CHAPTER 5

ON

THE AMENABILITY

OF

TENSOR PRODUCTS OF

BANACH ALGEBRAS
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This chapter is devoted to the study of amenability in the tensor product of two Banach algebras. The concept of amenability has its origin in the beginning of the modern measure theory. Since 1940s, amenability has become an important concept in abstract harmonic analysis. In 1972, Johnson introduced the notion of amenability as a cohomological property of Banach algebras. He showed that the amenability of a locally compact group $G$ can be characterized in terms of the Hochschild cohomology of its group algebra $L^1(G)$. Since then, amenability has become an important part in other branches of mathematics, such as von Neumann algebras, operator spaces, and even differential geometry. In [71], Johnson gave a characterization of amenability in terms of the approximate diagonals in a Banach algebra. In this chapter, we discuss these ideas in the generalized group algebra and also in the projective tensor product of group algebras, and we give a characterization of amenability in both of these algebras. If $X_1$ and $X_2$ are two unital Banach algebras, then we establish here that the amenability of $X_1 \otimes \gamma X_2$ ensures the amenability of each of the spaces $X_1$ and $X_2$. Investigating amenability in the injective tensor product, we prove that

This chapter is based on our research papers [31] and [55].
if $X_1$ and $X_2$ are unital Banach algebras having no non-trivial $M$-ideal, such that their injective tensor product, $X_1 \otimes X_2$ is an algebra, then $X_1 \otimes X_2$ is amenable if and only if for some suitable compact Hausdorff space $K$, every $M$-ideal in the function space $C(K, X_1 \otimes X_2)$ is amenable. With the help of this, we derive the conditions for amenability of the space of approximable operators on a Banach space $X$.

First we state the following Theorems and Lemmas which play an important role in establishing our results.

**Theorem 5.01** [10] For any tensor norm $\alpha$ the following two conditions are equivalent:

(i) For all Banach algebras $X_1$ and $X_2$ the natural algebra structure on $X_1 \otimes X_2$ induces a Banach algebra structure on $X_1 \otimes \alpha X_2$.

(ii) For all Banach spaces $E_1$, $E_2$, $F_1$, $F_2$ there is a natural continuous linear map $f: (E_1 \otimes_{\alpha} E_2) \otimes_{\gamma} (F_1 \otimes_{\alpha} F_2) \rightarrow (E_1 \otimes_{\gamma} F_1) \otimes_{\alpha} (E_2 \otimes_{\gamma} F_2)$ with

$$f((e_1 \otimes e_2) \otimes (f_1 \otimes f_2)) = (e_1 \otimes f_1) \otimes (e_2 \otimes f_2), \quad e_1 \in E_1, \ e_2 \in E_2, \ f_1 \in F_1, \ f_2 \in F_2.$$ 

Since the projective tensor norm $\gamma$ preserves the Banach algebra structure, so, applying the above Theorem for $\alpha = \| \| \gamma$ and taking $E_1 = E_2 = X_1$ and $F_1 = F_2 = X_2$, we get

**Theorem 5.02** If $X_1$ and $X_2$ are two Banach spaces, then there exists a natural continuous linear map

$$f: (X_1 \otimes_{\gamma} X_1) \otimes_{\gamma} (X_2 \otimes_{\gamma} X_2) \rightarrow (X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} (X_1 \otimes_{\gamma} X_2)$$

such that

$$f((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2), \quad a_1, a_2 \in X_1, \ b_1, b_2 \in X_2.$$
**Theorem 5.03** [70] A Banach algebra $X$ is amenable if and only if it has a virtual diagonal.

**Lemma 5.04** [70] A Banach algebra $X$ has an approximate diagonal if and only if it has a virtual diagonal.

So, by Lemma 5.03, Banach algebra $X$ is amenable if and only if it has an approximate diagonal.

It is known that many important Banach algebras can be expressed as the projective or injective tensor product of two component algebras. For example, if $X$ is a compact Hausdorff space and $B$ is a Banach algebra then the function algebra $C(X, B)$ can be identified with the injective tensor product $C(X)\otimes_{\gamma} B$ by putting $(f \otimes b)(x) = f(x)b$. Again if $\mu$, $\nu$ are positive finite measures on measure spaces $P$, $Q$ respectively, and $\mu \times \nu$ is the corresponding product measure on $P \times Q$, then $L^1(\mu \times \nu)$ can be identified with the projective tensor product $L^1(\mu)\otimes_{\pi} L^1(\nu)$. Similar is the case for the generalized group algebra $L^1(G, A)$.

**Lemma 5.05** [58] Let $G$ be a locally compact group and $A$, a Banach algebra. Then, $L^1(G, A) \cong L^1(G)\otimes_{\gamma} A$,

where $\cong$ denotes an isometric isomorphism.

In the first chapter, we have mentioned the concept of approximate identities. It is well known that an amenable algebra has a bounded two-sided approximate identity. The following Theorem gives us the conditions for the existence of bounded approximate identity in $L^1(G, A)$. 


**Theorem 5.06** Let $G$ be a locally compact group and $A$, a Banach algebra. Then $L^1(G, A)$ has an approximate identity bounded by a constant $K$ if and only if the algebra $A$ has an approximate identity bounded by the same constant $K$.

From [57], we get the following stability property of amenability.

