CHAPTER 4

COMPACT APPROXIMATION PROPERTY AND M-IDEALS OF COMPACT OPERATORS IN TENSOR PRODUCTS
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OF COMPACT OPERATORS IN
TENSOR PRODUCTS

This chapter deals with some illuminating properties of the compact approximation property (CAP) in tensor products and their applications to the $M$-ideals of operators and the quasi approximation property. In [99], Lima, Lima and Nygaard gave a characterization of the CAP in terms of ideals. Here, we extend these concepts to the tensor product of two Banach spaces. We prove that if $X_1$ and $X_2$ are two Banach spaces with CAP, then their projective tensor product $X_1 \otimes_p X_2$ has also the same, and conversely. Also, using the idea of trace mapping and the Hahn-Banach extension operators we establish similar results in case of the metric compact approximation property (MCAP) in the projective tensor product of two Banach spaces. As an application, we prove that if $K(X_1 \otimes_p X_2)$, the space of compact operators on $X_1 \otimes_p X_2$ is an $M$-ideal in $L(X_1 \otimes_p X_2)$, the space of bounded linear operators on $X_1 \otimes_p X_2$, then $K(X_i)$ is an $M$-ideal in $L(X_i)$ ($i=1,2$). Also, if $I_i$ is an ideal in $X_i$ ($i=1,2$), we investigate the conditions for which $K(I_1 \otimes_p I_2)$ is an $M$-ideal in $L(I_1 \otimes_p I_2)$. Moreover, we

This chapter is extracted from our research papers [26], [30] and [32].
prove that if \( X_1 \otimes_F X_2 \) has the quasi approximation property, then the compact approximation property in the spaces \( X_1 \) and \( X_2 \) implies the quasi approximation property in both the spaces.

It is already mentioned in our previous chapter that although AP is formally stronger than the CAP, but the example constructed by Willis [138] in 1992 shows that there is a separable Banach space with CAP, but fails to have the AP. In 1977, Lindenstrauss and Tzafriri gave a list of five equivalent formulations of AP for Banach spaces. The following analogous result characterizing spaces with CAP was given by Lima, Lima and Nygaard.

**Theorem 4.01** [99] Let \( X \) be a Banach space. The following statements are equivalent:

(i) \( X \) has the CAP.

(ii) For every Banach space \( Y \), for every compact operator \( T \) in \( K(Y, X) \), and for every \( \varepsilon > 0 \), there is a compact operator \( S \) in \( K(X) \) satisfying \( \|T - ST\| < \varepsilon \).

(iii) For every choice of the sequences \( \{x_n\}_{n=1}^{\infty} \subseteq X \), \( \{x_n^*\}_{n=1}^{\infty} \subseteq X^* \) such that \( \sum_{n=1}^{\infty} \|x_n^*\| \|x_n\| < \infty \) and \( \sum_{n=1}^{\infty} x_n^*(Sx_n) = 0 \) for all \( S \in K(X) \), we have, \( \sum_{n=1}^{\infty} x_n^*(x_n) = 0 \).

Before proceeding to our result on CAP, we give a brief discussion on compactness of tensor product operators.
Let $X_1$, $X_2$, $Y_1$, $Y_2$ be Banach spaces and $T_1: X_1 \to X_2$, $T_2: Y_1 \to Y_2$ be continuous linear operators. Then the operator defined on $X_1 \bigotimes Y_1$ by

$$(T_1 \bigotimes T_2)(\sum_{i=1}^{n} x_i \otimes y_i) = \sum_{i=1}^{n} T_1 x_i \otimes T_2 y_i,$$

is called the tensor product operator. If $\alpha$ is a tensor norm (in the terminology of Grothendieck [59]) then $T_1 \bigotimes T_2$ is continuous and may therefore be extended to the operator $T_1 \bigotimes_{\alpha} T_2$ from $X_1 \bigotimes_{\alpha} Y_1$ to $X_2 \bigotimes_{\alpha} Y_2$. In the study of tensor product spaces, it is also an important matter to know whether certain properties of the operators $T_1$ and $T_2$ are inherited to their tensor product. Discussing the compactness of tensor product operators, Holub obtained the following result:

**Theorem 4.02** [64] Let $T_1: X_1 \to X_2$ and $T_2: Y_1 \to Y_2$ be compact linear operators. Then $T_1 \bigotimes T_2: X_1 \bigotimes Y_1 \to X_2 \bigotimes Y_2$ is compact.

Now, regarding the existence of CAP in the tensor product, we give our first principal result as:

**Theorem 4.03** Let $X_1$ and $X_2$ be any two Banach spaces having CAP. Then their projective tensor product $X_1 \bigotimes_p X_2$ has also CAP.

**Proof.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $X_1 \bigotimes_p X_2$ and $\{z^*_n\}_{n=1}^{\infty}$ be a sequence in $(X_1 \bigotimes_p X_2)^*$ satisfying the conditions:

(i) $\sum_{n=1}^{\infty} \|z^*_n\| \|z_n\| < \infty$ and

(ii) $\sum_{n=1}^{\infty} z^*_n (S z_n) = 0$ for all $S \in K(X_1 \bigotimes_p X_2)$. 
We may assume that $\sum_{n=1}^{\infty} \| z_n \| < \infty$ and $1 \geq \| x_{1n} \| \to 0$ and $1 \geq \| x_{2n} \| \to 0$ as $k \to \infty$, where $z_n = \sum_{k=1}^{n} x_{nk} \otimes x_{2nk}$.

Let $K_1 = \text{conv}(x_{nk})_{k=1}^{\infty}$. Then $K_1$ is a compact subset of $B_{X_1}$. By the Davis-Figiel-Johnson-Pelczynski factorization procedure [20], there is a reflexive Banach space $Y_1$, an operator $T_1 \in K(Y_1, X_1)$ and a sequence $\{y_{1k}\}_{k=1}^{\infty} \subseteq B_{Y_1}$ such that $T_1(y_{1k}) = x_{1n}$ for all $k$.

Let $K_2 = \text{conv}(x_{nk})_{k=1}^{\infty}$. Then $K_2$ is a compact subset of $B_{X_2}$ and as in above, there is a reflexive Banach space $Y_2$, a compact operator $T_2 \in K(Y_2, X_2)$ and a sequence $\{y_{2k}\}_{k=1}^{\infty} \subseteq B_{Y_2}$ such that $T_2(y_{2k}) = x_{2n}$ for all $k$.

