \{e_i\}_i \text{ and } \{u_k\}_k \text{ being approximate identities, and}$$

$$\pi_1, \pi_2 \text{ being non-degenerate representations, both } \pi_1(e_i) \text{ and } \pi_2(u_k) \text{ tends strongly to } I \}$$

$$= \sum_j \|p_j\|^2 \|q_j\|^2$$

$\square$

**Deduction 6.29** For each of the positive forms $f_{1j}$ and $f_{2j}$ defined in the Theorem 6.28, let $N_{1j}$ and $N_{2j}$ denote the left ideals: $N_{1j} = \{x \in L^1(G_1, A; \theta): f_{1j}(x^* x) = 0\}$ and $N_{2j} = \{y \in L^1(G_2, A; \xi): f_{2j}(y^* y) = 0\}$. Then

$$\cap (N_{1j} \otimes N_{2j}) + \cap (N_{1j}^* \otimes N_{2j}^*) \subseteq \ker \tau.$$

**Proof.** For $\sum x_i \otimes y_i \in L^1(G_1, A; \theta) \otimes_y L^1(G_2, A; \xi)$, we have,

$$\left| \tau(\sum x_i \otimes y_i) \right| = \left| \sum_{i,j} f_{1j}(x_i) f_{2j}(y_i) \right| \leq \sum_{i,j} \left| f_{1j}(x_i) \right| \left| f_{2j}(y_i) \right|$$
\[ \left( \sum_{i,j} |f_{ij}(x_i)|^2 \right)^{1/2} \left( \sum_{i,j} |f_{2j}(y_i)|^2 \right)^{1/2} \]

\[ \Rightarrow \left| \tau(\sum_{i} x_i \otimes y_i) \right|^2 \leq \left( \sum_{i,j} |f_{ij}(x_i)|^2 \right) \left( \sum_{i,j} |f_{2j}(y_i)|^2 \right) \]

\[ \leq \left( \sum_{i,j} \|f_{ij}(x_i \ast x_i)\| \right) \left( \sum_{i,j} \|f_{2j}(y_i \ast y_i)\| \right) \] ..........................(6.6)

If \( u = \sum x_i \otimes y_i \in \cap(N_{ij} \otimes N_{2j}) \), then \( x_i \otimes y_i \in N_{ij} \otimes N_{2j} \) for each \( i \), and for each \( j \). So, \( f_{ij}(x_i \ast x_i) = 0 \) and \( f_{2j}(y_i \ast y_i) = 0 \) for each \( i \), and for each \( j \). By (6.6), \( u \in \ker \tau \), and thus, \( \cap(N_{ij} \otimes N_{2j}) \subseteq \ker \tau \). Again, \( \cap(N_{ij} \ast N_{2j}) \subseteq (\ker \tau)^\ast = \ker \tau \). Thus, \( \cap(N_{ij} \otimes N_{2j}) + \cap(N_{ij} \ast N_{2j}) \subseteq \ker \tau \). \( \square \)

We recall that given a C*-dynamical system \((A, G, \theta)\), the crossed product of \( A \) by the action \( \theta \) of \( G \), denoted by \( A \times_\theta G \), is the enveloping C*-algebra of \( L^1(G, A; \theta) \) (refer to [18]). Regarding representations of an involutive Banach algebra \( X \) and its enveloping C*-algebra \( Y \), we have,

**Lemma 6.30** [23] Let \( X \) be an involutive Banach algebra having an approximate identity, \( Y \) the enveloping C*-algebra of \( X \) and \( \iota \) the canonical map of \( X \) into \( Y \). Then

(i) If \( \pi \) is a representation of \( X \), there is exactly one representation \( \rho \) of \( Y \) such that \( \pi = \rho \circ \iota \), and \( \rho(Y) \) is the C*-algebra generated by \( \pi(X) \).

(ii) The map \( \pi \rightarrow \rho \) is a bijection of the set of representations of \( X \) onto the set of representations of \( Y \).

(iii) \( \rho \) is non-degenerate if and only if \( \pi \) is non-degenerate.
Now, from Theorem 6.28, corresponding to the non-degenerate representations $\pi_1$ and $\pi_2$ of $L^1(\mathcal{G}_1, A; \theta)$ and $L^1(\mathcal{G}_2, A; \xi)$, using above Lemma, we get the non-degenerate representations $\rho_1$ and $\rho_2$ of $A \times_\theta \mathcal{G}_1$ and $A \times_\xi \mathcal{G}_2$ respectively. So, analogous results as in Theorem 6.28 can be obtained in case of $(A \times_\theta \mathcal{G}_1) \otimes \gamma (A \times_\xi \mathcal{G}_2)$. 