**Lemma 5.07** [57] Let $A$ be a Banach algebra with a bounded two-sided approximate identity. If there is a closed left (algebraic) ideal $I$ in $A$ which is amenable and has a bounded left approximate identity for $A$, then $A$ is amenable.

Based on the above concepts, we can now give the ideal theoretic characterization of amenability for the generalized group algebra $L^1(G, A)$.

**Theorem 5.08** Let $G$ be a locally compact group and $A$, a Banach algebra. If

(i) $A$ has a bounded approximate identity,

(ii) Each of $L^1(G)$ and $A$ has an amenable closed (algebraic) ideal (which is also locally 1-complemented structure) having bounded left approximate identity,

then $L^1(G, A)$ is amenable.

**Proof.** Since $A$ has a bounded approximate identity, so, by Theorem 5.06, we have, $L^1(G, A)$ has also a bounded approximate identity. Let $X_1$ and $X_2$ be the amenable closed ideals (which are also given to be locally 1-complemented structures) in $L^1(G)$ and $A$ respectively, each having
bounded left approximate identity. Then $X_1 \otimes \gamma X_2$ is a subspace of $L^1(G) \otimes \gamma A$ and it is also a closed (algebraic) ideal in $L^1(G) \otimes \gamma A$. If $\{e_\lambda\}_{\lambda \in \mathcal{P}}$ and $\{h_v\}_{v \in \mathcal{Q}}$ are the bounded left approximate identities in $X_1$ and $X_2$ respectively, then from [100], we have, $\{e_\lambda \otimes h_v\}_{(\lambda, v) \in \mathcal{P} \times \mathcal{Q}}$ is a bounded left approximate identity for $X_1 \otimes \gamma X_2$.

Since $X_1$ is amenable, so, it has an approximate diagonal, say, $\{p_\alpha\}_{\alpha \in \mathcal{M}_0}$, where $p_\alpha = \sum a_{\alpha} \otimes a'_{\alpha} \in X_1 \otimes X_1$. Now, for any $\alpha \in X_1$, we have,

$$\lim_{\alpha \to \infty} \|a \ p_\alpha - p_\alpha \ a\| = 0$$

$$\Rightarrow \lim_{\alpha \to \infty} \left\| \sum a_{\alpha} \otimes a'_{\alpha} - \sum a_{\alpha} \otimes a'_{\alpha} \ a \right\| = 0 \quad \text{.................(5.1)}$$

and $\lim_{\alpha \to \infty} \|\Pi(p_\alpha) a - a\| = 0$

$$\Rightarrow \lim_{\alpha \to \infty} \left\| \sum a_{\alpha} a'_{\alpha} a - a \right\| = 0 \quad \text{.................(5.2)}$$

Since $X_2$ is amenable, so, it also has an approximate diagonal, say, $\{q_\beta\}_{\beta \in \mathcal{N}}$, where $q_\beta = \sum b_{\beta_j} \otimes b'_{\beta_j} \in X_2 \otimes X_2$. So, for any $b \in X_2$, we have,

$$\lim_{\beta \to \infty} \|b \ q_\beta - q_\beta \ b\| = 0$$

$$\Rightarrow \lim_{\beta \to \infty} \left\| \sum b_{\beta_j} b'_{\beta_j} - \sum b_{\beta_j} b'_{\beta_j} b \right\| = 0 \quad \text{...........(5.3)}$$

and $\lim_{\beta \to \infty} \|\Pi(q_\beta) b - b\| = 0$

$$\Rightarrow \lim_{\beta \to \infty} \left\| \sum b_{\beta_j} b'_{\beta_j} b - b \right\| = 0 \quad \text{.................(5.4)}$$
We define \( r_{\alpha\beta} = \sum_{i,j} (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j}) \in (X_1 \otimes \gamma X_2) \otimes \gamma (X_1 \otimes \gamma X_2) \). We show that this is an approximate diagonal for \( X_1 \otimes \gamma X_2 \).

Since \( \{p_{\alpha}\}_{\alpha \in M} \) and \( \{q_{\beta}\}_{\beta \in N} \) are bounded nets, so, \( \{r_{\alpha\beta}\}_{(\alpha,\beta) \in M \times N} \) is a bounded net in \( (X_1 \otimes \gamma X_2) \otimes \gamma (X_1 \otimes \gamma X_2) \).

For \( z = \sum_k a_k \otimes b_k \in X_1 \otimes \gamma X_2 \), we have,

\[
\| z r_{\alpha\beta} - r_{\alpha\beta} z \|
\]

\[
= \left\| \left( \sum_k a_k \otimes b_k \right) \left( \sum_{i,j} (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j})) \right) - \left( \sum_{i,j} (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j})) \right) \left( \sum_k a_k \otimes b_k \right) \right\|
\]

\[
= \left\| \left( \sum_{i,j} \left( \sum_k a_{\alpha_i} b_{\beta_j} a'_{\alpha_i} b'_{\beta_j} \right) \otimes (a'_{\alpha_i} \otimes b'_{\beta_j}) \right) - \left( \sum_{i,j} (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j})) \right) \left( \sum_k a_k \otimes b_k \right) \right\|
\]

\[
= \left\| \left( \sum_{i,j} \left( \sum_k a_k a_{\alpha_i} b_{\beta_j} a'_{\alpha_i} b'_{\beta_j} \right) \otimes (a'_{\alpha_i} \otimes b'_{\beta_j}) \right) - \left( \sum_{i,j} (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j})) \right) \left( \sum_k a_k \otimes b_k \right) \right\|
\]