We define an operator $T: Y_1 \otimes Y_2 \to X_1 \otimes Y_2$ by

$$T(\sum_{i=1}^{\infty} y_{1i} \otimes y_{2i}) = \sum_{i=1}^{\infty} T_1 y_{1i} \otimes T_2 y_{2i}.$$  

Applying Theorem 4.02, $T \in K(Y_1 \otimes Y_2, X_1 \otimes Y_2)$. Now, for $S \in K(X_1 \otimes Y_2)$ we get,

$$\sum_{n=1}^{\infty} z_n^* (Sz_n) = 0 \implies \sum_{n=1}^{\infty} z_n^* (S(\sum_{k=1}^{\infty} x_{nk} \otimes x_{2nk})) = 0$$

$$\implies \sum_{n=1}^{\infty} z_n^* (S(\sum_{k=1}^{\infty} T_1 y_{1k} \otimes T_2 y_{2k})) = 0$$

$$\implies \sum_{n=1}^{\infty} z_n^* (S(T(\sum_{k=1}^{\infty} y_{1k} \otimes y_{2k}))) = 0$$

$$\implies \sum_{n=1}^{\infty} z_n^* ((S \circ T)(\sum_{k=1}^{\infty} y_{1k} \otimes y_{2k})) = 0 \quad \ldots \ldots \quad (1)$$
We have, \( \sum_{n=1}^{\infty} z_n^* (\sum_{k=1}^{\infty} y_k \otimes y_{2_k}) \) \( \in (X_1 \otimes_{\gamma} X_2)^* \otimes_{\gamma} (Y_1 \otimes_{\gamma} Y_2) \). For \( f \in K(Y_1 \otimes_{\gamma} Y_2, X_1 \otimes_{\gamma} X_2) \), we define: \( \psi(f) = \sum_{n=1}^{\infty} z_n^* (f(\sum_{k=1}^{\infty} y_k \otimes y_{2_k})) \). Then, \( \psi \) is well defined and linear, and

\[
\|\psi(f)\| = \left\| \sum_{n=1}^{\infty} z_n^* (f(\sum_{k=1}^{\infty} y_k \otimes y_{2_k})) \right\| \leq \sum_{n=1}^{\infty} \left\| z_n^* \right\| \left\| f \right\| \left\| \sum_{k=1}^{\infty} y_k \otimes y_{2_k} \right\| \leq \sum_{n=1}^{\infty} \left\| z_n^* \right\| \left\| f \right\| \| u \| = M\|u\| \|f\|,
\]

[taking \( \sum_{n=1}^{\infty} \left\| z_n^* \right\| (= M, \text{say}) < \infty \), and \( u = \sum_{k=1}^{\infty} y_k \otimes y_{2_k} \in B_{Y_1} \otimes_{\gamma} B_{Y_2} \).]

Hence, \( \psi \) is bounded. So, \( \psi \in K(Y_1 \otimes_{\gamma} Y_2, X_1 \otimes_{\gamma} X_2)^* \). Now, from (1), \( \psi(SoT) = 0 \ \forall \ S \in K(X_1 \otimes_{\gamma} X_2) \).

Let \( M = \{ SoT : S \in K(X_1 \otimes_{\gamma} X_2) \} \). Then, \( \psi(J) = 0 \ \forall \ J \in M \).

Let \( \varepsilon > 0 \), \( T_1 \in K(Y_1, X_1) \). Since \( X_1 \) has the CAP, so, by Theorem 4.01, there is a compact operator \( S_1 \in K(X_1) \) such that \( \| T_1 - S_1 \| < \frac{\varepsilon}{2\|T_2\|} \).

Also, \( T_2 \in K(Y_2, X_2) \). Since \( X_2 \) has the CAP, so, there is a compact operator \( S_2 \in K(X_2) \) such that \( \| T_2 - S_2 \| < \frac{\varepsilon}{2\|T_1\|} \).

We define an operator \( J : X_1 \otimes_{\gamma} X_2 \to X_1 \otimes_{\gamma} X_2 \) by \( J(\sum_{n=1}^{\infty} x_n \otimes x_{2_n}) = \sum_{n=1}^{\infty} S_1 x_n \otimes S_2 x_{2_n} \). Then, \( J \in K(X_1 \otimes_{\gamma} X_2) \) and \( JoT \in K(Y_1 \otimes_{\gamma} Y_2, X_1 \otimes_{\gamma} X_2) \).

Let \( \alpha = \sum_{n=1}^{\infty} y_{1_n} \otimes y_{2_n} \in Y_1 \otimes_{\gamma} Y_2 \) be arbitrary.
Now, \( \|(T - J \circ T)(\alpha)\| = \|(T - J \circ T)(\sum_{n=1}^{\infty} y_{1n} \otimes y_{2n})\| \)

\[
= \left\| \sum_{n=1}^{\infty} T_{1}y_{1n} \otimes T_{2}y_{2n} - J\left(\sum_{n=1}^{\infty} T_{1}y_{1n} \otimes T_{2}y_{2n}\right) \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} T_{1}y_{1n} \otimes T_{2}y_{2n} - \sum_{n=1}^{\infty} S_{1}(T_{1}y_{1n}) \otimes S_{2}(T_{2}y_{2n}) \right\|
\]

\[
= \left\| \sum_{n=1}^{\infty} T_{1}y_{1n} \otimes T_{2}y_{2n} - \sum_{n=1}^{\infty} S_{1}(T_{1}y_{1n}) \otimes T_{2}y_{2n} + \sum_{n=1}^{\infty} S_{1}(T_{1}y_{1n}) \otimes S_{2}(T_{2}y_{2n}) - \sum_{n=1}^{\infty} S_{1}(T_{1}y_{1n}) \otimes S_{2}(T_{2}y_{2n}) \right\|
\]

\[
\leq \sum_{n=1}^{\infty} \left\| T_{1} - S_{1}T_{1} \right\| \left\| y_{1n} \right\| \left\| T_{2} \right\| \left\| y_{2n} \right\| + \sum_{n=1}^{\infty} \left\| S_{1} \right\| \left\| T_{1} \right\| \left\| y_{1n} \right\| \left\| T_{2} - S_{2}T_{2} \right\| \left\| y_{2n} \right\|
\]

\[
\leq \varepsilon \sum_{n=1}^{\infty} \left\| y_{1n} \right\| \left\| y_{2n} \right\|
\]

Thus, \( \|T - J \circ T\| \leq \varepsilon \); hence, \( T \in \overline{M} \). Since \( \psi \) is continuous, so,

\[
\psi(T) = 0 \Rightarrow \sum_{n=1}^{\infty} z_{n} \ast (T(\sum_{k=1}^{\infty} y_{1k} \otimes y_{2k})) = 0
\]

\[
\Rightarrow \sum_{n=1}^{\infty} z_{n} \ast (\sum_{k=1}^{\infty} T_{1}y_{1k} \otimes T_{2}y_{2k}) = 0
\]

\[
\Rightarrow \sum_{n=1}^{\infty} z_{n} \ast (\sum_{k=1}^{\infty} x_{1nk} \otimes x_{2nk}) = 0 \Rightarrow \sum_{n=1}^{\infty} z_{n} \ast (z_{n}) = 0.
\]

So, by Theorem 4.01, \( X_{1} \otimes_{\psi} X_{2} \) has the CAP. □

In [91], Lima and Oja proved that a Banach space \( X \) has the AP if and only if \( F(Y, X) \) is an ideal in \( K(Y, X) \) for all Banach spaces \( Y \). Extending this result, Lima, Nygaard and Oja in [94] proved that a Banach space \( X \) has the AP if and only if \( F(Y, X) \) is an ideal in \( W(Y, X) \) for all Banach spaces \( Y \) (\( W(Y, X) \) denotes the subspace of weakly compact operators). Discussing CAP in Banach spaces, Lima, Lima and Nygaard gave the following
characterization of CAP in terms of operator ideals, which plays an important role in proving the converse of Theorem 4.03.

**Theorem 4.04** [99] Let $X$ be a Banach space. The following statements are equivalent:

(i) $X$ has the CAP.

(ii) For every Banach space $Y$ and every weakly compact operator $T \in \mathcal{W}(Y, X)$, $U = \{S \circ T : S \in K(X)\}$ is an ideal in $V = \text{span} \{U, \{T\}\}$.

(iii) Same as the above condition (ii), but with the operator $T \in K(Y, X)$.

Considering ideal (i.e., locally complemented) structures in Banach spaces, from Lima, we gather the following results.

**Lemma 4.05** [98] Let $Y$ be a closed subspace of a Banach space $Z$. The following statements are equivalent:

(i) There exists a bounded linear extension operator $\phi: Y^* \to Z^*$ such that $\phi(y^*)(y) = y^*(y)$ for all $y \in Y$ and $y^* \in Y^*$.

(ii) The annihilator of $Y$, $Y^\perp$ is complemented in $Z^*$.

(iii) For every $\varepsilon > 0$ and finite dimensional subspace $F \subseteq Z$ there exists an operator $T: F \to Y$ satisfying the conditions:

(a) $\|T(x) - y\| \leq \varepsilon\|x\|$ for all $x \in F \cap Y$.

(b) $\|T\| \leq \lambda + \varepsilon$ for some $1 \leq \lambda < \infty$.

Let $Y$ be a closed subspace of a Banach space $Z$. $Y$ is said to be locally $(\lambda)$-complemented in $Z$ if $Y$ and $Z$ satisfy any of the statements in the above
Lemma. By [90], $Y$ is an ideal in $Z$ if and only if $Y$ is locally 1-complemented in $Z$.

**Lemma 4.06** [98] Let $Y$ be a subspace of a normed space $Z$. If for every $\varepsilon > 0$ and finite dimensional subspace $F$ of $Z$ there is an operator $T: F \to Y$ with $\|T\| \leq 1 + \varepsilon$ and $T(y) = y \ \forall y \in F \cap Y$, then $Y$ is an ideal in $\overline{Z}$.

We are now ready to prove the converse of Theorem 4.03.

**Theorem 4.07** Let $X_1$ and $X_2$ be two Banach spaces such that $X_1 \otimes X_2$ has the CAP. Then each of $X_1$ and $X_2$ has also CAP.

**Proof.** Let $Y$ be a reflexive Banach space. Let $T \in K(Y, X_1)$, $W = \{S \circ T : S \in K(X_1)\}$ and $V = \text{span}(W, \{T\})$. Let $g \in W^*$. Now $W$ is a subspace of $K(Y, X_1)$. Hence, by Hahn-Banach extension theorem, $g$ has a norm-preserving extension $g_1$ to the whole of $K(Y, X_1)$; i.e., there exists $g_1 \in K(Y, X_1)^*$ such that $\|g_1\| = \|g\|$.

Since $Y$ is reflexive, using the description of $K(Y, X_1)^*$ due to Feder and Saphar [42], we have, for $g_1 \in K(Y, X_1)^*$, there exists $u \in X_1^* \otimes_Y Y$ such that $g_1(S) = \text{trace}(Su)$, $S \in K(Y, X_1)$, and $\|g_1\| = \|u\|$. Let this $u = \sum_n x_n^* \otimes y_n$, $x_n^* \in X_1^*$, $y_n \in Y$. We may assume that $\sum \|x_n^*\| < \infty$ and $\|y_n\| \to 0$.

Let $x_2 \in X_2$ be such that $x_2 \neq 0$. Let $e_1 = \frac{x_2}{\|x_2\|}$. Then, $\|e_1\| = 1$. By Hahn-Banach theorem, there is $f \in X_2^*$ such that $\|f\| = 1$ and $f(x_2) = \|x_2\|$. Thus, $f(e_1) = 1$. Let $T_1 \in V = \text{span}(W, \{T\}) \subset K(Y, X_1)$. 


We define \( \tilde{T} : Y \to X_1 \otimes X_2 \) by \( \tilde{T}(y) = T_i(y) \otimes e_1 \), \( y \in Y \).

\[
\| \tilde{T}(y) \| = \| T_i(y) \otimes e_1 \| = \| T_i(y) \| \| e_1 \| = \| T_i(y) \|. \quad \text{Thus,} \quad \tilde{T} \in L(Y, X_1 \otimes X_2).
\]

Let \( \{ y_n \}_n \) be a bounded sequence in \( Y \) such that \( \| y_n \| \leq 1 \). Since \( T_i \) is compact, so, \( \{ T_i y_n \}_n \) has a convergent subsequence \( \{ T_i y_{n_k} \}_k \) which converges to \( u_k \), say.

\[
\| \tilde{T} y_{n_k} - u_k \otimes e_1 \|_r = \| T_i y_{n_k} \otimes e_1 - u_k \otimes e_1 \|_r = \| T_i y_{n_k} - u_k \| \| e_1 \| = \| T_i y_{n_k} - u_k \| \to 0 \quad \text{as} \quad k \to \infty.
\]

Hence, \( \{ \tilde{T} y_{n_k} \}_k \) is a convergent subsequence of \( \{ \tilde{T} y_n \}_n \). Thus, \( \tilde{T} \in K(Y, X_1 \otimes X_2) \). Now, \( \| y_n \| \to 0 \). So, \( \{ \tilde{T} y_1, \tilde{T} y_2, \ldots, \tilde{T} y_m, \ldots \} \) is a compact subset of \( X_1 \otimes X_2 \). Since \( X_1 \otimes X_2 \) has the CAP, there exists a net \( (S_a) \) in \( K(X_1 \otimes X_2) \) which converges uniformly to \( I_{X_1 \otimes X_2} \) on \( \{ \tilde{T} y_1, \tilde{T} y_2, \ldots, \tilde{T} y_m, \ldots \} \). Let \( \hat{T} : X_1 \otimes X_2 \to X_1 \) be an operator such that \( \hat{T}(\sum_{n} x_n \otimes x_{n'}) = \sum_{n} f(x_{n'}) x_n \). Then,

\[
\hat{T} \in L(X_1 \otimes X_2, X_1) \quad \text{and} \quad \| \hat{T} \| \leq 1.
\]