Using Theorem 5.02,

\[
f((a_{\alpha_i} \otimes a'_{\alpha_i}) \otimes (b_{\beta_j} \otimes b'_{\beta_j})) = (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} \otimes b'_{\beta_j})) \quad \text{and}
\]

\[
f((a_{\alpha_i} \otimes a'_{\alpha_i} b_{\beta_j}) \otimes (b_{\beta_j} \otimes b'_{\beta_j} b_k)) = (a_{\alpha_i} \otimes b_{\beta_j} \otimes (a'_{\alpha_i} b_{\beta_j} \otimes b'_{\beta_j} b_k)).
\]

So, the expression (*) is equal to

\[
\left\| \sum_{i,j,k} f((a_{\alpha_i} a_{\alpha_i} \otimes a'_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j}) \right\| - \sum_{i,j,k} f((a_{\alpha_i} a'_{\alpha_i} a_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j} b_k)) \right\|
\]

\[
\leq \|f\| \left\| \sum_{i,j,k} (a_{\alpha_i} a_{\alpha_i} \otimes a'_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j}) - \sum_{i,j,k} (a_{\alpha_i} a'_{\alpha_i} a_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j} b_k) \right\|
\]

since the mapping \( f \) is linear and bounded.

\[
= \|f\| \left\| \sum_{i,j,k} (a_{\alpha_i} a_{\alpha_i} - a_{\alpha_i} a'_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j}) + (a_{\alpha_i} a'_{\alpha_i} a_{\alpha_i}) \otimes (b_{\beta_j} b_{\beta_j} \otimes b'_{\beta_j} b_k) \right\|
\]
Substituting \( a = a_k, \ k = 1,2,... \) in (5.1), we get,

\[
\lim_{a \to \infty} \left| \sum_{i} a_i a_{i'} - \sum_{i} a_i \otimes a_{i'} a_k \right| = 0, \ \text{for all } k. \ \text{Also } \sum_{j} b_{j} \otimes b'_{j}b_k \text{ is bounded. So, the 1st term of the expression } (***) \text{ tends to 0 as } a \to \infty.
\]

Similarly, substituting \( b = b_k, \ k = 1,2,... \) in (5.3), and applying the boundedness of \( \left\{ \sum a_i \otimes a_{i'} \right\}_{i \in M} \) we get, the 2nd term of the expression (***) also tends to 0 as \( \beta \to \infty. \)

Thus, \( \lim_{(a,\beta) \to \infty} \| z r_{a\beta} - r_{a\beta} z \| = 0, \ z \in X_1 \otimes_{\varphi} X_2. \)

Now, \( \| \Pi(r_{a\beta})z - z \| = \left\| \sum_{i,j} (a_i \otimes b_{j})(a_{i'} \otimes b'_{j}) \sum_{k} a_k \otimes b_k - a_k \otimes b_k \right\| \)

\[
= \left\| \sum_{k} \sum_{i,j} a_i a_{i'} a_k \otimes b_{j} b'_{j} b_k - \sum_{k} a_k \otimes b_{j} b'_{j} b_k + \sum_{k} a_k \otimes b_{j} b'_{j} b_k - (a_k \otimes b_k) \right\|
\]

\[
= \left\| \sum_{k} \sum_{j} (a_i a_{i'} a_k - a_k) \otimes b_{j} b'_{j} b_k + a_k \otimes (\sum_{j} b_{j} b'_{j} b_k - b_k) \right\|
\]

\[
\leq \sum_{k,j} \left\| a_i a_{i'} a_k - a_k \right\| \| b_{j} b'_{j} b_k \| + \sum_{k} \| a_k \| \left\| \sum_{j} b_{j} b'_{j} b_k - b_k \right\|
\]

\( \to 0 \) as \( a, \beta \to \infty. \) (using (5.2) and (5.4)).
Thus, \( \lim_{(\alpha, \beta) \in M \times N} \| \Pi(r_{\alpha \beta}) z - z \| = 0, \quad z \in X_1 \otimes_p X_2 \).

So, \( \{ r_{\alpha \beta} \}_{(\alpha, \beta) \in M \times N} \) is an approximate diagonal for \( X_1 \otimes_p X_2 \). Hence, \( X_1 \otimes_p X_2 \) is amenable. Now, using Lemma 5.07, we get, \( L^1(G) \otimes_p A \) i.e., \( L^1(G, A) \) is amenable. □

**Remark 5.09** For a locally compact abelian group \( G \), let \( Z(I) \) denote the zero set of an ideal \( I \), where \( Z(I) = \cap \{ Z(\hat{f}) = \hat{f}^{-1}(0) : f \in I \} \). If \( \hat{G} \) denotes the dual of \( G \), let \( R(\hat{G}) \) be the ring of subsets of \( \hat{G} \) generated by the cosets of all subgroups of \( \hat{G} \). From [21], we have, an ideal \( I \) of \( L^1(G) \) has an approximate identity if and only if \( Z(I) \in R(\hat{G}) \). Thus the condition (ii) of Theorem 5.08 is very restrictive. Without using this condition, another criteria for amenability of \( L^1(G, A) \) can be given with the help of multiplier algebras (refer to [57]) of the Banach algebras \( L^1(G) \) and \( A \).