We define \( \hat{S}_a : X_1 \to X_1 \) by \( \hat{S}_a(x_1) = \hat{T}(S_a(x_1 \otimes e_1)) \), \( x_1 \in X_1 \). Clearly, \( \hat{S}_a \) is well defined and linear.

\[
\| \hat{S}_a(x_1) \| = \| \hat{T}(S_a(x_1 \otimes e_1)) \| \leq \| \hat{T} \| \cdot \| S_a(x_1 \otimes e_1) \| \leq \| S_a \| \cdot \| x_1 \|
\]
Let \( \{x_n\} \) be a bounded sequence in \( X_1 \) such that \( \|x_n\| \leq 1 \). Then, 
\( \{x_n \otimes e_1\} \) is a bounded sequence in \( X_1 \otimes \gamma X_2 \). We take \( t_n = x_n \otimes e_1, \ n = 1, 2, \ldots \). 
Since \( S_\alpha \) is compact, so, \( \{S_\alpha t_n\} \) has a convergent subsequence \( \{S_\alpha t_{n_k}\} \) which converges to \( v_k \in X_1 \otimes \gamma X_2 \), say.

Now, 
\[
\|S_\alpha (x_{n_k}) - \hat{T}(v_k)\| = \|\hat{T}(S_\alpha (x_{n_k} \otimes e_1)) - \hat{T}(v_k)\|
\]
\[
= \|\hat{T}(S_\alpha (t_{n_k})) - \hat{T}(v_k)\|
\]
\[
\leq \|\hat{T}\| \|S_\alpha (t_{n_k}) - v_k\|
\]
\[
\leq \|S_\alpha (t_{n_k}) - v_k\| \to 0 \text{ as } k \to \infty.
\]
So, \( \{S_\alpha x_{n_k}\} \) is a convergent subsequence of \( \{S_\alpha x_n\} \) and hence, \( S_\alpha \in K(X_1) \).

\[
\|T_1 y_n - \hat{S}_\alpha (T_1 y_n)\| = \|T_1 y_n - \hat{T}(S_\alpha (T_1 y_n \otimes e_1))\|
\]
\[
= \|T_1 y_n - \hat{T}(\bar{T}_1 y_n) + \hat{T}(\bar{T}_1 y_n - S_\alpha (\bar{T}_1 y_n))\|
\]
\[
= \|T_1 y_n - f(e_1) T_1 y_n + \hat{T}(\bar{T}_1 y_n - S_\alpha (\bar{T}_1 y_n))\|
\]
\[
= \|T_1 y_n - T_1 y_n + \hat{T}(\bar{T}_1 y_n - S_\alpha (\bar{T}_1 y_n))\|
\]
\[
\leq \|\hat{T}\| \|\bar{T}_1 y_n - S_\alpha (\bar{T}_1 y_n)\| \leq \|\bar{T}_1 y_n - S_\alpha (\bar{T}_1 y_n)\|
\]

Now, \( \hat{S}_\alpha \circ T_1 \in K(Y, X_1) \).
We define $\langle \gamma \rangle : W^* \to V^*$ by $(\langle \gamma \rangle g)(T_1) = \lim_{n \to \infty} g_1(S_n \circ T_1) = \trace(T_1 u)$, $g \in W^*$, $T_1 \in V$. Then, $\phi$ is well defined and linear, and

$$|\phi g(T_1)| = |\trace(T_1 u)| \leq \|T_1 u\| \leq \|T_1\| \cdot \|u\| = \|T_1\| \cdot \|g\| = \|T_1\| \cdot \|g\|$$

$$\Rightarrow \|\phi g\| \leq \|g\|.$$
can be factorized through $Z$ with $U$, i.e., $T$ can be written as $T = U_0T_0$, where the operator $T_0 : Y \to Z$ is defined by $T_0y = Ty \in Z \subseteq X_1$ (refer to [94]).

Let $W_U = \{SoU : S \in K(X_1)\}$, $V_U = \text{span}(W_U, \{U\})$. Since $Z$ is reflexive, by the first part of the proof, $W_U$ is an ideal in $V_U$.

Now, $W = \{SoT : S \in K(X_1)\}$, $V = \text{span}(W, \{T\})$.

Let $M$ be a finite dimensional subspace of $V$. Then there is finite dimensional subspace $N \subseteq V_U$ such that $M = \{QoT_0 : Q \in N\}$. Let $\varepsilon > 0$. Using the Lemma 4.06, we have, there exists a linear operator $S_N : N \to W_U$ such that $\|S_N\| \leq 1+\varepsilon$.

Now, $S_N(S_1) = S_1$ for all $S_1 \in N \cap W_U$. So, $S_N(SoU) = SoU$, where $SoU \in N$ and $S \in K(X_1)$.

We define: $S_M : M \to W$ by $S_M(QoT_0) = S_N(Q)oT_0$.

Then, $\|S_M\| \leq \|S_N\| \leq 1+\varepsilon$. If $S \in K(X_1)$ with $SoT \in M$, then $SoU \in N$.

$S_M(SoT) = S_N(SoU)oT_0 = SoUoT_0 = SoT$.

Using Lemma 4.06, $W$ is an ideal in $V$. So, by Theorem 4.04, $X_1$ has the CAP. Similarly, we can show that $X_2$ also has the CAP.

We now extend our study to the MCAP in the tensor product of two Banach spaces. In [14], Cho and Johnson proved that if $X$ is a reflexive Banach space with CAP, then $X$ has the MCAP. So, if $X_1 \otimes X_2$ is reflexive, then it follows from Theorem 4.03 that the existence of the CAP in $X_1$ and
$X_2$ implies the existence of the MCAP in $X_1 \otimes y X_2$. Also, if the Banach spaces $X_1$ and $X_2$ are reflexive then the CAP in $X_1 \otimes y X_2$ implies the existence of the MCAP in $X_1$ and $X_2$. Now, naturally the question arises:

Can we establish similar results regarding the existence of MCAP for non-reflexive spaces $X_1$, $X_2$ and $X_1 \otimes y X_2$?

Our next discussion gives an affirmative answer to this problem.

In terms of the trace mapping, the criteria for the existence of MCAP in Banach spaces is similar to that of MAP (which is already discussed in our previous chapter). Here, we mention this result collected from Lima and Oja [89], relating the trace mapping and the MCAP in Banach spaces:

**Theorem 4.08** [89] Let $X$ be a Banach space and $V: X^* \otimes y X \rightarrow K(X)^*$ be the trace mapping. Then $X$ has the MCAP if and only if $I_X \in V^*(B_{K(X)^*})$.

**Corollary 4.09** [89] For a Banach space $X$, if the trace mapping $V: X^* \otimes y X \rightarrow K(X)^*$ is isometric, then $X$ has the MCAP.

Regarding existence of MCAP in the projective tensor product we prove the following analogous result as in MAP.

**Theorem 4.10** Let $X_1$ and $X_2$ be two Banach spaces having MCAP. Then $X_1 \otimes y X_2$ has also the MCAP.

**Proof.** Using Corollary 4.09, we have to show that the trace mapping, $V: (X_1 \otimes y X_2)^* \otimes y (X_1 \otimes y X_2) \rightarrow K(X_1 \otimes y X_2)^*$ is isometric.

Replacing $F$ by $K$ in the proof of the Theorem 3.03 [Chapter-3], and applying Theorem 4.02, we get the desired result. □
In 2004, Lima and Lima gave a characterization of MCAP in terms of operator ideals by the following Theorem:

**Theorem 4.11** [95] A Banach space \( X \) has the MCAP if and only if for any Banach space \( Y \), \( K(Y, X) \) is an ideal in \( L(Y, X) \).

As a converse to Theorem 4.10, we prove:

**Theorem 4.12** Let \( X_1 \) and \( X_2 \) be two Banach spaces such that their projective tensor product \( X_2 \otimes_{\gamma} X_1 \) has the MCAP. Then each of \( X_1 \) and \( X_2 \) has also MCAP.

**Proof.** Since \( X_2 \otimes_{\gamma} X_1 \) has the MCAP, so, by Theorem 4.11 for any Banach space \( Y \), \( K(Y, X_2 \otimes_{\gamma} X_1) \) is an ideal in \( L(Y, X_2 \otimes_{\gamma} X_1) \). So, there exists a Hahn-Banach extension operator:

\[
\varphi: K(Y, X_2 \otimes_{\gamma} X_1)^* \to L(Y, X_2 \otimes_{\gamma} X_1)^*
\]

such that

\[
(\varphi f^*)(f) = f^*(f) \quad \text{and} \quad \|\varphi f^*\| = \|f^*\| \quad \forall f \in K(Y, X_2 \otimes_{\gamma} X_1), f^* \in K(Y, X_2 \otimes_{\gamma} X_1)^*.
\]

It can be proved that:

(i) *Each \( S \) in \( K(Y, X_2 \otimes_{\gamma} X_1) \) gives rise to an operator \( \hat{S} \) in \( K(Y, X_1) \) such that \( \|\hat{S}\| \leq \|S\| \).*

(ii) *Corresponding to each \( T \) in \( L(Y, X_1) \) there is an operator \( T_1 \) in \( L(Y, X_2 \otimes_{\gamma} X_1) \) such that \( \|T_1\| = \|T\| \) and if \( T \) is compact, then \( T_1 \) is also compact.*

(For the proof we refer to Theorem 3.09 [Chapter-3].)
For $f_i^* \in K(Y, X_1)^*$, we define $\hat{f}_i^*(S) = f_i^*(\hat{S})$ where $S \in K(Y, X_2 \otimes X_1)$. Then, $\hat{f}_i^* \in K(Y, X_2 \otimes X_1)^*$. Let $\psi : K(Y, X_1)^* \rightarrow L(Y, X_1)^*$ be such that $(\psi f_i^*)(T) = (\phi \hat{f}_i^*)(T_i)$ where $T \in L(Y, X_1)$, $f_i^* \in K(Y, X_1)^*$.

Then, $\psi \in HB(K(Y, X_1), L(Y, X_1))$.

So, for any Banach space $Y$, $K(Y, X_1)$ is an ideal in $L(Y, X_1)$. Hence, by Theorem 4.11, $X_1$ has the MCAP. Similarly, we can prove that $X_2$ also has the MCAP. □

To show some applications of our previous results regarding CAP, we want to concentrate on the study of $M$-ideals of compact operators in the tensor product. The notion of $M$-ideals was introduced by Alfsen and Effros in order to unify certain aspects of the theory of $C^*$-algebras (refer to [109]). In 1950, Dixmier proved that for any Hilbert space $H$, $K(H)$ is an $M$-ideal in $L(H)$. However, the question: for which Banach spaces $X$, $K(X)$ is an $M$-ideal in $L(X)$ has been of general interest since the early 70's.

In [105], involving the idea of MCAP, E. Oja gave a characterization of the Banach spaces $X$ for which the space $K(X)$ is an $M$-ideal in the space $L(X)$. In [109], Paya and Werner proved that for a Banach space $X$, $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ satisfies the following strong version of the MCAP:

*There is a net $(K_a)$ in the unit ball $B_{K(X)}$ of $K(X)$ converging to the identity strongly and satisfying $\limsup_a \|K_a T_1 + (I_d - K_a)T_2\| \leq 1$ for all $T_1, T_2 \in B_{L(X)}$.*
The following Theorem and the Lemma play a specific role in proving our next result.

**Theorem 4.13** [105] Let $X$ be a Banach space. Then $K(X)$ is an $M$-ideal in $L(X)$ if and only if (i) $X$ has the MCAP, and (ii) $K(Y)$ is an $M$-ideal in $L(Y)$ for all separable subspaces $Y$ of $X$ having MCAP.

**Lemma 4.14** Let $X_1$ and $X_2$ be two arbitrary separable Banach spaces. Then $X_1 \otimes \gamma X_2$ is separable.

**Proof.** Since $X_1$ and $X_2$ are separable, there are countable dense subsets $U = \{a_1, a_2, \ldots, a_n, \ldots\}$ in $X_1$ and $V = \{b_1, b_2, \ldots, b_n, \ldots\}$ in $X_2$. We consider the set $U \otimes \gamma V$ consisting of all finite sums of the form $\sum_{i=1}^{n} a_m \otimes b_k$. Clearly, $U \otimes \gamma V$ is countable. Now, let $p$ be an arbitrary element of $X_1 \otimes \gamma X_2$. For $\varepsilon > 0$, there exists an element $q = \sum_{i=1}^{n} x_i \otimes y_i$ such that $\|q - p\| < \frac{\varepsilon}{2}$ \hspace{1cm} (1)

Since $x_i \in X_1$, $y_i \in X_2$, so there exist elements $a_m \in X_1$ and $b_k \in X_2$ such that $\|x_i - a_m\| < \frac{\varepsilon}{4nM}$ and $\|y_i - b_k\| < \frac{\varepsilon}{4nM}$,

where $M = \max\{\|x_i\|, \|y_i\| : i = 1, 2, \ldots, n\}$.

Let $r = \sum_{i=1}^{n} a_m \otimes b_k \in U \otimes \gamma V$.