Let \( A \) be a Banach algebra. A **double multiplier** on \( A \) is a pair \( (T_L, T_R) \) of bounded linear operators on \( A \) satisfying the conditions:

(i) \( T_L T_R = T_R T_L \), and

(ii) for all \( x, y \in A \), \( T_L(xy) = T_L(x)y; T_R(xy) = x T_R(y) \); and \( x T_L(y) = T_R(x)y \).

Let \( M \) denote the collection of all double multipliers on \( A \). Then \( M \) is a Banach space and it becomes a Banach algebra with the product defined by \( (T_L, T_R)(S_L, S_R) = (T_L S_L, S_R T_R) \). If \( J_1 \) is an idempotent element of \( M \), let \( J_2 = I - J_1 \), where \( I \) is the identity element of \( M \). We take \( A_{ij} = J_i A J_j, \quad i, j = 1, 2; \quad P(A_{11}) = \Pi(A_{12} \otimes_p A_21), P(A_{22}) = \Pi(A_{21} \otimes_p A_{12}) \). Then the linear space \( P(A_{12}) \),
(i=1,2) is isomorphic to the quotient of $A_y \otimes_j A_{ji}, j \neq i$, by $(\ker \Pi) \cap (A_y \otimes_j A_{ji})$
(refer to [57]). Now, we get,

**Deduction 5.10** Let $G$ be a locally compact group and $A$, a Banach algebra. Let $X = L^1(G)$. If

(i) $A$ has a bounded approximate identity,

(ii) $X_{22} = P(X_{22}), A_{22} = P(A_{22})$, and $X_{11}$ and $A_{11}$ are amenable, (with the notations as above),

then $L^1(G, A)$ is amenable.

**Proof.** Since $A$ has a bounded approximate identity, and $A_{22} = P(A_{22})$, and $A_{11}$ is amenable, so, by a result of [57], we have, $A$ is amenable. Similarly, $L^1(G)$ is amenable. Now, as in Theorem 5.08, we can construct an approximate diagonal for $L^1(G) \otimes A$. Thus, $L^1(G, A)$ is amenable. \[\square\]

**Remark 5.11** If $A$ is an amenable Banach algebra then from [9], we have, $H^n(A, X^*) = \{0\}$ for every Banach $A$-bimodule $X$ and for every $n \in \mathbb{N}$. [$H^n(A, X)$ is the $n$th cohomology group of $A$ with coefficients in $X$.] So, if $L^1(G)$ and $A$ satisfy the conditions of Theorem 5.08 (or, of Deduction 5.10) then the $n$th cohomology group of $L^1(G, A)$ with coefficients in $X^*$ vanishes for $n = 1, 2, \ldots$, for every Banach $L^1(G, A)$-bimodule $X$. Now, we can raise the following problem:

*If $L^1(G)$ and $A$ satisfy the conditions of Theorem 5.08, then can we give a characterization of the $L^1(G, A)$-bimodules $X$ (other than the dual modules) which satisfy $H^n(L^1(G, A), X) = \{0\}$ for $n = 1, 2, \ldots$?*
A solution to this problem will help us to develop further interesting results in the cohomology of group algebras.

Next we turn our attention to the study of amenability in the projective tensor product of group algebras. First we mention the following important Lemma.

**Lemma 5.12** [136] Let $G_1$ and $G_2$ be two locally compact groups. Then,

$$L^1(G_1 \times G_2) \cong L^1(G_1) \otimes_r L^1(G_2).$$

For a locally compact group $G$, $L^1(G)$ has a two-sided approximate identity bounded by 1. So, using the above result, we get,

**Deduction 5.13** If $G_1$ and $G_2$ are two locally compact groups, then $L^1(G_1) \otimes_r L^1(G_2)$ has a two-sided approximate identity bounded by 1.

Now, we give the following condition for amenability of the algebra $L^1(G_1) \otimes_r L^1(G_2)$.

**Theorem 5.14** Let $G_1$ and $G_2$ be two locally compact groups. If each of $L^1(G_1)$ and $L^1(G_2)$ contains an amenable closed (algebraic) ideal (which is also locally 1-complemented structure) with bounded left approximate identity, then $L^1(G_1) \otimes_r L^1(G_2)$ is amenable.

**Proof.** If $X_1$ and $X_2$ are the amenable closed ideals in $L^1(G_1)$ and $L^1(G_2)$ respectively, each having bounded left approximate identity, then as in Theorem 5.08 it can be shown that $X_1 \otimes_r X_2$ is an amenable closed ideal in $L^1(G_1) \otimes_r L^1(G_2)$, having a bounded left approximate identity. Now, by Deduction 5.13, $L^1(G_1) \otimes_r L^1(G_2)$ has a two-sided approximate identity.
bounded by 1. So, an application of Lemma 5.07 gives that $L^1(G_1) \otimes L^1(G_2)$ is amenable.

The concept of amenable groups was introduced by J. von Neumann in 1929 in his investigation of the Banach-Tarski paradox. Amenability of a locally compact group is characterized in terms of many fundamental properties in harmonic analysis of the group. It can be described in terms of group representation, the fixed-point property, the Følner property and in many other ways. The class of amenable groups includes all compact groups and all abelian groups. However, a non-abelian free group, or any group which contains it, is not amenable. It is known that a locally compact group $G$ is amenable if and only if the group algebra $L^1(G)$ is amenable [9]. Since $L^1(G_1 \times G_2) \cong L^1(G_1) \otimes L^1(G_2)$, so, we get,

**Corollary 5.15** Let $G_1$ and $G_2$ be two locally compact groups. If $L^1(G_1)$ and $L^1(G_2)$ satisfy the condition of Theorem 5.14, then the topological product $G_1 \times G_2$ is amenable.

We recall that a locally compact group $G$ is called *inner amenable* [84] if there exists a state $m$ on $L^\infty(G)$, satisfying $m(\pi(a)f) = m(f)$ for all $a$ in $G$ and $f$ in $L^\infty(G)$, where $\pi(a)f(x) = f(a^{-1}xa)$, $x \in G$. Amenable locally compact groups are inner amenable (refer to [84]).

Let $V$ be a von Neumann algebra on a Hilbert space $H$ and let $V'$ be the commutant of $V$. If there exists a projection of norm one, $P$ from $L(H)$ (the space of all bounded linear operators on $H$) onto $V'$ with $P(I) = I$, then $V$ is
said to be injective [84]. Now let \( VN(G) \) denote the von Neumann algebra on \( L^2(G) \) generated by \( \{ l_x : x \in G \} \) where \( l_x h(t) = h(xt), \ t \in G \). Relating amenability and inner amenability of a locally compact group \( G \) and the property of injectivity of the group von Neumann algebra \( VN(G) \), Lau and Paterson, obtained the following important result.

**Lemma 5.16** [84] Let \( G \) be a locally compact group. Then \( G \) is amenable if and only if the group von Neumann algebra \( VN(G) \) is injective and \( G \) is inner amenable.

Now, from Theorem 5.14, we get,

**Corollary 5.17** Let \( G_1 \) and \( G_2 \) be two locally compact groups such that \( L^1(G_1) \) and \( L^1(G_2) \) satisfy the condition of Theorem 5.14. Then

(i) \( VN(G_1 \times G_2) \), the von Neumann algebra determined by the topological product \( G_1 \times G_2 \) is injective.

(ii) \( G_1 \times G_2 \) is inner amenable.

For every locally compact group, we can associate a \( C^* \)-algebra, and the representation theory of the group \( C^* \)-algebra coincides with the representation theory of the group (refer to [23]). The left regular representation \( t \mapsto \lambda_t \) of a locally compact group \( G \) is a unitary representation of \( G \) acting on \( L^2(G) \), defined by \( (\lambda_t f)(s) = f(t^{-1} s), \ f \in L^2(G), \ t, s \in G \).

From this representation, we get the left regular *-representation \( \lambda \) of \( L^1(G) \) on \( L^2(G) \), defined by \( \lambda(f) = \int f(t) \lambda_t \mathrm{d} \mu(t) \), for all \( f \in L^1(G) \). It is seen that \( \lambda : L^1(G) \rightarrow L(L^2(G)) \) is a faithful *-representation. The reduced group \( C^* \)-
algebra $C_r^*(G)$ is the closure of $\lambda(L^1(G))$ in $L(H)$, for a Hilbert space $H$. Considering C*-algebras, there is an interesting connection between amenability and the Banach space properties, viz., compact approximation property (CAP), metric compact approximation property (MCAP) etc. Here, we consider the projective tensor product of $C_r^*(G)$ with a Banach algebra $A$ to obtain the following.

**Theorem 5.18** Let $G$ be a locally compact group such that $L^1(G)$ has an amenable closed left ideal with a bounded left approximate identity. Then for any Banach algebra $A$, $C_r^*(G) \otimes_r A$ has the CAP (MCAP) if and only if $A$ has the CAP (MCAP).

**Proof.** Since $L^1(G)$ has a bounded two-sided approximate identity, and it satisfies the given condition, so, by Lemma 5.07, $L^1(G)$ is amenable. So, $G$ is amenable. From [82], we have, if a locally compact group $G$ is amenable, then its reduced group C*-algebra is nuclear. Again nuclear C*-algebras have the MAP. Thus, $C_r^*(G)$ has the MAP, and so, also has the MCAP.

In Chapter-4, we have proved (refer to Theorems 4.03, 4.07, 4.10 and 4.12) that for any two Banach algebras $X_1$ and $X_2$, $X_1 \otimes_r X_2$ has the CAP (MCAP) if and only if both $X_1$ and $X_2$ have the CAP (MCAP). So, applying this, the result now easily follows.

If $G$ is abelian then $C_r^*(G)$ can be identified with $C_0(\hat{G})$ (refer to [80]). Again all abelian groups are amenable. So, in this case, we get,
Corollary 5.19 Let $G$ be a locally compact abelian group. Then for any Banach algebra $A$, $C_0(\hat{G}) \otimes \gamma A$ has the CAP (MCAP) if and only if $A$ has the CAP (MCAP).

Our next aim is to give a characterization of the amenability in the injective tensor product of Banach algebras. It is known that in general, the injective tensor product does not preserve the Banach algebra structure. But some well-known Banach algebras can be expressed as the injective tensor product of two Banach spaces. For example, we have already mentioned about the function algebra $C(X, B)$ that can be identified with the injective tensor product $C(X) \otimes B$. Here, we want to give a characterization of the amenability in the injective tensor product in terms of the amenability in the $M$-ideal in function algebras. We are interested in the class of unital Banach algebras having no non-trivial $M$-ideal.