Then $\|q - r\| = \left\|\sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} a_m \otimes b_k\right\|$

$= \left\|\sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} x_i \otimes b_k + \sum_{i=1}^{n} x_i \otimes b_k - \sum_{i=1}^{n} a_m \otimes b_k\right\|$
\[
\frac{\sum_{i=1}^{n} x_i \otimes (y_i - b_k)}{nM} + \frac{\sum_{i=1}^{n} (x_i - a_{m_i}) \otimes b_k}{nM} \\
\leq \frac{\sum_{i=1}^{n} x_i \|y_i - b_k\|}{nM} + \frac{\sum_{i=1}^{n} x_i \|a_{m_i}\| \|b_k\|}{nM} \\
< nM \frac{\varepsilon}{4nM} + nM \frac{\varepsilon}{4nM} = \frac{\varepsilon}{2}.
\]

So, \( \|q - r\| < \frac{\varepsilon}{2} \) ……………………………………………………………… (2)

(1) and (2) imply that \( \|p - r\| < \varepsilon \). Therefore, \( U \otimes V \) is dense in \( X_1 \otimes \gamma X_2 \) and hence \( X_1 \otimes \gamma X_2 \) is separable.

Now, we are in a position to prove:

**Theorem 4.15** Let \( X_1 \) and \( X_2 \) be two Banach spaces such that \( K(X_1 \otimes \gamma X_2) \) is an \( M \)-ideal in \( L(X_1 \otimes \gamma X_2) \). Then \( K(X_1) \) is an \( M \)-ideal in \( L(X_1) \) and \( K(X_2) \) is an \( M \)-ideal in \( L(X_2) \).

**Proof.** Since \( K(X_1 \otimes \gamma X_2) \) is an \( M \)-ideal in \( L(X_1 \otimes \gamma X_2) \), so, by Theorem 4.13, \( X_1 \otimes \gamma X_2 \) has the MCAP and for every separable subspace \( Y \) of \( X_1 \otimes \gamma X_2 \) having MCAP, \( K(Y) \) is an \( M \)-ideal in \( L(Y) \).

By Theorem 4.12, \( X_1 \otimes \gamma X_2 \) has MCAP implies both \( X_1 \) and \( X_2 \) also have MCAP. We take \( Y_i \) as a separable subspace of \( X_i \) \( (i=1,2) \) having MCAP. Then there exists a separable ideal (locally 1-complemented structure) \( Z_i \) in \( X_i \) containing \( Y_i \) \( (i=1,2) \) (refer to [95]). Since \( X_i \) has MCAP, we have, \( Z_i \) being locally 1-complemented in \( X_i \) \( (i=1,2) \) has also MCAP. Now, \( Z_1 \otimes \gamma Z_2 \) is an ideal in \( X_1 \otimes \gamma X_2 \) having MCAP (The detailed study on the ideal structure
in tensor product will be taken up in our next topic of discussion in this chapter). Now, applying Lemma 4.14, we get, $Z_1 \otimes Y_2$ is separable. By Theorem 4.13, $K(Z_1 \otimes Y_2)$ is an $M$-ideal in $L(Z_1 \otimes Y_2)$. So, there exists a net $(S_a)$ in the unit ball $B_{K(Z_1 \otimes Y_2)}$ of $K(Z_1 \otimes Y_2)$ such that $S_a \to I_d$ strongly and satisfies:

$$\limsup_a \|S_a T_1 + (I_d - S_a)T_2\| \leq 1$$
for all $T_1, T_2 \in B_{K(Z_1 \otimes Y_2)}, \ldots \ldots \ldots \ldots (1)$

Let $z_2$ be a non-zero element in $Z_2$. Let $e_2 = \frac{z_2}{\|z_2\|}$. Then $\|e_2\| = 1$. By Hahn-Banach theorem, there exists $f$ in $Z_2^*$ such that $\|f\| = 1$ and $f(z_2) = \|z_2\|$. So, $f(e_2) = 1$.

Let $T : Z_1 \otimes Y_2 \to Y_1$ be defined by

$$T\left(\sum_{n=1}^{\infty} z_{1n} \otimes z_{2n}\right) = \begin{cases} \sum_{n=1}^{\infty} f(z_{2n}) z_{1n}, & \text{if } z_{1n} \in Y_1 \\ 0, & \text{otherwise} \end{cases}$$

Then, $T \in L(Z_1 \otimes Y_2, Y_1)$ and $\|T\| \leq 1$.

We define $\hat{S}_a : Y_1 \to Y_1$ by $\hat{S}_a(y_1) = T(S_a(y_1 \otimes e_2))$, $y_1 \in Y_1$.

Then it can be shown that $\hat{S}_a \in K(Y_1)$ and $\|\hat{S}_a\| \leq 1$. Thus, $(\hat{S}_a)$ is a net in the unit ball $B_{K(Y_1)}$. For $y_1 \in Y_1$,

$$\|\hat{S}_a(y_1) - y_1\| = \|T(S_a(y_1 \otimes e_2)) - y_1f(e_2)\| = \|T(S_a(y_1 \otimes e_2)) - T(y_1 \otimes e_2)\|$$

$$\leq \|T\|\|(S_a - I_d)(y_1 \otimes e_2)\|$$

$$\leq \|S_a - I_d\|\|y_1\|$$
Since $S_a \to I_d$ strongly, so, it follows that $\hat{S}_a \to I_d$ strongly.

Let $F_1, F_2 \in B_{L(Y_1)}$. We define $\hat{F}_i : Z_1 \otimes Z_2 \to Z_1 \otimes Z_2$ by

$$\hat{F}_i(\sum_i z_i \otimes z_{2i}) = \begin{cases} 
\sum_i F_i(z_i) \otimes z_{2i}, & \text{if } z_i \in Y_1 \\
0, & \text{otherwise}
\end{cases}$$

Then $\hat{F}_1 \in B_{L(Z_1 \otimes Z_2)}$, Similarly, $\hat{F}_2 \in B_{L(Z_1 \otimes Z_2)}$.

Now, $\| (\hat{S}_a F_1 + (I_d - \hat{S}_a)F_2)(y_1)\| = \| \hat{S}_a (F_1(y_1)) + (I_d - \hat{S}_a)(F_2(y_1))\|$

$$= \| T(S_a(F_1y_1 \otimes e_2)) + F_2(y_1) - T(S_a(F_2y_1 \otimes e_2))\|

= \| T(S_a(\hat{F}_1(y_1 \otimes e_2))) + F_2(y_1) - T(\hat{F}_2(y_1 \otimes e_2)) - T(S_a(F_2y_1 \otimes e_2))\|

= \| T(S_a(\hat{F}_1(y_1 \otimes e_2))) + F_2(y_1) - T(\hat{F}_2(y_1 \otimes e_2)) - T((I_d - S_a)(\hat{F}_2(y_1 \otimes e_2)))\|

\leq \| T\| \| S_a \hat{F}_1 + (I_d - S_a)\hat{F}_2 \| \| y_1 \|

\leq \| S_a \hat{F}_1 + (I_d - S_a)\hat{F}_2 \| \| y_1 \|

Thus, $\| \hat{S}_a F_1 + (I_d - \hat{S}_a)F_2\| \leq \| S_a \hat{F}_1 + (I_d - S_a)\hat{F}_2 \|

So, $\lim sup_a \| \hat{S}_a F_1 + (I_d - \hat{S}_a)F_2\| \leq \lim sup_a \| S_a \hat{F}_1 + (I_d - S_a)\hat{F}_2 \|

\leq 1 \text{ [using (1)]}

So, $K(Y_1)$ is an $M$-ideal in $L(Y_1)$. Similarly, we can show that $K(Y_2)$ is an $M$-ideal in $L(Y_2)$. 