Regarding amenability in the injective tensor product, we give the following result:

Theorem 5.20 Let $X_1$ and $X_2$ be two unital Banach algebras having no non-trivial $M$-ideal. Then $X_1 \otimes \gamma X_2$ is amenable if and only if every $M$-ideal in $C(K, X_1 \otimes \gamma X_2)$ is amenable for some suitable compact Hausdorff space $K$.

[Here, we consider the cases when $X_1 \otimes \gamma X_2$ is also an algebra.]

Before proceeding to the proof of this Theorem, first we present the following result, which shows that for unital algebras, the amenability of $X_1 \otimes \gamma X_2$ implies the amenability of both $X_1$ and $X_2$. 
Theorem 5.21 Let $X_1$ and $X_2$ be two unital Banach algebras such that their projective tensor product $X_1 \otimes_p X_2$ is amenable. Then both $X_1$ and $X_2$ are also amenable.

Proof. Let $\{d_a\}_{a \in A}$ be an approximate diagonal for $X_1 \otimes_p X_2$, where 
$$d_a = \sum_i x_{a_i} \otimes y_{a_i} \in (X_1 \otimes_p X_2) \otimes_p (X_1 \otimes_p X_2),$$
where $x_{a_i}, y_{a_i} \in X_1 \otimes_p X_2$.

We take, $x_{a_i} = \sum_m a_m \otimes b_{a_m}$, $y_{a_i} = \sum_n a_m' \otimes b_{a_m}'$.

Let $\phi$ be a complex homomorphism on $X_2$. We define $T : X_1 \otimes_p X_2 \rightarrow X_1$ by $T(\sum_i a_i \otimes b_i) = \sum_i \phi(b_i) a_i$. Then $T$ is also a homomorphism from $X_1 \otimes_p X_2$ to $X_1$ and $T \otimes_p T : (X_1 \otimes_p X_2) \otimes_p (X_1 \otimes_p X_2) \rightarrow X_1 \otimes_p X_1$ is defined by 
$$(T \otimes_p T)(\sum_i x_i \otimes y_i) = \sum_i T(x_i) \otimes T(y_i), \quad x_i, y_i \in X_1 \otimes_p X_2.$$

Let $f_1 : X_2 \otimes_p X_2 \rightarrow \mathbb{C}$ be such that $f_1(\sum_i b_i \otimes b_i') = \sum_i \phi(b_i) \phi(b_i')$. Then $f_1$ is well defined and linear and $\|f_1\| \leq \|\phi\|^2$.

We define, $R : (X_1 \otimes_p X_1) \otimes_p (X_2 \otimes_p X_2) \rightarrow X_1 \otimes_p X_1$ by 
$$R(\sum_i x_i \otimes y_i) = \sum_i f_1(y_i) x_i.$$

Then, $R$ is also well defined, linear and $\|R\| \leq \|f_1\| \leq \|\phi\|^2$.

In Theorem 5.01, taking $E_1 = F_1 = X_1$ and $E_2 = F_2 = X_2$ and applying it for $\alpha = \|\cdot\|_p$, we get, there exists a natural continuous linear map $f : (X_1 \otimes_p X_2) \otimes_p (X_1 \otimes_p X_2) \rightarrow (X_1 \otimes_p X_1) \otimes_p (X_2 \otimes_p X_2)$ such that 
$$f((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)) = (a_1 \otimes a_2) \otimes (b_1 \otimes b_2), \quad a_1, a_2 \in X_1, \quad b_1, b_2 \in X_2.$$
Since \( \{d_{\alpha} \}_{\alpha \in \Lambda} \) is an approximate diagonal for \( X_1 \otimes_\gamma X_2 \), so, for any \( z \in X_1 \otimes_\gamma X_2 \),

\[
\lim_{\alpha \to \infty} \|z \cdot d_{\alpha} - d_{\alpha} \cdot z\| = 0 \quad \text{.......................... (5.5)}
\]

and \( \lim_{\alpha \to \infty} \|\Pi(d_{\alpha})z - z\| = 0 \quad \text{.......................... (5.6)} \)

Now, \((T \otimes_\gamma T)(d_{\alpha}) \in X_1 \otimes_\gamma X_1 \ \forall \ \alpha \in \Lambda \). Also \( \{(T \otimes_\gamma T)(d_{\alpha})\}_{\alpha \in \Lambda} \) is bounded.