Now, by an application of Theorem 4.13, we have, $K(X_i)$ is an $M$-ideal in $L(X_i)$ ($i=1,2$).

**Remark 4.16** We recall that a Banach space $X$ is said to have the property $(M)$, (refer to [75], [105]) if the following condition is satisfied:

$$\limsup \|x + x_n\| = \limsup \|y + x_n\|$$

whenever $\|x\| = \|y\|$, and $(x_n)$ is a weakly null sequence in $X$. In [105], it is proved that for a Banach space $X$, $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has MCAP, contains no subspace isomorphic to $l_1$, and has property $(M)$. Combining this result with Theorem 4.15, we obtain:

**Corollary 4.17** Let $X_1$ and $X_2$ be two Banach spaces such that $K(X_1 \otimes X_2)$ is an $M$-ideal in $L(X_1 \otimes X_2)$. Then both $X_1$ and $X_2$ have MCAP, contain no subspace isomorphic to $l_1$, and have the property $(M)$.

Our next aim is to study the $M$-ideals of compact operators in $I_i \otimes I_2$, where $I_i$ is an ideal in $X_i$ ($i=1, 2$). It is known that in general, the subspace structure is not preserved when taking the projective tensor product. In 2005, Lima proved that with the subspaces that are locally complemented structures, the corresponding tensor products preserve the subspace structure, and similar results hold in case of tensor products of ideals (i.e., locally $1$-complemented structures). Regarding the projective tensor product of ideals, here we mention the following result (refer to [98]).

**Lemma 4.18** Let $X_1$ and $X_2$ be two Banach spaces and $I_1$ be an ideal in $X_1$ and $I_2$ be an ideal in $X_2$. Then $I_1 \otimes I_2$ is an ideal in $X_1 \otimes X_2$. 
Proof. Let $F$ be a finite dimensional subspace of $X_1 \otimes X_2$ and we take 
\{z_1, z_2, \ldots, z_m\} as a basis for $F \cap (I_1 \otimes I_2)$. We extend it to a basis \{z_1, z_2, \ldots, z_m\}
for $F$. Let $z_i = \sum_{j=1}^{n} u_{ij} \otimes v_{ij}, i = 1, 2, \ldots, n$ where $u_{ij} \in I_1, v_{ij} \in I_2$.

We choose finite dimensional subspaces $F_1$ of $X_1$ and $F_2$ of $X_2$ with 
\{u_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\} \subseteq F_1
and \{v_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\} \subseteq F_2.

Let $\epsilon > 0$. Let $\delta > 0$ be such that $\delta^2 + 2\delta \leq \epsilon$. Since $I_1$ is an ideal (i.e.,
locally 1-complemented) in $X_1$ and $F_1$ is a finite dimensional subspace of $X_1$, so, there is an operator $T_1 : F_1 \to I_1$ with $\|T_1\| \leq 1 + \delta$ and $T_1(a) = a$ for all $a \in F_1 \cap I_1$.

Similarly, since $I_2$ is an ideal in $X_2$, so, there exists $T_2 : F_2 \to I_2$ with
\[\|T_2\| \leq 1 + \delta\] and $T_2(b) = b$ for all $b \in F_2 \cap I_2$.

We define: $T_1 \otimes T_2 : F_1 \otimes F_2 \to I_1 \otimes I_2$ by
\[(T_1 \otimes T_2)(\sum_k a_{1k} \otimes a_{2k}) = \sum_k T_1(a_{1k}) \otimes T_2(a_{2k}).\]

Let $T = (T_1 \otimes T_2) |_F$. Then $T : F \to I_1 \otimes I_2$ is a linear operator.

Let $z \in F \cap (I_1 \otimes I_2)$. Then $z = \sum_{i=1}^{m} \alpha_i z_i = \sum_{i=1}^{m} \alpha_i (\sum_{j=1}^{n_i} u_{ij} \otimes v_{ij})$.

\[T(z) = T(\sum_{i=1}^{m} \alpha_i (\sum_{j=1}^{n_i} u_{ij} \otimes v_{ij})) = \sum_{i=1}^{m} \alpha_i (\sum_{j=1}^{n_i} T_1(u_{ij}) \otimes T_2(v_{ij}))\]
\[= \sum_{i=1}^{m} \alpha_i (\sum_{j=1}^{n_i} u_{ij} \otimes v_{ij}) = z.\]
Again, for $y = \sum x_i \otimes x_j \in F$, we have,

$$\|Ty\| = \left\| \sum_{j} T_j(x_i) \otimes T_j(x_j) \right\| \leq \sum_i \|T_i\| \|x_i\| \|T_j\| \|x_j\| = \|T_1\| \|T_2\| \sum_i \|x_i\| \|x_j\|.$$

So, $\|T\| \leq \|T_1\| \|T_2\| \leq (1 + \delta)^2 \leq 1 + \varepsilon$.

Using Lemma 4.06, $I_1 \otimes I_2$ is an ideal in $X_1 \otimes_{\gamma} X_2$.

In [14], Cho and Johnson obtained a characterization of subspaces of $l_p$ ($1 < p < \infty$) for which $K(X)$ is an $M$-ideal in $L(X)$.

**Lemma 4.19** [14] Let $X$ be a closed subspace of $(\sum X_n)_p$, $(\dim X_n < \infty)$, $1 < p < \infty$. Then $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the CAP.

Now, we give the conditions for which $K(I_1 \otimes I_2)$ is an $M$-ideal in $L(I_1 \otimes I_2)$ by the following:

**Theorem 4.20** Let $X_1$ and $X_2$ be two Banach spaces such that $X_1 \otimes_{\gamma} X_2$ has a finite dimensional Schauder decomposition with $l_p$-norm. If $I_1$ and $I_2$ are two closed ideals in $X_1$ and $X_2$ respectively such that $K(I_i)$ is an $M$-ideal in $L(I_i)$ ($i=1,2$), then $K(I_1 \otimes I_2)$ is an $M$-ideal in $L(I_1 \otimes I_2)$.