For \( \alpha \in X_1 \), we have,

\[
\|a \cdot (T \otimes_\gamma T)(d_{\alpha}) - (T \otimes_\gamma T)(d_{\alpha}) \cdot a\|
\]

\[
= \|a \left( \sum_i \phi(b_{\alpha_n})a_{\alpha_n} \otimes \sum_n \phi(b'_{\alpha_n})a'_{\alpha_n} \right) - \left( \sum_i \phi(b_{\alpha_n})a_{\alpha_n} \otimes \sum_n \phi(b'_{\alpha_n})a'_{\alpha_n} \right) a\|
\]

\[
= \left| \sum_i f_i(b_{\alpha_n} \otimes b'_{\alpha_n})(a_{\alpha_n} \otimes a'_{\alpha_n}) - \sum_i f_i(b_{\alpha_n} \otimes b'_{\alpha_n})(a_{\alpha_n} \otimes a'_{\alpha_n}) \right| a
\]

\[
= \left| \sum_{i,n,m} R((a_{\alpha_n} \otimes a'_{\alpha_n}) \otimes (b_{\alpha_n} \otimes b'_{\alpha_n})) - R((a_{\alpha_n} \otimes a'_{\alpha_n})a \otimes (b_{\alpha_n} \otimes b'_{\alpha_n})) \right|
\]

\[
\leq R \|f\| \left| \sum_{i,m,n} ((a_{\alpha_n} \otimes a'_{\alpha_n}) \otimes (b_{\alpha_n} \otimes b'_{\alpha_n})) - (a_{\alpha_n} \otimes a'_{\alpha_n}) \otimes (b_{\alpha_n} \otimes b'_{\alpha_n})) \right|
\]

\[
= R \|f\| \left| \sum_{i|m,n} ((a \otimes e_2)(a_{\alpha_n} \otimes b_{\alpha_n}) \otimes (b_{\alpha_n} \otimes b'_{\alpha_n}) - (a_{\alpha_n} \otimes b_{\alpha_n}) \otimes (b_{\alpha_n} \otimes b'_{\alpha_n})(a \otimes e_2)) \right|
\]

where \( e_2 \) is the unit in \( X_2 \).

\[
\rightarrow 0 \text{ as } \alpha \to \infty, \text{ using (5.5)}. \]
Again, \( \| \Pi((T \otimes T)(d_a)) a - a \| = \| \Pi(\sum_{i} T(x_{a_i}) \otimes T(y_{a_i})) a - a \| \)

\[ = \left\| \sum_{i} T(x_{a_i}y_{a_i}) a - a \right\| \]

\[ = \left\| \sum_{i} T(x_{a_i}y_{a_i}) (a \otimes e_2) - T(a \otimes e_2) \right\|, \text{ since } T(a \otimes e_2) = \phi(e_2) a = a. \]

\[ \leq \| T \| . \| \Pi(d_a)(a \otimes e_2) - (a \otimes e_2) \| \]

\( \rightarrow 0 \) as \( \alpha \rightarrow \infty \), using (5.6).

Thus, \( \{(T \otimes T)(d_a)\}_{\alpha \in \Lambda} \) is an approximate diagonal for \( X_1 \). Hence, \( X_1 \) is amenable. Similarly, we can show that \( X_2 \) is also amenable. \( \square \)

The following result from [127] gives the characterization of the \( M \)-ideals in function algebras.

**Theorem 5.22** [127] Let \( X \) be a function algebra contained in \( C(K) \), for some compact Hausdorff space \( K \). Then the \( M \)-ideals in \( X \) are precisely the closed ideals having a bounded approximate identity.

**Theorem 5.23** [134] If \( X \) and \( Y \) are Banach spaces without non-trivial \( M \)-ideal, then \( X \otimes \alpha Y \) is also without non-trivial \( M \)-ideal.

**Proposition 5.24** [134] Let \( X \) and \( Y \) be Banach spaces. If \( P \) is an \( M \)-projection on \( Y \), then \( I_\phi \otimes P \) is an \( M \)-projection on \( X \otimes \alpha Y \).

**Proposition 5.25** [134] Let \( \alpha \) be a tensor norm having the following property for all pairs of Banach spaces \( X \) and \( Y \):
If $P$ is an $M$-projection on $Y$, then $I$ is an $M$-projection on $X \otimes \alpha Y$.

Then $\alpha$ also satisfies:

*If $E$ is an $M$-ideal in $Y$, then the closure of $X \otimes E$ with respect to $\alpha$ is an $M$-ideal in $X \otimes \alpha Y$.*

**Corollary 5.26** [134] For Banach spaces $X$ and $Y$, if $E$ is an $M$-ideal in $Y$, then $X \otimes \lambda E$ is an $M$-ideal in $X \otimes \alpha Y$.

[Here, the closure of $X \otimes E$ with respect to $\lambda$ is denoted by $X \otimes \lambda E$.

**Theorem 5.27** [134] Let $X$ and $Y$ be Banach spaces, $X$ having no non-trivial $M$-ideal. If $Z$ is an $M$-ideal in $X \otimes \lambda Y$, then $Z$ can be expressed in the form $Z = X \otimes \lambda J$ with some $M$-ideal $J$ in $Y$.

**Proof of Theorem 5.20:**

Let $X_1$ and $X_2$ be unital Banach algebras having no non-trivial $M$-ideal such that $X_1 \otimes \lambda X_2$ is amenable. By Theorem 5.23, $X_1 \otimes \lambda X_2$ is also without non-trivial $M$-ideal.

We have, $C(K, X_1 \otimes \lambda X_2) = (X_1 \otimes \lambda X_2) \otimes \lambda C(K)$. Let $Z$ be an $M$-ideal in $C(K, X_1 \otimes \lambda X_2)$. By Theorem 5.27, $Z$ is of the form $(X_1 \otimes \lambda X_2) \otimes \lambda J$, for some $M$-ideal $J$ in $C(K)$. Since $K$ is a compact Hausdorff space, so, $C(K)$ is amenable. By Theorem 5.22, every $M$-ideal in $C(K)$ is a closed ideal with bounded approximate identity.

From [57], we have, if $X$ is an amenable Banach algebra and $I$ is a closed left ideal of $X$ having a bounded approximate identity, then $I$ is amenable. So, an application of this result gives that every $M$-ideal in $C(K)$
is amenable and thus, $J$ is amenable. Hence, $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ is amenable. [As in Theorem 5.08, we can construct an approximate diagonal for $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ from the approximate diagonals of $X_1 \otimes_{\gamma} X_2$ and $J$.]

Since the $M$-ideal $J$ of $C(K)$ is having approximation property, so, the canonical mapping $\psi: (X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J \to (X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ is one-to-one [74]. Thus, $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ is amenable, as $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ is amenable. Hence, $Z$ is amenable, i.e., every $M$-ideal in $C(K, X_1 \otimes_{\gamma} X_2)$ is amenable.

For the converse, let $J$ be an $M$-ideal in $C(K)$. Since $X_1 \otimes_{\gamma} X_2$ is without non-trivial $M$-ideal, by Werner [134], $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J = Z$, say, is an $M$-ideal in $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} C(K) = C(K, X_1 \otimes_{\gamma} X_2)$. By the assumption, $Z$ is amenable. $J$ having approximation property, as in the previous part, we can show that $(X_1 \otimes_{\gamma} X_2) \otimes_{\gamma} J$ is amenable. Now, an application of Theorem 5.21 implies that $X_1 \otimes_{\gamma} X_2$ is amenable. \hfill $\square$

**Remark 5.28** The algebra of approximable operators, denoted by $A(X)$ (i.e., uniform closure of finite rank operators, or, $\overline{F(X)}$), in a Banach algebra $X$ can be identified with the injective tensor product of $X$ and its dual space $X^*$, i.e., $A(X) = X \otimes_{\gamma} X^*$. It is known that if $A(X)$ is amenable, then $X^*$ and so, $X$ has the bounded approximation property. But for $A(X)$ to be amenable, it is not sufficient that $X$ has the bounded approximation property. For example, the algebra $A(l_p \otimes_{\gamma} l_q)$, where $1/p + 1/q \geq 1$, is not amenable although $l_p \otimes_{\gamma} l_q$ has the bounded approximation property.
Now, naturally the question arises:

*What are the other conditions to be satisfied by $X$ which are sufficient for $A(X)$ to be amenable?*

Using the concept of amenability of $M$-ideal structure, we can now give a solution to this problem as:

**Theorem 5.29** Let $X$ be a unital Banach algebra. If

(i) $X$ and $X^*$ have no non-trivial $M$-ideal, and

(ii) for some compact Hausdorff space $K$, every $M$-ideal in $C(K, A(X))$ is amenable,

then, $A(X)$ is amenable.

**Corollary 5.30** Let $X_1$ and $X_2$ be two Banach algebras and let $Z = X_1 \oplus X_2$. If $Z$ satisfies the conditions of Theorem 5.29, then at least one of $A(X_1)$ and $A(X_2)$ is amenable.

**Proof.** By Theorem 5.29, $A(Z)$ is amenable. From [57], we have, if $A(X_1 \oplus X_2)$ is amenable, then at least one of $A(X_1)$ and $A(X_2)$ is amenable. □

**Corollary 5.31** Let $X$ be a Banach algebra such that $Z = X^*$ has the Radon-Nikodym property. If $Z$ satisfies all the conditions of Theorem 5.29, then $A(A(X))$ is amenable.

**Proof.** Clearly, $A(X^*)$ is amenable. Now, since $X^*$ has the Radon-Nikodym property, so, by [57], $A(A(X))$ is amenable. □

**Corollary 5.32** Let $X$ be a Banach algebra satisfying all the conditions of Theorem 5.29. Then $K(X \oplus \mathbb{C})$ is amenable.
Proof. Clearly, $A(X)$ is amenable. So, $A(X) = K(X)$. From [57], we have, $K(X)$ is amenable if and only if $K(X \oplus \mathbb{C})$ is amenable. Hence the result. □

Remark 5.33 Since the amenable Banach algebras have bounded approximate identities, so, if $K(X)$ is amenable, then it has the bounded approximate identity. From [122], we have, for any Banach algebra $X$, $K(X)$ admits a bounded right approximate identity if and only if $X^*$ has the *-bounded compact approximation property.

[For $\mu \geq 1$, the dual $X^*$ of a Banach space $X$ is said to have the *$\mu$-bounded compact approximation property (*$\mu$-BCAP) (refer to [122]) if for every finite subset $F$ of $X^*$ and for every $\varepsilon > 0$, there is an operator $T \in K(X)$ such that $\|T\| \leq \mu$ and $\|T^*(x^*) - x^*\| \leq \varepsilon$ for all $x^* \in F$. $X^*$ is said to have the *-$\mu$-bounded compact approximation property (*-BCAP) if it has the *$\mu$-BCAP for some $\mu \geq 1$.]

Now, if $A(X)$ is amenable, then $A(X) = K(X)$, and so, for all the spaces discussed in the above results, we have, their dual spaces have the *-BCAP. However, the problem: whether the amenability of $K(X)$ implies the approximation property of $X$, still remains open.