**Proof.** $K(I_i)$ is an $M$-ideal in $L(I_i)$ ($i=1, 2$). By Theorem 4.13, $I_1$ and $I_2$ have MCAP. Now, using Lemma 4.18, $I_1 \otimes I_2$ is a closed ideal in $X_1 \otimes_{\gamma} X_2$. Since $I_1$ and $I_2$ have MCAP, so, by Theorem 4.10, $I_1 \otimes I_2$ has also the MCAP. By the given condition, $X_1 \otimes_{\gamma} X_2 = (\sum Z_n)_p$, where each $Z_n$ is a finite dimensional subspace of $X_1 \otimes_{\gamma} X_2$. So, by Lemma 4.19, $K(I_1 \otimes I_2)$ is an $M$-ideal in $L(I_1 \otimes I_2)$. □
Remark 4.21 The condition: $X_1 \otimes \gamma X_2$ has a finite dimensional Schauder decomposition in the above Theorem can be replaced by the condition: $X_1 \otimes \gamma X_2$ is a $c_0$-sum of finite dimensional spaces. If $\{Z_i : i \in I\}$ is a family of Banach spaces, the $c_0$-sum $\left(\sum Z_i\right)_{c_0}$ of $\{Z_i\}$ is the Banach space of all functions $z$ on $I$ with the properties that for $i \in I$, $z(i) \in Z_i$ and for any $\varepsilon > 0$ there exists a finite set $A \subseteq I$ such that $|z(i)| < \varepsilon$ for $i \in I - A$. The norm on $\left(\sum Z_i\right)_{c_0}$ is the supremum norm (refer to [15]). In 1989, D. Werner obtained the same conclusion as in Lemma 4.19, for a closed subspace $X$ of a $c_0$-sum of finite dimensional spaces, $X$ having the MCAP. So, using this, we obtain the following analogue of Theorem 4.20.

Theorem 4.22 Let $X_1$ and $X_2$ be two Banach spaces such that $X_1 \otimes \gamma X_2$ is a $c_0$-sum of finite dimensional spaces. If $I_1$ and $I_2$ are two closed ideals in $X_1$ and $X_2$ such that $K(I_i)$ is an $M$-ideal in $L(I_i)$ ($i=1,2$), then $K(I_1 \otimes \gamma I_2)$ is an $M$-ideal in $L(I_1 \otimes \gamma I_2)$.

Our next goal is to discuss the relation between the CAP and the QAP in Banach spaces. The Theorem below establishes a relation between the AP and QAP in Banach spaces.

Theorem 4.23 [17] Let $X$ be a Banach space. Then the following statements are equivalent.

(i) $X$ has the AP.

(ii) $X$ has both the CAP and the QAP.
The following example shows that the existence of CAP does not imply the existence of QAP in general.

**Example 4.24** We consider the Willis space $Z$, which is a separable reflexive Banach space having the CAP, but failing to have the AP [138]. Now, if $Z$ has QAP, then by Theorem 4.23, it must have also the AP. So, $Z$ cannot have QAP. Thus the Willis space has CAP, but it is without QAP.

Now, the problem arises:

*Under which conditions, the existence of CAP in a Banach space implies the existence of QAP?*

Our next result gives a partial solution to this problem:

**Theorem 4.25** Let $X_1$ and $X_2$ be two Banach spaces such that their projective tensor product $X_1 \otimes_r X_2$ has the QAP. Then the CAP in $X_1$ and $X_2$ implies the QAP in both the spaces.

To prove this Theorem, first we state some enlightening results regarding AP in Banach spaces.

**Lemma 4.26** [94] Let $X$ be a Banach space. The following statements are equivalent.

(i) $X$ has the AP.

(ii) For all Banach spaces $Y$, $F(Y, X)$ is an ideal in $W(Y, X)$.

(iii) For all separable reflexive Banach spaces $Y$, $F(Y, X)$ is an ideal in $W(Y, X)$.

(iv) For all closed subspaces $Y \subset c_0, F(Y, X)$ is an ideal in $W(Y, X)$. 
(v) For all Banach spaces $Y, F(Y, X)$ is an ideal in $K(Y, X)$.

(vi) For all separable reflexive Banach spaces $Y, F(Y, X)$ is an ideal in $K(Y, X)$.

(vii) For all closed subspaces $Y \subset c_0, F(Y, X)$ is an ideal in $K(Y, X)$.

**Lemma 4.27** Let $X_1$ and $X_2$ be two Banach spaces such that their projective tensor product $X_2 \otimes_p X_1$ has the AP. Then each of $X_1$ and $X_2$ has also the AP.

**Proof.** Since $X_2 \otimes_p X_1$ has AP, so, by Lemma 4.26, for any Banach space $Y, F(Y, X_2 \otimes_p X_1)$ is an ideal in $K(Y, X_2 \otimes_p X_1)$. So, there exists a Hahn-Banach extension operator,

$$\varphi : F(Y, X_2 \otimes_p X_1)^* \to K(Y, X_2 \otimes_p X_1)^*$$

such that

$$(\varphi f^*)(f) = f^*(f) \text{ and } \|\varphi f^*\| = \|f^*\| \forall f \in F(Y, X_2 \otimes_p X_1), f^* \in F(Y, X_2 \otimes_p X_1)^*.$$ 

Now, replacing $K$ by $F$ and $L$ by $K$ in the proof of Theorem 4.12, we get the desired result. \[\square\]

**Proof of Theorem 4.25:**

Since $X_1$ and $X_2$ have CAP, so, using Theorem 4.03, $X_1 \otimes_p X_2$ has the CAP. By the given condition, $X_1 \otimes_p X_2$ has the QAP.

So, applying Theorem 4.23, $X_1 \otimes_p X_2$ has also the AP. This implies that both $X_1$ and $X_2$ have the AP (by Lemma 4.27). Now, again an application of Theorem 4.23 enables us to say that $X_1$ and $X_2$ have the QAP. \[\square\]
Remark 4.28 In [17], it is proved that if $X$ is a reflexive Banach space, then $X$ has the QAP if and only if $X^*$ has the QAP. So, in the above Theorem, if $X_1$ and $X_2$ are reflexive Banach spaces, then the CAP in $X_1$ and $X_2$ also implies the QAP in the dual spaces of $X_1$ and $X_2$.

As noted in [17], a Banach space $X$ is said to have the bounded weak approximation property (BWAP), if for every operator $T \in K(X)$, there exists a $\lambda_T > 0$, such that for every compact set $K$ in $X$ and every $\varepsilon > 0$, there is an operator $S \in F(X)$ such that $\|S\| \leq \lambda_T$ and $\|Sx - Tx\| < \varepsilon$ for all $x \in K$. If $X$ is a separable reflexive Banach space, then $X$ has the QAP if and only if $X$ has the BWAP [17]. So, in Theorem 4.25, if $X_1$ and $X_2$ are two separable reflexive Banach spaces, then the existence of CAP in $X_1$ and $X_2$ also implies the existence of BWAP in both the spaces.